# Exercises for Theorists

## (Topics in perturbative higher-order calculations)

Stefan Dittmaier (part I) and Thomas Hahn (part II)

MPI München

## 1 Introduction

### Comparison of the Standard Model (SM) and QED:

| | QED | SM |
|---|---|---|
| matter fields (spin $\frac{1}{2}$) | $\mathrm{e}^{\pm}$ | leptons + quarks |
| gauge symmetry | $\mathrm{U(1)_{em}}$ | $\mathrm{SU(2)\times U(1)}$ |
| $\rightarrow$ gauge bosons (spin 1) | $\gamma$ | $\gamma, \mathrm{Z}^0, \mathrm{W}^{\pm}$ |

### Differences to QED:

- non-abelian gauge group
  $\rightarrow$ gauge-boson self-interactions
- spontaneous symmetry breaking $\mathrm{SU(2)\times U(1)\rightarrow U(1)_{em}}$
  $\rightarrow$ massive gauge bosons $\mathrm{Z}^0$, $\mathrm{W}^{\pm}$
  Higgs boson $\mathrm{H}$ (spin 0)

$\Rightarrow$ Common description of electromagnetic and weak interactions

## 1 Introduction

Comparison of the Standard Model (SM) and QED:

| | QED | SM |
|---|---|---|
| matter fields (spin $\frac{1}{2}$) | $e^\pm$ | leptons + quarks |
| gauge symmetry | $U(1)_{em}$ | $SU(3) \times SU(2) \times U(1)$ |
| $\rightarrow$ gauge bosons (spin 1) | $\gamma$ | $\gamma$, $Z^0$, $W^\pm$ , g |

Differences to QED:

- non-abelian gauge group

  $\rightarrow$ gauge-boson self-interactions
- spontaneous symmetry breaking $SU(2) \times U(1) \rightarrow U(1)_{em}$

  $\rightarrow$ massive gauge bosons $Z^0$, $W^\pm$

  Higgs boson $H$ (spin 0)

$\Rightarrow$ Common description of electromagnetic and weak interactions as well as strong interactions

## Starting point: model formulated as quantum field theory

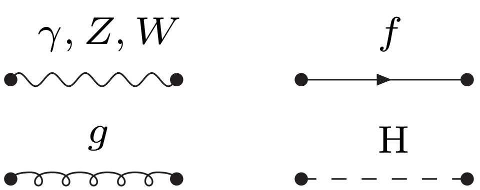
- each particle corresponds to a field $\phi_i$
- Lagrangian $\mathcal{L}(\phi_i)$ for free motion & interactions

## Perturbative evaluation of quantum field theories

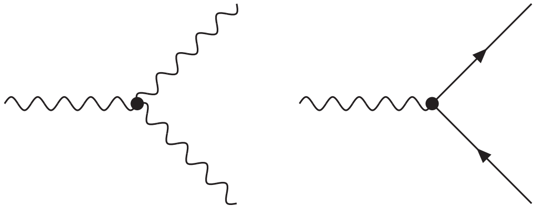
Transition amplitude $\langle f|S|i\rangle \;=\; \sum$ Feynman graphs for $|i\rangle \;\to\; |f\rangle$

Form graphs following Feynman rules:

free propagators:




vertices = elementary interactions:



etc.

propagators & vertices $\quad\longleftrightarrow\quad$ terms in $\mathcal{L}(\phi_i)$

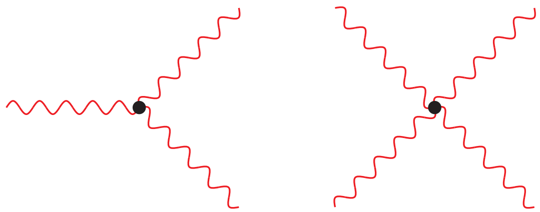
Perturbative series for $g \to 0$

= power series in $g^n$
= power series in $\hbar^m$
= expansion in # loops in diagrams

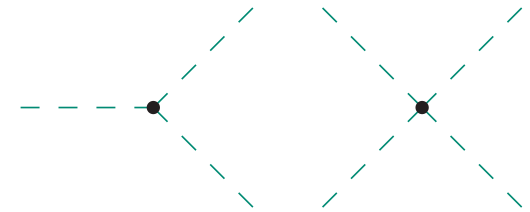
$\Rightarrow$ loop diagrams = quantum corrections

# Elementary couplings of electroweak interactions:

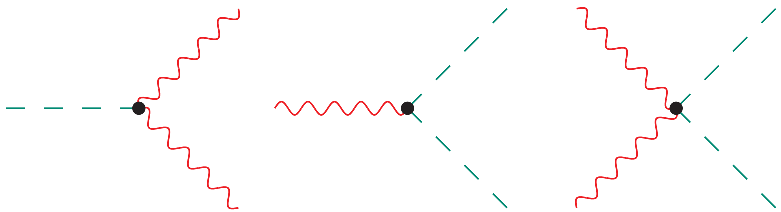
gauge-boson self-couplings:



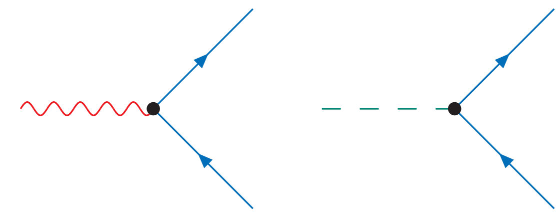
Higgs self-couplings:



gauge-boson–Higgs couplings:

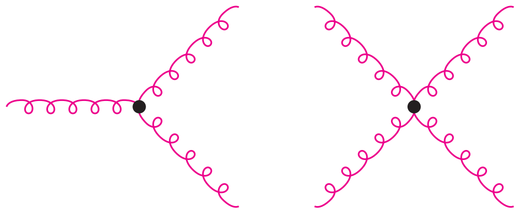


fermion couplings:

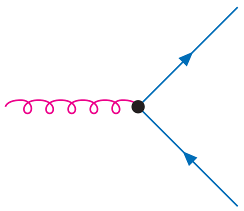


# Elementary couplings of strong interactions:

gluon self-couplings:



quark–gluon coupling:



## Aim of the exercise block

$\hookrightarrow$ give some idea about higher-order calculations via simple examples

Specifically: Z-boson decay $\mathrm{Z} \to f\bar{f}$

Exercises include

- LO prediction (tree level)
- NLO QED correction
  - ◇ real photon bremsstrahlung
    $\hookrightarrow$ soft and collinear singularities
  - ◇ virtual one-loop correction
    $\hookrightarrow$ UV singularities and renormalization
- NLO QCD correction (derived from the QED case)

## 2 Fermion–gauge-boson sector of the SM and Z decay in lowest order

### Fermion content of the SM:

(ignoring possible right-handed neutrinos)

| | | | | | $T_{\mathrm{I}}^3$ | $Q$ |
|---|---|---|---|---|---|---|
| leptons: | $\Psi_L^{\mathrm{L}} =$ | $\begin{pmatrix} \nu_{\mathrm{e}}^{\mathrm{L}} \\ \mathrm{e}^{\mathrm{L}} \end{pmatrix},$ | $\begin{pmatrix} \nu_{\mu}^{\mathrm{L}} \\ \mu^{\mathrm{L}} \end{pmatrix},$ | $\begin{pmatrix} \nu_{\tau}^{\mathrm{L}} \\ \tau^{\mathrm{L}} \end{pmatrix},$ | $+\frac{1}{2}$ <br> $-\frac{1}{2}$ | $0$ <br> $-1$ |
| | $\psi_l^{\mathrm{R}} =$ | $\mathrm{e}^{\mathrm{R}},$ | $\mu^{\mathrm{R}},$ | $\tau^{\mathrm{R}},$ | $0$ | $-1$ |
| quarks: (Each quark exists in 3 colours!) | $\Psi_Q^{\mathrm{L}} =$ | $\begin{pmatrix} \mathrm{u}^{\mathrm{L}} \\ \mathrm{d}^{\mathrm{L}} \end{pmatrix},$ | $\begin{pmatrix} \mathrm{c}^{\mathrm{L}} \\ \mathrm{s}^{\mathrm{L}} \end{pmatrix},$ | $\begin{pmatrix} \mathrm{t}^{\mathrm{L}} \\ \mathrm{b}^{\mathrm{L}} \end{pmatrix},$ | $+\frac{1}{2}$ <br> $-\frac{1}{2}$ | $+\frac{2}{3}$ <br> $-\frac{1}{3}$ |
| | $\psi_u^{\mathrm{R}} =$ | $\mathrm{u}^{\mathrm{R}},$ | $\mathrm{c}^{\mathrm{R}},$ | $\mathrm{t}^{\mathrm{R}},$ | $0$ | $+\frac{2}{3}$ |
| | $\psi_d^{\mathrm{R}} =$ | $\mathrm{d}^{\mathrm{R}},$ | $\mathrm{s}^{\mathrm{R}},$ | $\mathrm{b}^{\mathrm{R}},$ | $0$ | $-\frac{1}{3}$ |

Left- and right-handed parts of fermions interact differently:

$\psi^{\mathrm{L}} = \omega_-\psi$, $\psi^{\mathrm{R}} = \omega_+\psi$, $\qquad \omega_\pm = \frac{1}{2}(1 \pm \gamma_5)$

- $\psi^{\mathrm{L}}$ couple to $\mathrm{W}^\pm$ $\quad\rightarrow$ group $\psi^{\mathrm{L}}$ into SU(2)$_{\mathrm{I}}$ doublets, weak isospin $T_{\mathrm{I}}^a = \frac{\sigma^a}{2}$
- $\psi^{\mathrm{R}}$ do not couple to $\mathrm{W}^\pm$ $\quad\rightarrow$ $\psi^{\mathrm{R}}$ are SU(2)$_{\mathrm{I}}$ singlets, weak isospin $T_{\mathrm{I}}^a = 0$
- $\psi^{\mathrm{L/R}}$ couple to $\gamma$ in the same way

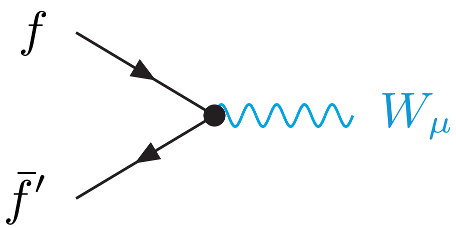
# Fermion–gauge-boson interaction:

$$\mathcal{L}_{\mathrm{ferm,YM}} = \frac{e}{\sqrt{2}s_\mathrm{W}} \overline{\Psi^\mathrm{L}_F} \begin{pmatrix} 0 & \not{W}^+ \\ \not{W}^- & 0 \end{pmatrix} \Psi^\mathrm{L}_F + \frac{e}{2c_\mathrm{W}s_\mathrm{W}} \overline{\Psi^\mathrm{L}_F} \sigma^3 \not{Z} \Psi^\mathrm{L}_F$$

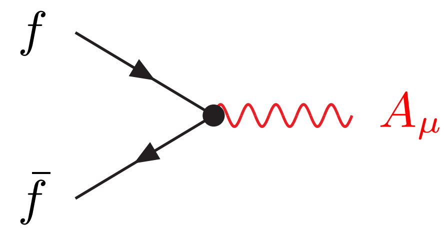
$$- e\frac{s_\mathrm{W}}{c_\mathrm{W}} Q_f \overline{\psi_f} \not{Z} \psi_f - eQ_f \overline{\psi_f} \not{A} \psi_f$$

($f$=all fermions, $F$= all doublets)

