MPI Young Scientists Workshop
October 27^{th}-29^{th}, 2004

Deformed Spaces and Symmetries

Frank Meyer

Max-Planck-Institut für Physik München

Universität München, Fakultät für Physik

Joined work with *Marija Dimitrijević* and *Julius Wess*

Why Noncommutative Geometry ?

- Divergences in QFT:
  Discrete space-time may lead to a finite or at least renormalisable theory (natural cutoff)

- Poor understanding of physics at short distances/high energies:
  Rich mathematical structures of deformed symmetries (Hopf algebras) give rise to new features

- Localization with extreme precision cause gravitational collapse $\Rightarrow$ space-time below Planck scale has no operational meaning
  ($\rightarrow$ K. Fredenhagen, S. Doplicher, J. E. Roberts)

- String Theory:
  Open strings in a magnetic background field
  $\longrightarrow$ endpoints of open strings move on noncommutative D-branes
  ($\rightarrow$ N. Seiberg, E. Witten)

- Loop Quantum Gravity:
  Discretization of space-time in spin-foam models

Noncommutative Spaces

- *Underlying idea:* Noncommutative Coordinates
  At very short distances: coordinates do not commute (Heisenberg 1930)

$$[\hat{x}^i, \hat{x}^j] = C^{ij}(\hat{x}) \neq 0 \tag{1}$$

- This could be valid at very short distances

- At large distance we do *not* experience Noncommutative coordinates

- $\Longrightarrow C^{ij}(\hat{x})$ must be complicated functions that vanishes at large distances

- We do not know such a function $C^{ij}(\hat{x})$ which leads to a mathematical meaningful commutation relations

- $\Longrightarrow$ Consider power series expansion of the unknown functions $C^{ij}(\hat{x})$:

$$C^{ij}(\hat{x}) = \theta^{ij} + iC^{ij}{}_k\hat{x}^k + R^{ij}{}_{kl}\hat{x}^k\hat{x}^l + \ldots$$

- Look at processes that take place at very short distances where the first terms of such an expansion lead to a good approximation for $C^{ij}(\hat{x})$

- $\Longrightarrow$ Understand in detail commutation relations which are constant or linear/quadratic in $\hat{x}^i$

The Algebra of Functions

- give up differentiable space-time **manifold**
  $\longrightarrow$ consider **algebra** of noncommutative coordinates:

$$\widehat{\mathcal{A}}_{\hat{x}} = \mathbb{C}\langle\langle \hat{x}^1, \ldots, \hat{x}^n \rangle\rangle / ([\hat{x}^i, \hat{x}^j] - C^{ij}(\hat{x}))$$

- Roughly speaking, these are "functions" that respect the commutation relations (1)

- The commutative analogue: Take all formal power series generated by the coordinates x^i:

$$\mathbb{C}\langle\langle x^1, \ldots, x^n \rangle\rangle \,.$$

- In the commutative setting, the coordinates satisfy the commutation relations

$$[x^i, x^j] = 0 \,.$$

- Devide by the Ideal generated by the above commutation relations

$$A_x = \mathbb{C}\langle\langle x^1, \ldots, x^n \rangle\rangle / ([x^i, x^j] - 0) = \mathbb{C}[[x^1, \ldots, x^n]] \,.$$

- "Good" examples (Poincare Birkhoff Witt property): Freely generated algebras with the following commutation relations

1. canonical structure: $[\hat{x}^i, \hat{x}^j] = i\theta^{ij}$
2. Lie algebra structure: $[\hat{x}^i, \hat{x}^j] = iC^{ij}{}_k \hat{x}^k$
   (e.g. $\kappa-$*Deformed Minkowski Spacetime*)
3. Quantum Space structure: $\hat{x}^i \hat{x}^j = q\hat{R}^{ij}{}_{kl} \hat{x}^k \hat{x}^l$

Symmetries

- ✘

  over to noncommutative algebras

- ✘ $^{i}, \hat{x}^{j}] = i\theta^{ij}$ breaks Lorentz symmetry.

$$\Downarrow$$

- ✘

Commutative Space + *Commutative* Symmetry	$\overset{\text{Deformation}}{\Longrightarrow}$	*Deformed* Space + *Deformed* Symmetry

- ✘

  on a Lie-group within the category of *Hopf algebras.*

- ✘

  metries, i.e. we have a deformed symmetry acting on the deformed space

Canonically Deformed Spaces

✘

$$[\hat{x}^i, \hat{x}^j] = i\theta^{ij},\ \theta^{ij} = \text{const}$$

breaks Lorentz-symmetry.

✘

deformed space was known.

✘

$\theta-$DEFORMED POINCARÉ BIALGEBRA

$$\begin{aligned}
[\hat{\partial}_\mu, \hat{\partial}_\nu] &= 0, \quad [\hat{\delta}_\omega, \hat{\partial}_\rho] = \omega_\rho{}^\mu \hat{\partial}_\mu, \\
[\hat{\delta}_\omega, \hat{\delta}'_\omega] &= \hat{\delta}_{\omega \times \omega'}, \quad (\omega \times \omega)'_\mu{}^\nu = -\left(\omega_\mu{}^\sigma \omega'_\sigma{}^\nu - \omega'_\mu{}^\sigma \omega_\sigma{}^\nu\right), \\
\Delta \hat{\partial}_\mu &= \hat{\partial}_\mu \otimes \mathbf{1} + \mathbf{1} \otimes \hat{\partial}_\mu, \\
\Delta \hat{\delta}_\omega &= \hat{\delta}_\omega \otimes \mathbf{1} + \mathbf{1} \otimes \hat{\delta}_\omega + \frac{i}{2}\left(\theta^{\mu\nu}\omega_\nu{}^\rho - \theta^{\rho\nu}\omega_\nu{}^\mu\right)\hat{\partial}_\rho \otimes \hat{\partial}_\mu.
\end{aligned}$$

Derivatives

✘

$$\hat{\partial}^{\widehat{\mathcal{A}}} : \widehat{\mathcal{A}}_{\hat{x}} \to \widehat{\mathcal{A}}_{\hat{x}} \,.$$

Thus, they have to be consistent with commutation relations of the coordinates.

✘

$$[\hat{\partial}^{\widehat{\mathcal{A}}}_\mu, \hat{x}^\nu] = \delta^\nu_\mu + \sum_j A_\mu^{\;\nu\rho_1\ldots\rho_j} \hat{\partial}_{\rho_1} \ldots \hat{\partial}_{\rho_j} \,.$$

✘

duced by requiring that the derivatives should be a module with respect to the deformed background symmetry

✘

$$\hat{\partial}_\mu(\hat{f}\hat{g}) = \hat{\partial}_\mu(\hat{f})\hat{g} + \hat{O}_\mu{}^\nu(\hat{f})(\hat{\partial}_\nu \hat{g}) \,.$$

✘ $\hat{\partial}_i$ act as in the commutative setting and satisfy the usual Leibniz rule

$$\begin{aligned} [\hat{\partial}_\mu, \hat{x}^\nu] &= \delta_\mu^\nu \\ \hat{\partial}_\mu(\hat{f}\hat{g}) &= \hat{\partial}_\mu(\hat{f})\hat{g} + \hat{f}\hat{\partial}_\mu(\hat{g}) \,. \end{aligned}$$

Towards a Physical Theory

✘

abstract algebra to *complex numbers.*

✘

1. Study Representations of the NC algebra (cp. QM)

2. Star product approach and Seiberg-Witten map:

Represent the noncommutative algebra on the algebra of commutative functions by a *star product* (next slide) and express noncommutative fields in terms of commutative ones by *Seiberg-Witten map*

Star Products

- Vector space of formal power series in commutative coordinates is isomorphic to the vector space of formal power series in noncommutative coordinates

$$\begin{aligned} \rho : \mathbb{R}[[x^0, \ldots, x^n]] &\rightarrow \widehat{\mathcal{A}} \\ f(x^\mu) &\mapsto \hat{f}(\hat{x}^\mu) \end{aligned}$$

- To transmit the noncommutativity to the algebra of commutative functions we define a *new product*, called **star product** by pulling back the product of the noncommutative algebra:

$$f(x^\mu) \star g(x^\mu) := \rho^{-1}(\hat{f}(\hat{x}^\mu) \cdot \hat{g}(\hat{x}^\mu))$$

- The star product is *not* unique since the isomorphism ρ (called *ordering prescription*) is not unique

- *Example:* Canonical Structure $[\hat{x}^i, \hat{x}^j] = i\theta^{ij}$
  MOYAL-WEYL PRODUCT

$$f \star g = \mu \circ e^{i\theta^{ij}\partial_i \otimes \partial_j}(f \otimes g) = fg + \frac{i}{2}\theta^{ij}(\partial_i f)(\partial_j g) + \ldots,$$

where $\mu(f \otimes g) := fg$ is just the multiplication map.

Seiberg-Witten Map

✘ in terms of commutative ones

$$A_i = A_i[a_j],\ \psi = \psi[\phi, a_i],\ \Lambda = \Lambda_\alpha[a_i]\,.$$

✘

✘ explicitly read off in orders of h where in the limit $h \to 0$ the commutative theory is obtained

1. Requirement: commutative gauge transformations induce noncommutative ones, i.e.

$$\begin{aligned} A_i[a_i] + \delta A_i[a_i] &= A_i[a_i + \delta a_i] \\ \psi[\phi, a_i] + \delta\psi[\phi, a_i] &= \psi[\phi + \delta\phi, a_i + \delta a_i] \end{aligned}$$

2. Requirement: consistency condition

$$\begin{aligned} & (\delta_\alpha \delta_\beta - \delta_\beta \delta_\alpha)\psi &&= \delta_{-i[\alpha,\beta]}\psi \\ \Leftrightarrow\ & i\delta_\alpha \Lambda_\beta - i\delta_\beta \Lambda_\alpha + [\Lambda_\alpha \stackrel{\star}{,} \Lambda_\beta] &&= i\Lambda_{-i[\alpha,\beta]} \end{aligned}$$

Summary

- Concept of deformed spaces:
  spacetime is *discrete* at very *short distances* resp. at *high energies*

- differentiable manifold $\Longrightarrow$noncommutative algebra
  commutative product $\Longrightarrow$ star product
  Leibniz rule $\Longrightarrow$ deformed Leibniz rule
  (important for gauge theory via Seiberg-Witten map)

- We found a $\theta-$Deformed Poincaré Algebra as symmtery for the canonical case where $[\hat{x}^i, \hat{x}^j] = i\theta^{ij}$

Outlook

- Study implications of the new symmetry for canonically deformed spaces for Noncommutative Gauge Field Theories
- Formulate a theory of Gravity on Noncommutative Spaces
- Study phenomenological consequences of noncommutativity

Noncommutative Spaces ⇒ Interesting concepts for a better understanding of physics at short distances