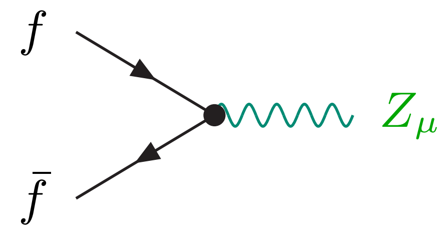
Feynman rules:




$$\frac{\mathrm{i}e}{\sqrt{2}s_\mathrm{W}} \gamma_\mu \omega_-$$




$$-\mathrm{i}Q_f e \gamma_\mu$$



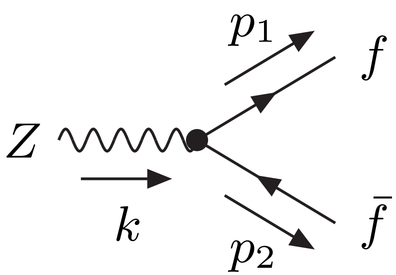

$$\mathrm{i}e\gamma_\mu (g_f^+ \omega_+ + g_f^- \omega_-)$$

with $g_f^+ = -\frac{s_\mathrm{W}}{c_\mathrm{W}} Q_f, \quad g_f^- = -\frac{s_\mathrm{W}}{c_\mathrm{W}} Q_f + \frac{T^3_{\mathrm{I},f}}{c_\mathrm{W} s_\mathrm{W}},$

$$c_\mathrm{W} = \frac{M_\mathrm{W}}{M_\mathrm{Z}} \approx 0.88, \qquad s_\mathrm{W} = \sqrt{1 - c_\mathrm{W}^2} \approx 0.47$$

## Z-boson decay in lowest order

Tree-level diagram:




Born amplitude $\mathcal{M}_0$ (approximation of massless $f$)

$$\mathrm{i}\mathcal{M}_0 = \bar{u}_f(p_1,\sigma_f)\,\mathrm{i}eg_f^\sigma\gamma_\mu\omega_\sigma\, v_{\bar{f}}(p_2,\sigma_{\bar{f}})\,\varepsilon_{\mathrm{Z}}^\mu(k,\lambda_{\mathrm{Z}})$$

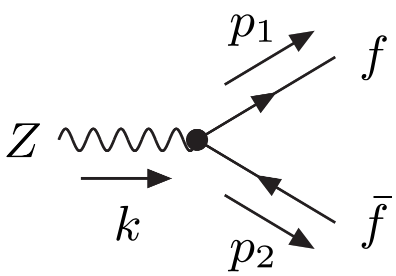
with $k^2=(p_1+p_2)^2=M_{\mathrm{Z}}^2,\quad p_1^2=p_2^2=0$

### Spin-/colour-averaged amplitude squared:

$$\langle|\mathcal{M}_0|^2\rangle = \frac{1}{3}\sum_{\lambda_{\mathrm{Z}}=0,\pm1}\;\sum_{\sigma_f=\pm\frac{1}{2}}\;\sum_{\sigma_{\bar{f}}=\pm\frac{1}{2}}\;\sum_{\mathrm{colour}}|\mathcal{M}_0|^2$$

## Z-boson decay in lowest order

Tree-level diagram:




Born amplitude $\mathcal{M}_0$ (approximation of massless $f$)

$$\mathrm{i}\mathcal{M}_0 \;=\; \bar{u}_f(p_1,\sigma_f)\,\mathrm{i}eg_f^\sigma\gamma_\mu\omega_\sigma\; v_{\bar{f}}(p_2,\sigma_{\bar{f}})\;\varepsilon_{\mathrm{Z}}^\mu(k,\lambda_{\mathrm{Z}})$$

with $\quad k^2=(p_1+p_2)^2=M_{\mathrm{Z}}^2, \quad p_1^2=p_2^2=0$

### Spin-/colour-averaged amplitude squared:

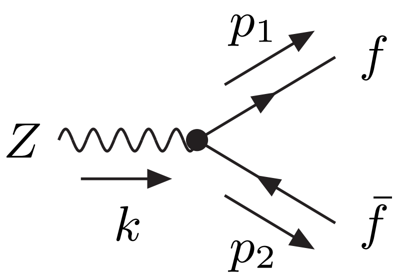
$$\langle|\mathcal{M}_0|^2\rangle \;=\; \frac{1}{3}\sum_{\lambda_{\mathrm{Z}}=0,\pm1}\;\sum_{\sigma_f=\pm\frac{1}{2}}\;\sum_{\sigma_{\bar{f}}=\pm\frac{1}{2}}\;\sum_{\mathrm{colour}}|\mathcal{M}_0|^2$$

colour factors: $N_{\mathrm{c}}^l=1,\; N_{\mathrm{c}}^q=3$

$$=\; \frac{1}{3}\,N_{\mathrm{c}}^f\,e^2 g_f^\sigma g_f^\tau \sum_{\sigma_f}\bar{u}_f\;\gamma_\mu\omega_\sigma\;\underbrace{\sum_{\sigma_{\bar{f}}} v_{\bar{f}}\,\bar{v}_{\bar{f}}}_{=\not{p}_2}\;\omega_{-\tau}\;\gamma_\nu\;u_f\qquad\underbrace{\sum_{\lambda}\varepsilon_{\mathrm{Z}}^\mu\varepsilon_{\mathrm{Z}}^{*\nu}}_{=-g^{\mu\nu}+k^\mu k^\nu/M_{\mathrm{Z}}^2}$$

## Z-boson decay in lowest order

Tree-level diagram:




Born amplitude $\mathcal{M}_0$ (approximation of massless $f$)

$$\mathrm{i}\mathcal{M}_0 = \bar{u}_f(p_1,\sigma_f)\,\mathrm{i}eg_f^\sigma\gamma_\mu\omega_\sigma\, v_{\bar{f}}(p_2,\sigma_{\bar{f}})\,\varepsilon_\mathrm{Z}^\mu(k,\lambda_\mathrm{Z})$$

with $k^2 = (p_1+p_2)^2 = M_\mathrm{Z}^2, \quad p_1^2 = p_2^2 = 0$

### Spin-/colour-averaged amplitude squared:

$$\langle|\mathcal{M}_0|^2\rangle = \frac{1}{3}\sum_{\lambda_\mathrm{Z}=0,\pm1}\sum_{\sigma_f=\pm\frac{1}{2}}\sum_{\sigma_{\bar{f}}=\pm\frac{1}{2}}\sum_{\text{colour}}|\mathcal{M}_0|^2$$

colour factors: $N_\mathrm{c}^l = 1,\; N_\mathrm{c}^q = 3$

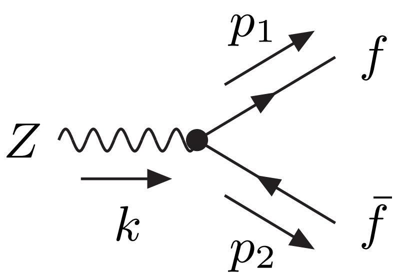
$$= \frac{1}{3}N_\mathrm{c}^f\, e^2 g_f^\sigma g_f^\tau \sum_{\sigma_f}\bar{u}_f\,\gamma_\mu\omega_\sigma \underbrace{\sum_{\sigma_{\bar{f}}} v_{\bar{f}}\,\bar{v}_{\bar{f}}}_{=\not{p}_2}\,\omega_{-\tau}\,\gamma_\nu\,u_f \quad \underbrace{\sum_\lambda \varepsilon_\mathrm{Z}^\mu\varepsilon_\mathrm{Z}^{*\nu}}_{=-g^{\mu\nu}+k^\mu k^\nu/M_\mathrm{Z}^2}$$

only $g^{\mu\nu}$ term contributes

$$= -\frac{1}{3}N_\mathrm{c}^f\, e^2 g_f^\sigma g_f^\tau\, \mathrm{Tr}\{\not{p}_1\,\gamma_\mu\,\omega_\sigma\,\not{p}_2\,\omega_{-\tau}\,\gamma^\mu\}$$

## Z-boson decay in lowest order

Tree-level diagram:




Born amplitude $\mathcal{M}_0$ (approximation of massless $f$)

$$\mathrm{i}\mathcal{M}_0 = \bar u_f(p_1,\sigma_f)\, \mathrm{i}e g_f^\sigma \gamma_\mu \omega_\sigma\, v_{\bar f}(p_2,\sigma_{\bar f})\, \varepsilon_{\mathrm{Z}}^\mu(k,\lambda_{\mathrm{Z}})$$

with $\quad k^2 = (p_1+p_2)^2 = M_{\mathrm{Z}}^2, \quad p_1^2 = p_2^2 = 0$

### Spin-/colour-averaged amplitude squared:

$$\langle|\mathcal{M}_0|^2\rangle = \frac{1}{3} \sum_{\lambda_{\mathrm{Z}}=0,\pm1} \sum_{\sigma_f=\pm\frac12} \sum_{\sigma_{\bar f}=\pm\frac12} \sum_{\mathrm{colour}} |\mathcal{M}_0|^2$$

colour factors: $N_{\mathrm{c}}^l = 1,\ N_{\mathrm{c}}^q = 3$

$$= \frac{1}{3} N_{\mathrm{c}}^f\, e^2 g_f^\sigma g_f^\tau \sum_{\sigma_f} \bar u_f\, \gamma_\mu \omega_\sigma \underbrace{\sum_{\sigma_{\bar f}} v_{\bar f}\, \bar v_{\bar f}}_{=\not p_2}\, \omega_{-\tau}\, \gamma_\nu\, u_f \quad \underbrace{\sum_{\lambda} \varepsilon_{\mathrm{Z}}^\mu \varepsilon_{\mathrm{Z}}^{*\nu}}_{=-g^{\mu\nu}+k^\mu k^\nu/M_{\mathrm{Z}}^2}$$

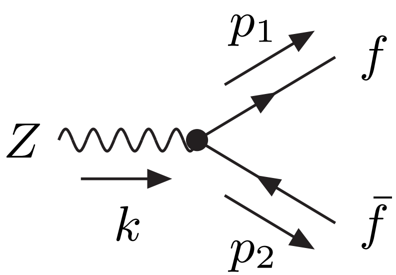
only $g^{\mu\nu}$ term contributes

$$= -\frac{1}{3} N_{\mathrm{c}}^f\, e^2 g_f^\sigma g_f^\tau\, \mathrm{Tr}\{\not p_1\, \gamma_\mu\, \omega_\sigma\, \not p_2\, \omega_{-\tau}\, \gamma^\mu\}$$

$$= \frac{2}{3} N_{\mathrm{c}}^f\, e^2 (g_f^\sigma)^2\, \mathrm{Tr}\{\not p_1\, \not p_2\, \omega_{-\sigma}\}$$

## Z-boson decay in lowest order

Tree-level diagram:




Born amplitude $\mathcal{M}_0$ (approximation of massless $f$)

$$\mathrm{i}\mathcal{M}_0 = \bar u_f(p_1,\sigma_f)\, \mathrm{i}eg_f^\sigma \gamma_\mu \omega_\sigma\, v_{\bar f}(p_2,\sigma_{\bar f})\, \varepsilon_Z^\mu(k,\lambda_Z)$$

with $k^2 = (p_1+p_2)^2 = M_Z^2, \quad p_1^2 = p_2^2 = 0$

### Spin-/colour-averaged amplitude squared:

$$\langle|\mathcal{M}_0|^2\rangle = \frac{1}{3} \sum_{\lambda_Z=0,\pm1} \sum_{\sigma_f=\pm\frac12} \sum_{\sigma_{\bar f}=\pm\frac12} \sum_{\text{colour}} |\mathcal{M}_0|^2$$

colour factors: $N_c^l = 1, \; N_c^q = 3$

$$= \frac{1}{3} N_c^f\, e^2 g_f^\sigma g_f^\tau \sum_{\sigma_f} \bar u_f\, \gamma_\mu \omega_\sigma \underbrace{\sum_{\sigma_{\bar f}} v_{\bar f}\, \bar v_{\bar f}}_{=\not p_2}\, \omega_{-\tau}\, \gamma_\nu\, u_f \quad \underbrace{\sum_\lambda \varepsilon_Z^\mu \varepsilon_Z^{*\nu}}_{=-g^{\mu\nu}+k^\mu k^\nu/M_Z^2}$$

only $g^{\mu\nu}$ term contributes

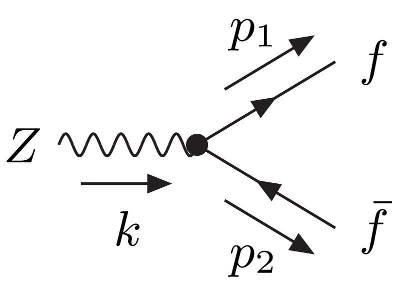
$$= -\frac{1}{3} N_c^f\, e^2 g_f^\sigma g_f^\tau\, \mathrm{Tr}\{\not p_1\, \gamma_\mu\, \omega_\sigma\, \not p_2\, \omega_{-\tau}\, \gamma^\mu\}$$

$$= \frac{2}{3} N_c^f\, e^2 (g_f^\sigma)^2\, \mathrm{Tr}\{\not p_1\, \not p_2\, \omega_{-\sigma}\}$$

$$= \frac{4}{3} N_c^f\, e^2 (g_f^\sigma)^2\, (p_1\cdot p_2)$$

## Z-boson decay in lowest order

Tree-level diagram:




Born amplitude $\mathcal{M}_0$ (approximation of massless $f$)

$$\mathrm{i}\mathcal{M}_0 = \bar u_f(p_1,\sigma_f)\,\mathrm{i}eg_f^\sigma\gamma_\mu\omega_\sigma\, v_{\bar f}(p_2,\sigma_{\bar f})\,\varepsilon_Z^\mu(k,\lambda_Z)$$

with $k^2=(p_1+p_2)^2=M_Z^2, \quad p_1^2=p_2^2=0$

### Spin-/colour-averaged amplitude squared:

$$\langle|\mathcal{M}_0|^2\rangle = \frac{1}{3}\sum_{\lambda_Z=0,\pm1}\sum_{\sigma_f=\pm\frac12}\sum_{\sigma_{\bar f}=\pm\frac12}\sum_{\text{colour}}|\mathcal{M}_0|^2$$

colour factors: $N_c^l=1,\ N_c^q=3$

$$= \frac13 N_c^f\, e^2 g_f^\sigma g_f^\tau \sum_{\sigma_f}\bar u_f\,\gamma_\mu\omega_\sigma \underbrace{\sum_{\sigma_{\bar f}} v_{\bar f}\bar v_{\bar f}}_{=\not p_2}\,\omega_{-\tau}\,\gamma_\nu\,u_f \underbrace{\sum_\lambda \varepsilon_Z^\mu\varepsilon_Z^{*\nu}}_{=-g^{\mu\nu}+k^\mu k^\nu/M_Z^2}$$

only $g^{\mu\nu}$ term contributes

$$= -\frac13 N_c^f\, e^2 g_f^\sigma g_f^\tau\, \mathrm{Tr}\{\not p_1\,\gamma_\mu\,\omega_\sigma\,\not p_2\,\omega_{-\tau}\,\gamma^\mu\}$$

$$= \frac23 N_c^f\, e^2 (g_f^\sigma)^2\, \mathrm{Tr}\{\not p_1\,\not p_2\,\omega_{-\sigma}\}$$

$$= \frac43 N_c^f\, e^2 (g_f^\sigma)^2\,(p_1\cdot p_2)$$

$$= \frac23 N_c^f\, e^2\left[(g_f^+)^2+(g_f^-)^2\right] M_Z^2$$

## Lowest-order partial Z-decay width

$$\Gamma_{Z\to f\bar{f},0} = \frac{1}{2M_Z}\int d\Phi_{1\to 2}\,\langle|\mathcal{M}_0|^2\rangle$$

with the $1 \to 2$ phase-space integral

$$\int d\Phi_{1\to 2} = \int \frac{d^3\mathbf{p}_1}{(2\pi)^3 2p_1^0}\int \frac{d^3\mathbf{p}_2}{(2\pi)^3 2p_2^0}(2\pi)^4\,\delta^{(4)}(k-p_1-p_2)$$

$$= \frac{1}{8(2\pi)^2}\underbrace{\int d\Omega_{p_1}}_{\hookrightarrow\, 4\pi}$$

### Final result:

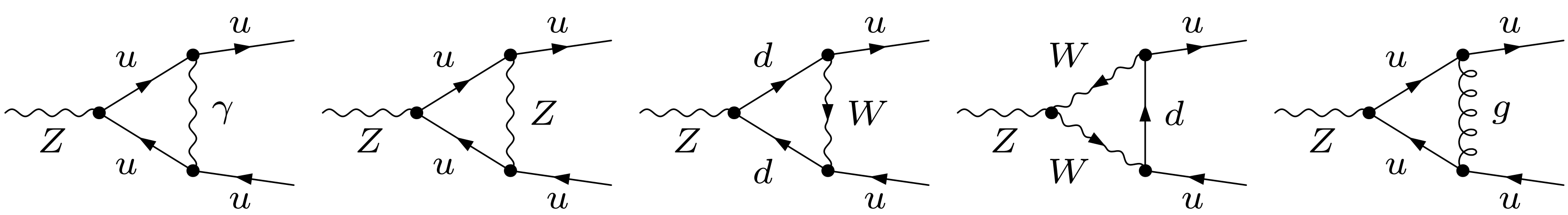
$$\Gamma_{Z\to f\bar{f},0} = \frac{N_c^f\, e^2}{24\pi}\left[(g_f^+)^2+(g_f^-)^2\right]M_Z = \frac{N_c^f\,\alpha}{6}\left[(g_f^+)^2+(g_f^-)^2\right]M_Z$$

Comments:

- LO result has theoretical uncertainty of some $\%$
- LEP accuracy for $\Gamma_{Z,\mathrm{tot}}$ is $\sim 0.1\%$ !
  $\hookrightarrow$ calculation of higher orders indispensable

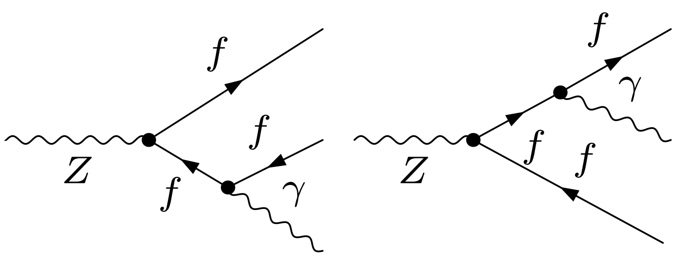
## Survey of NLO corrections

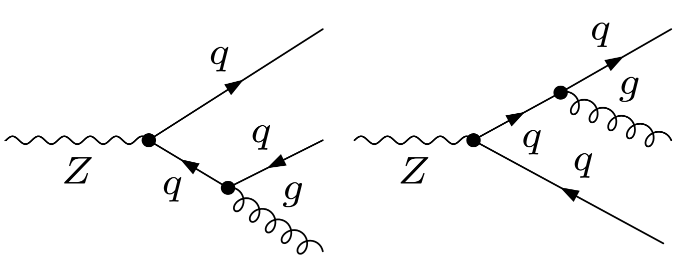
Virtual one-loop corrections (example $f = u$-quark)




+ many self-energy diagrams needed in the renormalization

Real corrections from photon or gluon emission




Total NLO corrections

$\hookrightarrow$ virtual $\oplus$ real contributions yield corrections of relative orders $\mathcal{O}(\alpha)$ and $\mathcal{O}(\alpha_{\rm s})$

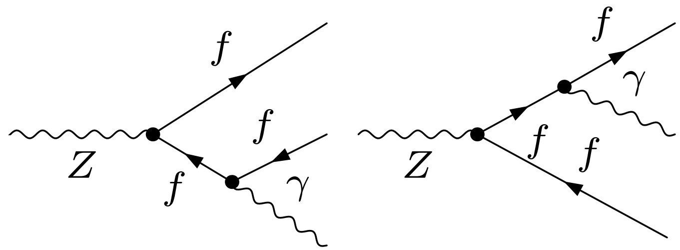
Comment:

To match the aimed $0.1\%$ accuracy even corrections beyond NLO are required.

## 3 Real photon corrections

### 3.1 Non-singular contributions

Amplitudes for $\mathrm{Z}(k,\lambda_\mathrm{Z}) \;\to\; f(p_1,\sigma_f) \,+\, \bar{f}(p_2,\sigma_{\bar{f}}) \,+\, \gamma(q,\lambda_\gamma)$ in lowest order




$$k^2 = (p_1+p_2+q)^2 = M_\mathrm{Z}^2, \qquad p_1^2 = p_2^2 = q^2 = 0$$

$$\mathrm{i}\mathcal{M}_{\gamma,1} \;=\; \bar{u}_f \;\mathrm{i}e g_f^\sigma \not{\varepsilon}_\mathrm{Z}\omega_\sigma \,\frac{-\mathrm{i}(\not{p}_2+\not{q})}{(p_2+q)^2}\,(-\mathrm{i}Q_f e)\not{\varepsilon}_\gamma^* \; v_{\bar{f}}$$

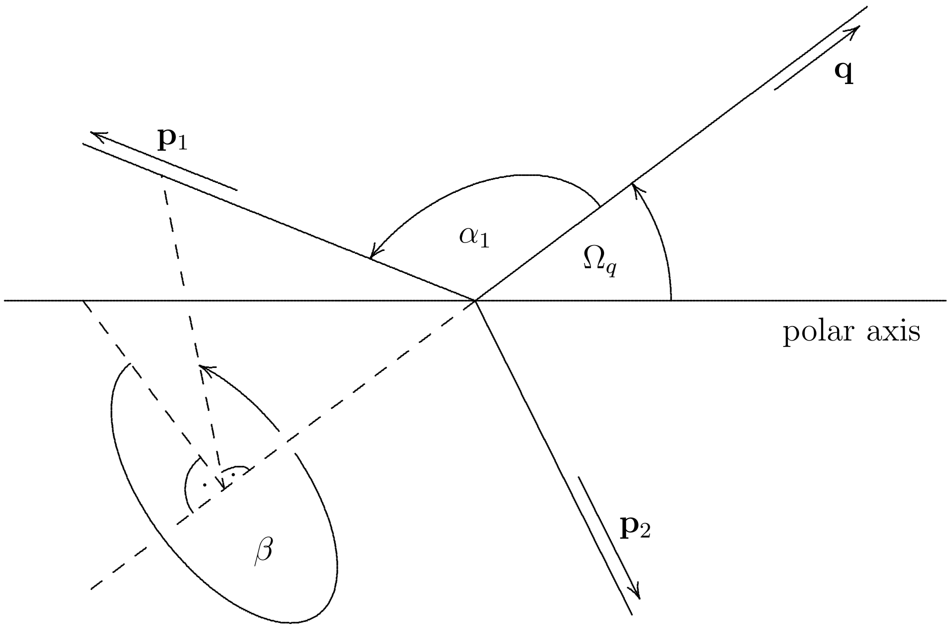
$$\mathrm{i}\mathcal{M}_{\gamma,2} \;=\; \bar{u}_f \;(-\mathrm{i}Q_f e)\not{\varepsilon}_\gamma^* \,\frac{\mathrm{i}(\not{p}_1+\not{q})}{(p_1+q)^2}\,\mathrm{i}e g_f^\sigma \not{\varepsilon}_\mathrm{Z}\omega_\sigma \; v_{\bar{f}}$$

Spin-/colour-averaged amplitude squared:

$$\langle|\mathcal{M}_\gamma|^2\rangle \;=\; \frac{1}{3}\sum_{\lambda_\mathrm{Z}=0,\pm1}\;\sum_{\sigma_f=\pm\frac{1}{2}}\;\sum_{\sigma_{\bar{f}}=\pm\frac{1}{2}}\;\sum_{\lambda_\gamma=\pm1}\;\sum_{\mathrm{colour}} |\mathcal{M}_{\gamma,1}+\mathcal{M}_{\gamma,2}|^2$$

$$= \frac{4}{3}\,N_\mathrm{c}^f\,Q_f^2 e^4\left[(g_f^+)^2+(g_f^-)^2\right]\left[\frac{(p_1p_2)M_\mathrm{Z}^2}{(qp_1)(qp_2)} + \frac{qp_2}{qp_1} + \frac{qp_1}{qp_2}\right]$$

# 3-particle phase space




CMS of Z boson:

$$k^\mu = (M_\mathrm{Z}, \mathbf{0}) = p_1^\mu + p_2^\mu + q^\mu$$

$$q^0 = |\mathbf{q}|, \quad p_i^0 = |\mathbf{p}_i|$$

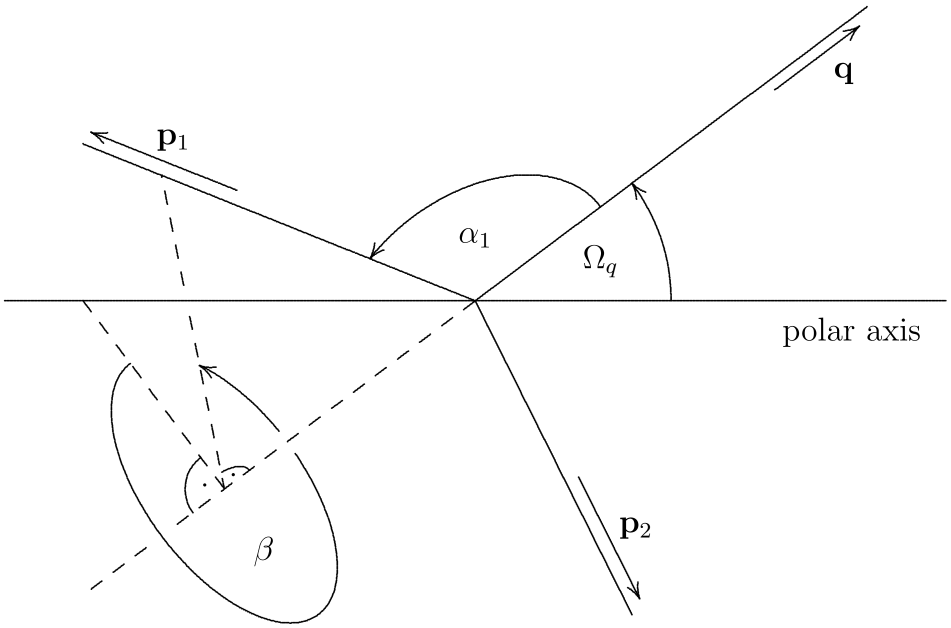
$$\int \mathrm{d}\Phi_{1\to3} \; = \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3 2q^0} \int \frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3 2p_1^0} \int \frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3 2p_2^0} \; (2\pi)^4 \, \delta^{(4)}(k - q - p_1 - p_2)$$

# 3-particle phase space

CMS of Z boson:

$$k^\mu = (M_\mathrm{Z}, \mathbf{0}) = p_1^\mu + p_2^\mu + q^\mu$$

$$q^0 = |\mathbf{q}|, \quad p_i^0 = |\mathbf{p}_i|$$




$$\int \mathrm{d}\Phi_{1\to3} = \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3 2q^0} \int \frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3 2p_1^0} \int \frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3 2p_2^0} \, (2\pi)^4 \, \delta^{(4)}(k-q-p_1-p_2)$$

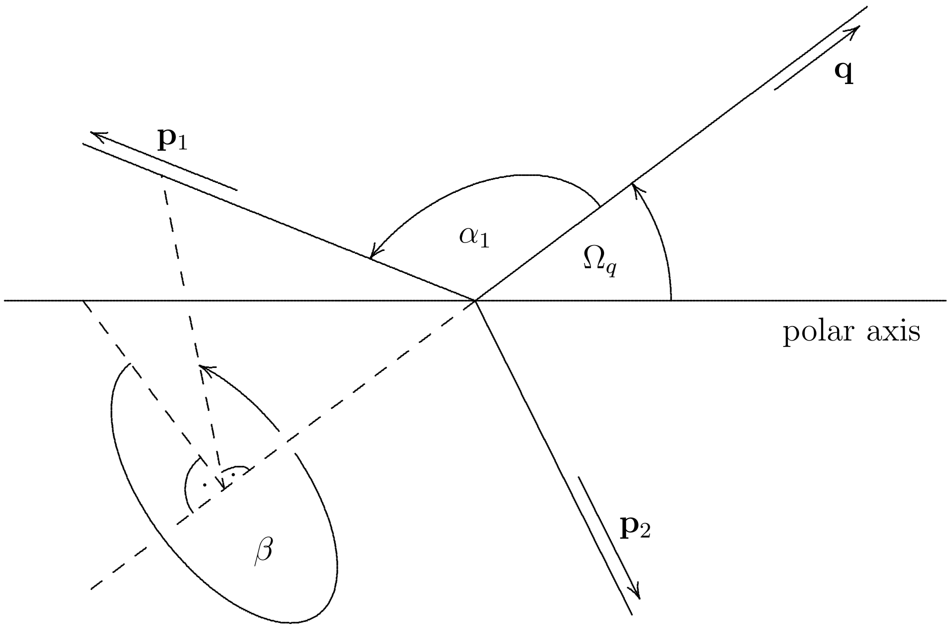
$$= \frac{1}{8(2\pi)^5} \int \mathrm{d}^3\mathbf{q} \int \mathrm{d}^3\mathbf{p}_1 \, \frac{1}{q^0\, p_1^0\, p_2^0} \, \delta(M_\mathrm{Z} - q^0 - p_1^0 - p_2^0), \qquad p_2^0 = |\mathbf{q} + \mathbf{p}_1|$$

# 3-particle phase space

CMS of Z boson:

$$k^\mu = (M_Z, \mathbf{0}) = p_1^\mu + p_2^\mu + q^\mu$$

$$q^0 = |\mathbf{q}|, \quad p_i^0 = |\mathbf{p}_i|$$




$$\int \mathrm{d}\Phi_{1\to 3} = \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3 2q^0} \int \frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3 2p_1^0} \int \frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3 2p_2^0} (2\pi)^4 \delta^{(4)}(k - q - p_1 - p_2)$$

$$= \frac{1}{8(2\pi)^5} \int \mathrm{d}^3\mathbf{q} \int \mathrm{d}^3\mathbf{p}_1 \frac{1}{q^0 p_1^0 p_2^0} \delta(M_Z - q^0 - p_1^0 - p_2^0), \qquad p_2^0 = |\mathbf{q} + \mathbf{p}_1|$$

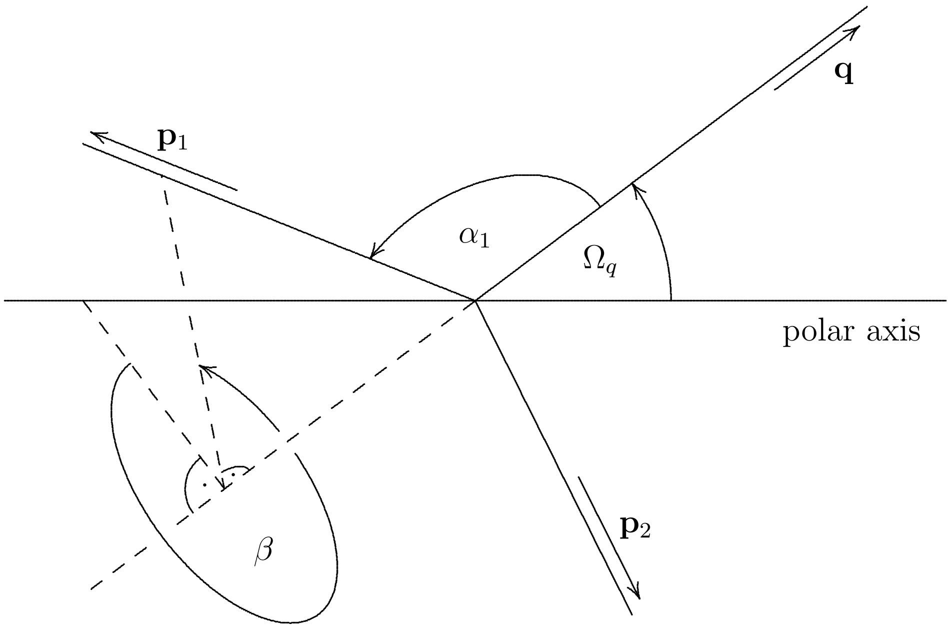
$$= \frac{1}{8(2\pi)^5} \int \mathrm{d}q^0 \int \mathrm{d}\Omega_q \int \mathrm{d}p_1^0 \int \mathrm{d}\beta \int \mathrm{d}\cos\alpha_1 \frac{q^0 p_1^0}{p_2^0} \delta(M_Z - q^0 - p_1^0 - p_2^0)$$

# 3-particle phase space

CMS of Z boson:

$k^\mu = (M_Z, \mathbf{0}) = p_1^\mu + p_2^\mu + q^\mu$

$q^0 = |\mathbf{q}|, \quad p_i^0 = |\mathbf{p}_i|$




$$\int \mathrm{d}\Phi_{1\to 3} = \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3 2q^0} \int \frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3 2p_1^0} \int \frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3 2p_2^0} \, (2\pi)^4 \, \delta^{(4)}(k - q - p_1 - p_2)$$

$$= \frac{1}{8(2\pi)^5} \int \mathrm{d}^3\mathbf{q} \int \mathrm{d}^3\mathbf{p}_1 \, \frac{1}{q^0 \, p_1^0 \, p_2^0} \, \delta(M_Z - q^0 - p_1^0 - p_2^0), \qquad p_2^0 = |\mathbf{q} + \mathbf{p}_1|$$

$$= \frac{1}{8(2\pi)^5} \int \mathrm{d}q^0 \int \mathrm{d}\Omega_q \int \mathrm{d}p_1^0 \int \mathrm{d}\beta \int \mathrm{d}\cos\alpha_1 \, \underbrace{\frac{q^0 \, p_1^0}{p_2^0}}_{= \left. \frac{\partial p_2^0}{\partial \cos\alpha_1} \right|_{q^0, p_1^0}} \, \delta(M_Z - q^0 - p_1^0 - p_2^0)$$

$$= \frac{1}{8(2\pi)^5} \int \mathrm{d}\Omega_q \int \mathrm{d}\beta \int \mathrm{d}q^0 \int \mathrm{d}p_1^0$$

## Phase-space boundary:

$$p_2^0 = |\mathbf{q} + \mathbf{p}_1| \quad \Leftrightarrow \quad (p_2^0)^2 = (q^0)^2 + (p_1^0)^2 + 2q^0 p_1^0 \cos\alpha_1$$

Implications:

- $(p_2^0)^2 < (q^0)^2 + (p_1^0)^2 + 2q^0 p_1^0 = (q^0 + p_1^0)^2 = (M_\mathrm{Z} - p_2^0)^2$

  $\Rightarrow \; p_2^0 < \frac{1}{2} M_\mathrm{Z},$ i.e. $q^0 + p_1^0 < \frac{1}{2} M_\mathrm{Z}$
- $p_2^0|_{\mathrm{min}} = 0,$ since $\mathbf{q} = -\mathbf{p}_1$ possible

  $\Rightarrow \; q^0|_{\mathrm{min}} = p_1^0|_{\mathrm{min}} = 0$ by symmetry

## Phase-space integral:

$$\int \mathrm{d}\Phi_{1\to 3} \;=\; \frac{1}{8(2\pi)^5} \int \mathrm{d}\Omega_q \int_0^{2\pi} \mathrm{d}\beta \int_0^{\frac{1}{2}M_\mathrm{Z}} \mathrm{d}q^0 \int_{\frac{1}{2}M_\mathrm{Z} - q^0}^{\frac{1}{2}M_\mathrm{Z}} \mathrm{d}p_1^0$$

## Kinematics and singular regions:

Scalar products can be expressed in terms of energies $p_1^0$, $p_2^0$, and $q^0$:

$$
\begin{aligned}
p_1 p_2 &= \tfrac{1}{2}(p_1+p_2)^2 = \tfrac{1}{2}(k-q)^2 = \tfrac{1}{2}k^2 - kq = \tfrac{1}{2}M_Z^2 - M_Z q^0 = M_Z\left(\tfrac{1}{2}M_Z - q^0\right), \\
q p_1 &= \ldots = M_Z\left(\tfrac{1}{2}M_Z - p_2^0\right) = M_Z\left(q^0 + p_1^0 - \tfrac{1}{2}M_Z\right), \\
q p_2 &= \ldots = M_Z\left(\tfrac{1}{2}M_Z - p_1^0\right).
\end{aligned}
$$

## Behaviour of phase-space integrals:

- $$\int \mathrm{d}q^0 \int \mathrm{d}p_1^0 \, \frac{(qp_1)}{(qp_2)} = \int_0^{\frac{1}{2}M_Z} \mathrm{d}q^0 \int_{\frac{1}{2}M_Z - q^0}^{\frac{1}{2}M_Z} \mathrm{d}p_1^0 \left[ \frac{q^0}{\frac{1}{2}M_Z - p_1^0} - 1 \right]$$

  $\hookrightarrow$ logarithmically divergent for $p_1^0 \to \frac{1}{2}M_Z$

- $$\int \mathrm{d}q^0 \int \mathrm{d}p_1^0 \, \frac{(p_1p_2)}{(qp_1)(qp_2)}$$

  $$= \underbrace{\int_0^{\frac{1}{2}M_Z} \mathrm{d}q^0}_{\text{log. divergent}} \underbrace{\frac{\frac{1}{2}M_Z - q^0}{M_Z\, q^0}}_{\text{for } q^0 \to 0} \int_{\frac{1}{2}M_Z - q^0}^{\frac{1}{2}M_Z} \mathrm{d}p_1^0 \left[ \underbrace{\frac{1}{q^0 + p_1^0 - \frac{1}{2}M_Z}}_{\text{for } p_2^0 \to \frac{1}{2}M_Z} + \underbrace{\frac{1}{\frac{1}{2}M_Z - p_1^0}}_{\text{for } p_1^0 \to \frac{1}{2}M_Z} \right]$$

## General origin of singularities:

- **Collinear singularities for $(qp_i) \to 0$**

  $(qp)$ for nearly collinear $\gamma$ emission off light particle with mom. $p$ ($p^2 = m^2 \to 0$)

$$qp \;=\; q_0(p_0 - |\mathbf{p}|\cos\alpha) \underset{\substack{m\to 0\\ \alpha\to 0}}{\widetilde{\quad}} q_0 p_0 \left[\frac{m^2}{2p_0^2} + 1 - \cos\alpha\right]$$

  Integration over collinear region $\alpha \sim \mathcal{O}(m^2/p_0^2)$ yields

$$\int_{-1}^{+1} \mathrm{d}\cos\alpha \, \frac{1}{(qp)} \;\propto\; \ln(m) \;+\; \text{(finite for } m \to 0\text{)}$$

- **Soft singularity for $q \to 0$**

  $(qp_1)(qp_2)$ for arbitrary on-shell momenta $p_i$ ($p_i^2 = m_i^2$):

$$(qp_1)(qp_2) \;=\; q_0^2\,(p_1^0 - |\mathbf{p}_1|\cos\alpha_1)(p_2^0 - |\mathbf{p}_2|\cos\alpha_2)$$

  Integration over photon momentum

$$I \;=\; \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3 2q_0}\,\frac{1}{(qp_1)(qp_2)} \;\propto\; \int \frac{\mathrm{d}q_0}{q_0} \;\to\; \infty$$

  Regularization by infinitesimally small photon mass $m_\gamma$: $\quad I_{\mathrm{div}} \propto \ln m_\gamma$

  or via dimensional regularization: $\quad I_{\mathrm{div}} \propto 1/(D-4)$

## Simple technical procedure: phase-space slicing

Decomposition of photon pase space into 3 different regions: $(\Delta E, \Delta\alpha \to 0)$

- Non-singular region: $q_0 > \Delta E, \quad \alpha_i > \Delta\alpha, \quad i = 1, 2$

  $\hookrightarrow$ Integration of $\langle|\mathcal{M}_\gamma|^2\rangle$ possible without regulators, i.e. $m_f = m_\gamma = 0$

- Soft-photon region: $m_\gamma < q_0 < \Delta E$

  $\langle|\mathcal{M}_\gamma(q)|^2\rangle \underset{q\to 0}{\widetilde{\quad}} \langle|\mathcal{M}_0|^2\rangle \times f_{\mathrm{soft}}(q)$, where $f_{\mathrm{soft}}(q) =$ universal factor

  $\hookrightarrow$ process-independent integral $\int \mathrm{d}q\, f_{\mathrm{soft}}(q)$, e.g., with infinitesimal $m_\gamma$

- Collinear region: $q_0 > \Delta E, \quad 0 < \alpha_i < \Delta\alpha, \quad i = 1, 2$

  $\langle|\mathcal{M}_\gamma(p_i)|^2\rangle \underset{\alpha_i\to 0}{\widetilde{\quad}} \langle|\mathcal{M}_0|^2\rangle \times f_{\mathrm{coll},i}(\alpha_i, m_i)$,

  $\hookrightarrow$ process-independent integral $\int \mathrm{d}\alpha_i\, f_{\mathrm{coll},i}(\alpha_i, m_i)$ with finite $m_i$

Comment: case of collinear initial-state radiation somewhat more complicated,
because $\langle|\mathcal{M}_0|^2\rangle$ is boosted
$\hookrightarrow$ process-independent integral involves convolution over $\langle|\mathcal{M}_0|^2\rangle$

## Cutoff parameters $\Delta E$ and $\Delta\alpha$ for $\mathrm{Z} \to f\bar{f}\gamma$

Impose cuts $q^0 > \Delta E$ and $\alpha_i > \Delta\alpha$ on $\displaystyle\int_0^{\frac{1}{2}M_\mathrm{Z}} \mathrm{d}q^0 \int_{\frac{1}{2}M_\mathrm{Z}-q^0}^{\frac{1}{2}M_\mathrm{Z}} \mathrm{d}p_1^0$

## Cutoff parameters $\Delta E$ and $\Delta\alpha$ for $\mathrm{Z} \to f\bar{f}\gamma$

Impose cuts $q^0 > \Delta E$ and $\alpha_i > \Delta\alpha$ on $\displaystyle\int_0^{\frac{1}{2}M_\mathrm{Z}} \mathrm{d}q^0 \int_{\frac{1}{2}M_\mathrm{Z}-q^0}^{\frac{1}{2}M_\mathrm{Z}} \mathrm{d}p_1^0$

- $qp_2 \;=\; q^0 p_2^0(1-\cos\alpha_2) \;=\; q^0(M_\mathrm{Z}-q^0-p_1^0)\,\underbrace{(1-\cos\alpha_2)}_{>\frac{1}{2}\Delta\alpha^2} \;=\; M_\mathrm{Z}\left(\frac{1}{2}M_\mathrm{Z}-p_1^0\right)$

## Cutoff parameters $\Delta E$ and $\Delta\alpha$ for $\mathrm{Z} \to f\bar{f}\gamma$

Impose cuts $q^0 > \Delta E$ and $\alpha_i > \Delta\alpha$ on $\displaystyle\int_0^{\frac{1}{2}M_\mathrm{Z}} \mathrm{d}q^0 \int_{\frac{1}{2}M_\mathrm{Z}-q^0}^{\frac{1}{2}M_\mathrm{Z}} \mathrm{d}p_1^0$

- $$qp_2 = q^0 p_2^0 (1-\cos\alpha_2) = q^0 (M_\mathrm{Z} - q^0 - p_1^0) \underbrace{(1-\cos\alpha_2)}_{>\frac{1}{2}\Delta\alpha^2} = M_\mathrm{Z}\left(\tfrac{1}{2}M_\mathrm{Z} - p_1^0\right)$$

$\hookrightarrow \; p_1^0 < \frac{1}{2}M_\mathrm{Z} - \epsilon(q^0)$ with $\displaystyle \epsilon(q^0) = \frac{q^0(M_\mathrm{Z} - 2q^0)}{4M_\mathrm{Z}}\Delta\alpha^2$

## Cutoff parameters $\Delta E$ and $\Delta\alpha$ for $\mathrm{Z} \to f\bar{f}\gamma$

Impose cuts $q^0 > \Delta E$ and $\alpha_i > \Delta\alpha$ on $\quad \int_0^{\frac{1}{2}M_\mathrm{Z}} \mathrm{d}q^0 \int_{\frac{1}{2}M_\mathrm{Z}-q^0}^{\frac{1}{2}M_\mathrm{Z}} \mathrm{d}p_1^0$

- $$qp_2 = q^0 p_2^0(1-\cos\alpha_2) = q^0(M_\mathrm{Z}-q^0-p_1^0)\underbrace{(1-\cos\alpha_2)}_{>\frac{1}{2}\Delta\alpha^2} = M_\mathrm{Z}\left(\tfrac{1}{2}M_\mathrm{Z}-p_1^0\right)$$

$\hookrightarrow \quad p_1^0 < \frac{1}{2}M_\mathrm{Z} - \epsilon(q^0) \qquad$ with $\qquad \epsilon(q^0) = \dfrac{q^0(M_\mathrm{Z}-2q^0)}{4M_\mathrm{Z}}\,\Delta\alpha^2$

- $$qp_1 = q^0 p_1^0 \underbrace{(1-\cos\alpha_1)}_{>\frac{1}{2}\Delta\alpha^2} = M_\mathrm{Z}\left(q^0+p_1^0-\tfrac{1}{2}M_\mathrm{Z}\right)$$

# Cutoff parameters $\Delta E$ and $\Delta\alpha$ for $\mathrm{Z} \to f\bar{f}\gamma$

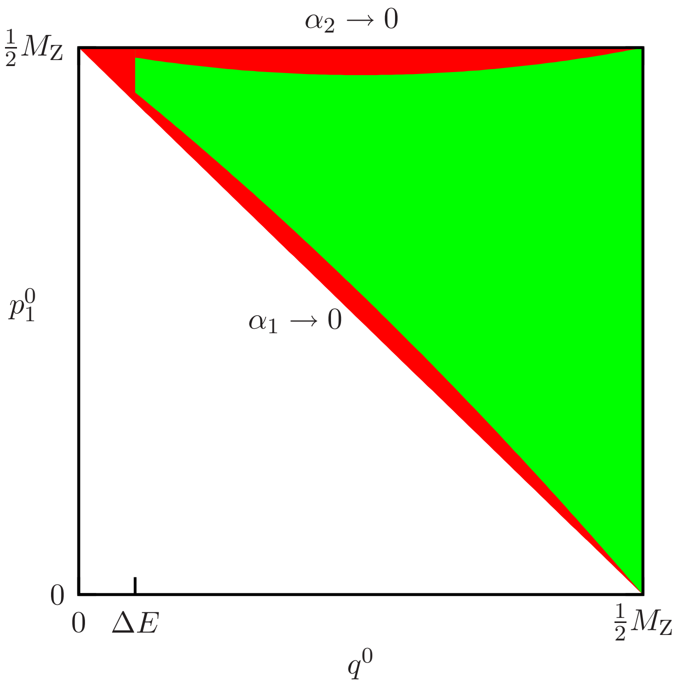
Impose cuts $q^0 > \Delta E$ and $\alpha_i > \Delta\alpha$ on $\displaystyle\int_0^{\frac{1}{2}M_\mathrm{Z}} \mathrm{d}q^0 \int_{\frac{1}{2}M_\mathrm{Z}-q^0}^{\frac{1}{2}M_\mathrm{Z}} \mathrm{d}p_1^0$

- $$qp_2 = q^0 p_2^0 (1-\cos\alpha_2) = q^0 (M_\mathrm{Z} - q^0 - p_1^0) \underbrace{(1-\cos\alpha_2)}_{>\frac{1}{2}\Delta\alpha^2} = M_\mathrm{Z}\left(\tfrac{1}{2}M_\mathrm{Z} - p_1^0\right)$$

$\hookrightarrow$ $p_1^0 < \frac{1}{2}M_\mathrm{Z} - \epsilon(q^0)$ with $\epsilon(q^0) = \dfrac{q^0(M_\mathrm{Z}-2q^0)}{4M_\mathrm{Z}}\Delta\alpha^2$

- $$qp_1 = q^0 p_1^0 \underbrace{(1-\cos\alpha_1)}_{>\frac{1}{2}\Delta\alpha^2}$$
$$= M_\mathrm{Z}\left(q^0 + p_1^0 - \tfrac{1}{2}M_\mathrm{Z}\right)$$

$\hookrightarrow$ $p_1^0 > \left(\frac{1}{2}M_\mathrm{Z} - q^0\right) + \epsilon(q^0)$



## Integration over non-singular region

$$\Gamma_{Z\to f\bar f}\Big|_{\rm hard} = \frac{1}{2M_Z}\int_{\rm non-sing} d\Phi_{1\to3}\,\langle|\mathcal{M}_\gamma|^2\rangle$$

$$= \frac{1}{2M_Z}\,\tfrac{4}{3}\,N_c^f\,Q_f^2 e^4\left[(g_f^+)^2+(g_f^-)^2\right]\frac{1}{8(2\pi)^5}\underbrace{\int d\Omega_q\int_0^{2\pi}d\beta}_{\to\,8\pi^2}$$

$$\times\int_{\Delta E}^{\frac12 M_Z}dq^0\int_{\frac12 M_Z-q^0+\epsilon}^{\frac12 M_Z-\epsilon}dp_1^0\left[\frac{(p_1p_2)M_Z^2}{(qp_1)(qp_2)}+\underbrace{\frac{qp_2}{qp_1}+\frac{qp_1}{qp_2}}_{\text{integrals equal by symmetry}}\right]$$

$$= \Gamma_{Z\to f\bar f,0}\,\frac{Q_f^2\alpha}{\pi}\int_{\Delta E}^{\frac12 M_Z}dq^0\int_{\frac12 M_Z-q^0+\epsilon}^{\frac12 M_Z-\epsilon}dp_1^0\left[\frac{2(p_1p_2)}{(qp_1)(qp_2)}+\frac{4}{M_Z^2}\,\frac{qp_1}{qp_2}\right]$$

$$\equiv \Gamma_{Z\to f\bar f,0}\,\frac{Q_f^2\alpha}{\pi}\,(J_{12}+J_2)$$

$$\equiv \Gamma_{Z\to f\bar f,0}\,\delta_{\rm hard}$$

# Evaluation of integrals

$$J_2 = \frac{4}{M_Z^2} \int_{\underbrace{\Delta E}_{\to 0 \text{ possible}}}^{\frac{1}{2}M_Z} dq^0 \int_{\frac{1}{2}M_Z - q^0 + \epsilon}^{\frac{1}{2}M_Z - \epsilon} dp_1^0 \, \frac{qp_1}{qp_2}$$

$$= \frac{4}{M_Z^2} \int_0^{\frac{1}{2}M_Z} dq^0 \int_{\frac{1}{2}M_Z - q^0 + \epsilon}^{\frac{1}{2}M_Z - \epsilon} dp_1^0 \, \frac{q^0 + p_1^0 - \frac{1}{2}M_Z}{\frac{1}{2}M_Z - p_1^0} \; + \; \mathcal{O}\left(\frac{\Delta E}{M_Z}\right)$$

$$= \frac{4}{M_Z^2} \int_0^{\frac{1}{2}M_Z} dq^0 \left[ -q^0 \ln\left(\tfrac{1}{2}M_Z - p_1^0\right) - p_1^0 \right]_{\frac{1}{2}M_Z - q^0 + \epsilon}^{\frac{1}{2}M_Z - \epsilon} \; + \; \ldots$$

$$= \frac{4}{M_Z^2} \int_0^{\frac{1}{2}M_Z} dq^0 \, q^0 \left[ -\ln(\epsilon) + \ln(q^0) - 1 \right] \; + \; \mathcal{O}(\Delta\alpha) \; + \; \ldots$$

$$= \frac{4}{M_Z^2} \int_0^{\frac{1}{2}M_Z} dq^0 \, q^0 \left[ -\ln\left( \frac{(M_Z - 2q^0)}{4M_Z} \, \Delta\alpha^2 \right) - 1 \right] \; + \; \ldots$$

$$= \ln\left(\frac{2}{\Delta\alpha}\right) + \frac{1}{4} \; + \; \ldots$$

# Evaluation of integrals (continued)

$$
\begin{aligned}
J_{12} &= \int_{\Delta E}^{\frac{1}{2}M_\mathrm{Z}} \mathrm{d}q^0 \int_{\frac{1}{2}M_\mathrm{Z}-q^0+\epsilon}^{\frac{1}{2}M_\mathrm{Z}-\epsilon} \mathrm{d}p_1^0 \, \frac{2(p_1p_2)}{(qp_1)(qp_2)} \\
&= \int_{\Delta E}^{\frac{1}{2}M_\mathrm{Z}} \mathrm{d}q^0 \int_{\frac{1}{2}M_\mathrm{Z}-q^0+\epsilon}^{\frac{1}{2}M_\mathrm{Z}-\epsilon} \mathrm{d}p_1^0 \, \frac{M_\mathrm{Z}-2q^0}{M_\mathrm{Z}\left(q^0+p_1^0-\frac{1}{2}M_\mathrm{Z}\right)\left(\frac{1}{2}M_\mathrm{Z}-p_1^0\right)} \\
&= \int_{\Delta E}^{\frac{1}{2}M_\mathrm{Z}} \mathrm{d}q^0 \, \frac{M_\mathrm{Z}-2q^0}{M_\mathrm{Z}\,q^0} \int_{\frac{1}{2}M_\mathrm{Z}-q^0+\epsilon}^{\frac{1}{2}M_\mathrm{Z}-\epsilon} \mathrm{d}p_1^0 \left[\frac{1}{\frac{1}{2}M_\mathrm{Z}-p_1^0} + \frac{1}{q^0+p_1^0-\frac{1}{2}M_\mathrm{Z}}\right] \\
&= 2\int_{\Delta E}^{\frac{1}{2}M_\mathrm{Z}} \mathrm{d}q^0 \left(\frac{1}{q^0}-\frac{2}{M_\mathrm{Z}}\right)\left[-\ln(\epsilon)+\ln(q^0)\right] \;+\; \dots \\
&= -2\int_{\Delta E}^{\frac{1}{2}M_\mathrm{Z}} \mathrm{d}q^0 \left(\frac{1}{q^0}-\frac{2}{M_\mathrm{Z}}\right) \ln\left(\frac{M_\mathrm{Z}-2q^0}{4M_\mathrm{Z}}\,\Delta\alpha^2\right) \;+\; \dots \\
&= 4\ln\left(\frac{2}{\Delta\alpha}\right)\int_{\Delta E}^{\frac{1}{2}M_\mathrm{Z}} \frac{\mathrm{d}q^0}{q^0} - 2\int_0^1 \frac{\mathrm{d}t}{t}\,\ln(1-t) + 2\int_0^1 \mathrm{d}t\,\ln\left((1-t)\frac{\Delta\alpha^2}{4}\right) \;+\; \dots \\
&= 4\ln\left(\frac{2}{\Delta\alpha}\right)\ln\left(\frac{M_\mathrm{Z}}{2\Delta E}\right) - 4\ln\left(\frac{2}{\Delta\alpha}\right) - 2 + \frac{\pi^2}{3} \;+\; \dots
\end{aligned}
$$

## Final results for the non-singular contribution:

$$\delta_{\mathrm{hard}} = \frac{Q_f^2\alpha}{\pi}\,(J_{12}+J_2)$$

$$= \frac{Q_f^2\alpha}{\pi}\left[4\ln\left(\frac{2}{\Delta\alpha}\right)\ln\left(\frac{M_{\mathrm{Z}}}{2\Delta E}\right) - 3\ln\left(\frac{2}{\Delta\alpha}\right) - \frac{7}{4} + \frac{\pi^2}{3}\right] + \mathcal{O}(\Delta\alpha) + \mathcal{O}\left(\frac{\Delta E}{M_{\mathrm{Z}}}\right)$$

## Comment on more complicated processes: (scattering processes, etc.)

Analytical evaluation of the phase-space integrals in general not possible

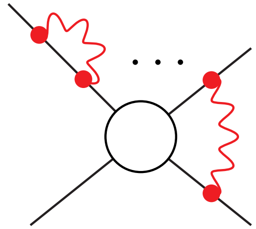
(and not sufficient because of experimental cuts, etc.)

$\hookrightarrow$ application of Monte Carlo integration / event generators techniques

## 3.2 Soft-singular contributions

General considerations about soft-photon singularities:

- Virtual corrections: loop diagrams



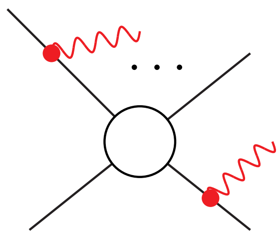
IR divergences from soft virtual photons ($q \to 0$)

$$\int \frac{\mathrm{d}^4 q \ldots}{(q^2 - \underbrace{m_\gamma^2}_{})(2qp_1)(2qp_2)} \to C \ln(m_\gamma)$$

fictitious infinitesimal photon mass as regulator

- "Real" corrections: photon bremsstrahlung

$\int \frac{\mathrm{d}^3\mathbf{q}}{2q_0}$ |



$|^2$

IR divergences from soft real photons ($\mathbf{q} \to 0$)

$$\int \frac{\mathrm{d}^3\mathbf{q} \ldots}{\sqrt{\mathbf{q}^2 + m_\gamma^2}(2qp_1)(2qp_2)} \to -C \ln(m_\gamma)$$

Bloch–Nordsieck theorem:

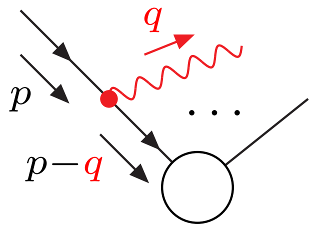
IR divergences of virtual and real corrections cancel in the sum

$\hookrightarrow$ virtual and soft-photonic corrections cannot be discussed separately

$\leftrightarrow$ related to limited experimental resolution of soft photons

$\Rightarrow$ Predictions necessarily depend on treatment of photon emission

## General calculation of soft-photon factor:




$$= \; A(p-q)\,\frac{\mathrm{i}(\not{p}-\not{q}+m_f)}{(p-q)^2-m_f^2}(-\mathrm{i}Q_f e)\not{\varepsilon}^*_\gamma u_f(p)$$

$$\underset{q\to 0}{\sim} \; -Q_f e\,\frac{\varepsilon^*_\gamma p}{qp}A(p)u_f(p) \; = \; -Q_f e\,\frac{\varepsilon^*_\gamma p}{qp}\mathcal{M}_{\mathrm{Born}}$$

"Eikonal factorization" holds for all charged particles (spin $0, \frac{1}{2}, 1$)

$$\Rightarrow \; \delta_{\mathrm{soft}} \; = \; -\frac{\alpha}{2\pi^2}\int_{m_\gamma<q_0<\Delta E}\frac{\mathrm{d}^3\mathbf{q}}{2q_0}\sum_{i,j}\frac{(\pm Q_i)(\pm Q_j)(p_ip_j)}{(qp_i)(qp_j)}$$

($i$ = particle with charge $Q_i$ incoming($+$) or outgoing ($-$))

## Application to $\mathrm{Z}\to f\bar{f}\gamma$:

$$\delta_{\mathrm{soft}} \; = \; \frac{Q_f^2\alpha}{2\pi^2}\int_{m_\gamma<q_0<\Delta E}\frac{\mathrm{d}^3\mathbf{q}}{2q_0}\left[\frac{2(p_ip_j)}{(qp_i)(qp_j)}-\frac{m_f^2}{(qp_1)^2}-\frac{m_f^2}{(qp_2)^2}\right]$$

## Note:

- fermion $f$ cannot be taken massless in soft region
  $\hookrightarrow$ hierarchy of limits: $M_{\mathrm{Z}}, p_i^0 \gg m_f \gg \Delta E \gg m_\gamma \to 0$
- $\delta_{\mathrm{soft}}$ is not Lorentz invariant
  $\hookrightarrow$ evaluation in CMS of decaying Z boson

## Evaluation of soft-photon integrals

Allowed approximation: $k = p_1 + p_2$, i.e. $p_{1,2}^{\mu} = (p_0, \pm\mathbf{p})$, $p_0 = \frac{1}{2} M_{\mathrm{Z}}$ in CMS

### Integral from squared diagrams:

$$I_{11} = \int_{m_\gamma < q_0 < \Delta E} \frac{\mathrm{d}^3\mathbf{q}}{2q_0} \frac{m_f^2}{(qp_1)^2}, \qquad qp_1 = q_0 p_0 - |\mathbf{q}|\,|\mathbf{p}|\,\cos\theta$$

## Evaluation of soft-photon integrals

Allowed approximation: $k = p_1 + p_2$, i.e. $p^{\mu}_{1,2} = (p_0, \pm\mathbf{p})$, $p_0 = \frac{1}{2} M_\mathrm{Z}$ in CMS

### Integral from squared diagrams:

$$I_{11} = \int_{m_\gamma < q_0 < \Delta E} \frac{\mathrm{d}^3\mathbf{q}}{2q_0} \frac{m_f^2}{(qp_1)^2}, \qquad qp_1 = q_0 p_0 - |\mathbf{q}|\,|\mathbf{p}|\,\cos\theta$$

$$= 2\pi \int_{m_\gamma < q_0 < \Delta E} \frac{\mathrm{d}|\mathbf{q}|\,\mathbf{q}^2}{2q_0} \int_{-1}^{+1} \mathrm{d}\cos\theta \, \frac{m_f^2}{(q_0 p_0 - |\mathbf{q}|\,|\mathbf{p}|\,\cos\theta)^2}$$

## Evaluation of soft-photon integrals

Allowed approximation: $k = p_1 + p_2,$ i.e. $p^\mu_{1,2} = (p_0, \pm\mathbf{p}),\ \ p_0 = \frac{1}{2} M_\mathrm{Z}$ in CMS

### Integral from squared diagrams:

$$I_{11} = \int_{m_\gamma < q_0 < \Delta E} \frac{\mathrm{d}^3\mathbf{q}}{2q_0} \frac{m_f^2}{(qp_1)^2}, \qquad qp_1 = q_0 p_0 - |\mathbf{q}|\,|\mathbf{p}|\,\cos\theta$$

$$= 2\pi \int_{m_\gamma < q_0 < \Delta E} \frac{\mathrm{d}|\mathbf{q}|\,\mathbf{q}^2}{2q_0} \int_{-1}^{+1} \mathrm{d}\cos\theta\, \frac{m_f^2}{(q_0 p_0 - |\mathbf{q}|\,|\mathbf{p}|\,\cos\theta)^2}$$

$$= 2\pi \int_{m_\gamma < q_0 < \Delta E} \frac{\mathrm{d}|\mathbf{q}|\,\mathbf{q}^2}{q_0} \frac{m_f^2}{q_0^2 p_0^2 - \mathbf{q}^2\mathbf{p}^2}, \qquad \mathbf{q}^2 = q_0^2 - m_\gamma^2,\ \ \mathbf{p}^2 = p_0^2 - m_f^2$$

## Evaluation of soft-photon integrals

Allowed approximation: $k = p_1 + p_2$, i.e. $p^\mu_{1,2} = (p_0, \pm\mathbf{p})$, $p_0 = \frac{1}{2}M_\mathrm{Z}$ in CMS

### Integral from squared diagrams:

$$\begin{aligned}
I_{11} &= \int_{m_\gamma<q_0<\Delta E} \frac{\mathrm{d}^3\mathbf{q}}{2q_0} \frac{m_f^2}{(qp_1)^2}, \qquad\qquad qp_1 = q_0p_0 - |\mathbf{q}|\,|\mathbf{p}|\,\cos\theta \\
&= 2\pi \int_{m_\gamma<q_0<\Delta E} \frac{\mathrm{d}|\mathbf{q}|\,\mathbf{q}^2}{2q_0} \int_{-1}^{+1} \mathrm{d}\cos\theta\, \frac{m_f^2}{(q_0p_0 - |\mathbf{q}|\,|\mathbf{p}|\,\cos\theta)^2} \\
&= 2\pi \int_{m_\gamma<q_0<\Delta E} \frac{\mathrm{d}|\mathbf{q}|\,\mathbf{q}^2}{q_0} \frac{m_f^2}{q_0^2p_0^2 - \mathbf{q}^2\mathbf{p}^2}, \qquad\qquad \mathbf{q}^2 = q_0^2 - m_\gamma^2, \quad \mathbf{p}^2 = p_0^2 - m_f^2 \\
&= 2\pi \int_{m_\gamma}^{\Delta E} \mathrm{d}q_0\, \frac{m_f^2\sqrt{q_0^2 - m_\gamma^2}}{[q_0^2 - m_\gamma^2]m_f^2 + p_0^2m_\gamma^2}, \qquad\qquad m_\gamma\sqrt{z} \equiv \sqrt{q_0^2 - m_\gamma^2} + q_0
\end{aligned}$$

## Evaluation of soft-photon integrals

Allowed approximation: $k = p_1 + p_2$, i.e. $p^\mu_{1,2} = (p_0, \pm\mathbf{p})$, $p_0 = \frac{1}{2} M_\mathrm{Z}$ in CMS

### Integral from squared diagrams:

$$I_{11} = \int_{m_\gamma < q_0 < \Delta E} \frac{\mathrm{d}^3\mathbf{q}}{2q_0} \frac{m_f^2}{(qp_1)^2}, \qquad qp_1 = q_0 p_0 - |\mathbf{q}|\,|\mathbf{p}|\,\cos\theta$$

$$= 2\pi \int_{m_\gamma < q_0 < \Delta E} \frac{\mathrm{d}|\mathbf{q}|\,\mathbf{q}^2}{2q_0} \int_{-1}^{+1} \mathrm{d}\cos\theta \, \frac{m_f^2}{(q_0 p_0 - |\mathbf{q}|\,|\mathbf{p}|\,\cos\theta)^2}$$

$$= 2\pi \int_{m_\gamma < q_0 < \Delta E} \frac{\mathrm{d}|\mathbf{q}|\,\mathbf{q}^2}{q_0} \frac{m_f^2}{q_0^2 p_0^2 - \mathbf{q}^2\mathbf{p}^2}, \qquad \mathbf{q}^2 = q_0^2 - m_\gamma^2, \quad \mathbf{p}^2 = p_0^2 - m_f^2$$

$$= 2\pi \int_{m_\gamma}^{\Delta E} \mathrm{d}q_0 \, \frac{m_f^2 \sqrt{q_0^2 - m_\gamma^2}}{[q_0^2 - m_\gamma^2] m_f^2 + p_0^2 m_\gamma^2}, \qquad m_\gamma \sqrt{z} \equiv \sqrt{q_0^2 - m_\gamma^2} + q_0$$

$$= \pi \int_1^{4\Delta E^2/m_\gamma^2} \mathrm{d}z \, \frac{m_f^2 (z-1)^2}{z[m_f^2 (z-1)^2 + 4 z p_0^2]} + \ldots$$

# Evaluation of soft-photon integrals

Allowed approximation: $k = p_1 + p_2$, i.e. $p_{1,2}^{\mu} = (p_0, \pm\mathbf{p})$, $p_0 = \frac{1}{2} M_{\mathrm{Z}}$ in CMS

## Integral from squared diagrams:

$$I_{11} = \int_{m_\gamma < q_0 < \Delta E} \frac{\mathrm{d}^3\mathbf{q}}{2q_0} \frac{m_f^2}{(qp_1)^2}, \qquad qp_1 = q_0 p_0 - |\mathbf{q}|\,|\mathbf{p}|\,\cos\theta$$

$$= 2\pi \int_{m_\gamma < q_0 < \Delta E} \frac{\mathrm{d}|\mathbf{q}|\,\mathbf{q}^2}{2q_0} \int_{-1}^{+1} \mathrm{d}\cos\theta\, \frac{m_f^2}{(q_0 p_0 - |\mathbf{q}|\,|\mathbf{p}|\,\cos\theta)^2}$$

$$= 2\pi \int_{m_\gamma < q_0 < \Delta E} \frac{\mathrm{d}|\mathbf{q}|\,\mathbf{q}^2}{q_0} \frac{m_f^2}{q_0^2 p_0^2 - \mathbf{q}^2\mathbf{p}^2}, \qquad \mathbf{q}^2 = q_0^2 - m_\gamma^2, \quad \mathbf{p}^2 = p_0^2 - m_f^2$$

$$= 2\pi \int_{m_\gamma}^{\Delta E} \mathrm{d}q_0\, \frac{m_f^2 \sqrt{q_0^2 - m_\gamma^2}}{[q_0^2 - m_\gamma^2] m_f^2 + p_0^2 m_\gamma^2}, \qquad m_\gamma \sqrt{z} \equiv \sqrt{q_0^2 - m_\gamma^2} + q_0$$

$$= \pi \int_{1}^{4\Delta E^2/m_\gamma^2} \mathrm{d}z\, \frac{m_f^2 (z-1)^2}{z[m_f^2 (z-1)^2 + 4z p_0^2]} + \dots$$

$$= \pi \int_{1}^{4\Delta E^2/m_\gamma^2} \mathrm{d}z \left[\frac{1}{z} - \frac{1}{z - z_1} + \frac{1}{z - z_2}\right] + \dots, \qquad z_1 = \frac{1}{z_2} = -\frac{m_f^2}{4p_0^2} + \dots$$

## Evaluation of soft-photon integrals

Allowed approximation: $k = p_1 + p_2$, i.e. $p^\mu_{1,2} = (p_0, \pm\mathbf{p})$, $p_0 = \frac{1}{2} M_Z$ in CMS

Integral from squared diagrams:

$$
\begin{aligned}
I_{11} &= \int_{m_\gamma < q_0 < \Delta E} \frac{\mathrm{d}^3\mathbf{q}}{2q_0} \frac{m_f^2}{(qp_1)^2}, & qp_1 &= q_0 p_0 - |\mathbf{q}|\,|\mathbf{p}|\,\cos\theta \\
&= 2\pi \int_{m_\gamma < q_0 < \Delta E} \frac{\mathrm{d}|\mathbf{q}|\,\mathbf{q}^2}{2q_0} \int_{-1}^{+1} \mathrm{d}\cos\theta\, \frac{m_f^2}{(q_0 p_0 - |\mathbf{q}|\,|\mathbf{p}|\,\cos\theta)^2} \\
&= 2\pi \int_{m_\gamma < q_0 < \Delta E} \frac{\mathrm{d}|\mathbf{q}|\,\mathbf{q}^2}{q_0} \frac{m_f^2}{q_0^2 p_0^2 - \mathbf{q}^2\mathbf{p}^2}, & \mathbf{q}^2 &= q_0^2 - m_\gamma^2, \quad \mathbf{p}^2 = p_0^2 - m_f^2 \\
&= 2\pi \int_{m_\gamma}^{\Delta E} \mathrm{d}q_0\, \frac{m_f^2 \sqrt{q_0^2 - m_\gamma^2}}{[q_0^2 - m_\gamma^2] m_f^2 + p_0^2 m_\gamma^2}, & m_\gamma \sqrt{z} &\equiv \sqrt{q_0^2 - m_\gamma^2} + q_0 \\
&= \pi \int_1^{4\Delta E^2/m_\gamma^2} \mathrm{d}z\, \frac{m_f^2 (z-1)^2}{z[m_f^2 (z-1)^2 + 4 z p_0^2]} + \dots \\
&= \pi \int_1^{4\Delta E^2/m_\gamma^2} \mathrm{d}z \left[\frac{1}{z} - \frac{1}{z - z_1} + \frac{1}{z - z_2}\right] + \dots, & z_1 &= \frac{1}{z_2} = -\frac{m_f^2}{4p_0^2} + \dots \\
&= \pi \left[\ln\left(\frac{4\Delta E^2}{m_\gamma^2}\right) + \ln\left(\frac{m_f^2}{M_Z^2}\right)\right] + \dots, & \text{note: } \frac{\Delta E}{m_\gamma} &\gg \frac{M_Z}{m_f} \gg 1 \\
&= I_{22}
\end{aligned}
$$

## Integral from interference diagrams:

$$I_{12} = \int_{m_\gamma < q_0 < \Delta E} \frac{\mathrm{d}^3\mathbf{q}}{2q_0} \frac{(p_1 p_2)}{(qp_1)(qp_2)}, \qquad qp_{1,2} = q_0 p_0 \pm |\mathbf{q}|\,|\mathbf{p}|\,\cos\theta$$

$$= 2\pi \int_{m_\gamma < q_0 < \Delta E} \frac{\mathrm{d}|\mathbf{q}|\mathbf{q}^2}{2q_0} \int_{-1}^{+1} \mathrm{d}\cos\theta \, \frac{(p_1 p_2)}{q_0^2 p_0^2 - \mathbf{q}^2\mathbf{p}^2\,\cos^2\theta}$$

$$= 2\pi \int_{m_\gamma}^{\Delta E} \frac{\mathrm{d}q_0}{q_0} \, \ln\left(\frac{q_0 p_0 + |\mathbf{q}|\,|\mathbf{p}|}{q_0 p_0 - |\mathbf{q}|\,|\mathbf{p}|}\right)$$

$$\vdots$$

$$= \pi\left[4\ln\left(\frac{2\Delta E}{m_\gamma}\right)\ln\left(\frac{M_\mathrm{Z}}{m_f}\right) - 2\ln^2\left(\frac{m_f}{M_\mathrm{Z}}\right) - \frac{\pi^2}{3}\right] + \dots$$

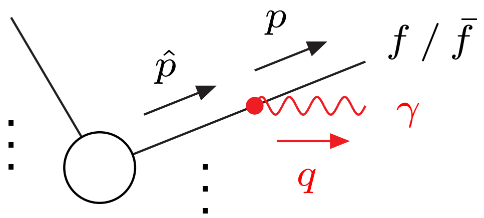
## Soft-photon factor for $\mathrm{Z} \to f\bar{f}\gamma$:

$$\delta_\mathrm{soft} = \frac{Q_f^2\alpha}{2\pi^2}\left[2I_{12} - I_{11} - I_{22}\right]$$

$$= -\frac{Q_f^2\alpha}{\pi}\left\{2\ln\left(\frac{2\Delta E}{m_\gamma}\right)\left[1 + 2\ln\left(\frac{m_f}{M_\mathrm{Z}}\right)\right] + 2\ln^2\left(\frac{m_f}{M_\mathrm{Z}}\right) + 2\ln\left(\frac{m_f}{M_\mathrm{Z}}\right) + \frac{\pi^2}{3}\right\}$$

## 3.3 Collinear-singular contributions

General collinear factorization for final-state radiation:




$$\langle|\mathcal{M}_\gamma(p,q)|^2\rangle \underset{qp\to 0}{\sim} \frac{Q_f^2\, e^2}{qp}\left[\frac{1+z^2}{1-z} - \frac{m_f^2}{qp}\right]\langle|\mathcal{M}_0(\hat{p})|^2\rangle$$

with $z = \dfrac{p^0}{\hat{p}^0}$

Comments:

- asymptotics valid both for outgoing $f$ and $\bar{f}$

  (can be transferred to incoming fermions via appropriate substitutions)

- for “non-exceptional” photon gauges ($|\hat{p}n| \gg m_f$ if $\varepsilon^* n = 0$) only subgraphs of shown type contribute to leading asymptotics

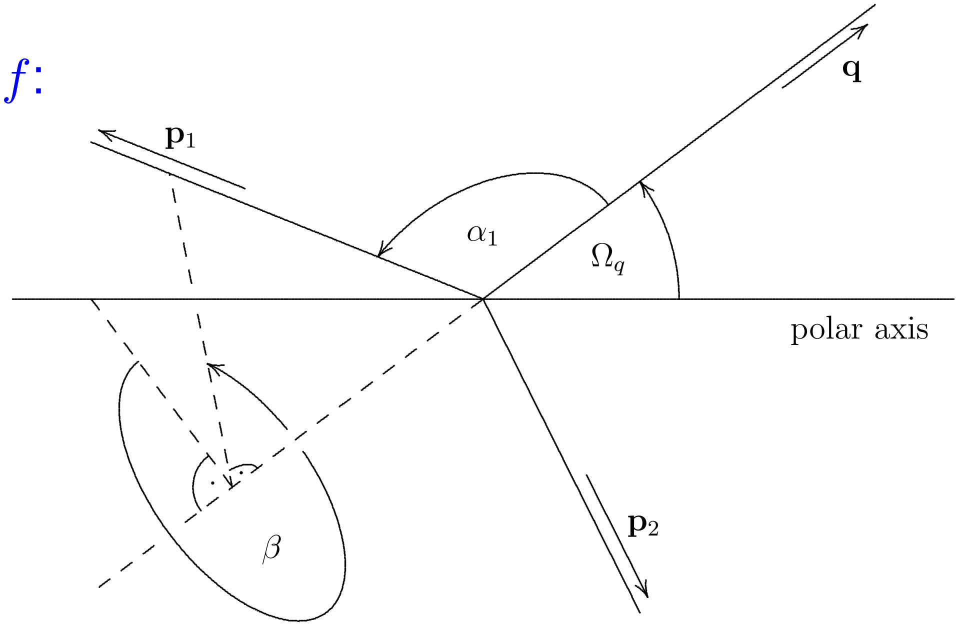
- kinematics in collinear regime:

$$\begin{aligned} q^\mu &= (1-z)\hat{p}^\mu + q_\perp^\mu + q_r^\mu, \\ p^\mu &= z\hat{p}^\mu - q_\perp^\mu - q_r^\mu, \end{aligned}$$

with $\hat{p}q_\perp = 0$, $\quad \mathbf{q}_r = \mathbf{0}$, $\quad \mathbf{q}_\perp^2 = \mathcal{O}(m_f^2)$, $\quad q_r^0 = -q_\perp^0 = \mathcal{O}(m_f^2/\hat{p}^0)$

## Phase-space for collinear region of $\gamma$ and $f$:

$$\hat{p}_1^0 = \tfrac{1}{2} M_Z$$
$$q^0 = \tfrac{1}{2} M_Z(1-z)$$
$$q p_1 = q^0 p_1^0 (1-\cos\alpha_1) = M_Z(q^0 + p_1^0 - \tfrac{1}{2} M_Z)$$




$$\int_{\alpha_1<\Delta\alpha} \mathrm{d}\Phi_{1\to 3} = \frac{1}{8(2\pi)^5} \int \underbrace{\mathrm{d}\Omega_q}_{=\mathrm{d}\Omega_{\hat{p}}} \int \mathrm{d}q^0 \int \mathrm{d}\beta \int \mathrm{d}p_1^0\, \theta(\Delta\alpha - \alpha_1)$$

for $\alpha_1 \to 0$

$$= \frac{1}{8(2\pi)^5} \int \mathrm{d}\Omega_{\hat{p}} \int_{\Delta E}^{\frac{1}{2}M_Z} \mathrm{d}q^0 \int_0^{2\pi} \mathrm{d}\beta \int_{1-\frac{1}{2}\Delta\alpha^2}^{1} \mathrm{d}\cos\alpha_1 \left| \frac{\partial p_1^0}{\partial \cos\alpha_1} \right|$$

$$\left. \frac{\partial p_1^0}{\partial\cos\alpha_1} \right|_{q^0} = \frac{M_Z q^0 (2q^0 - M_Z)}{2[M_Z - q^0(1-\cos\alpha_1)]^2} \underset{\alpha_1\to 0}{\sim} \frac{q^0(2q^0-M_Z)}{2M_Z} = -\frac{1}{4} M_Z\, z(1-z)$$

$$\Rightarrow \int_{\alpha_1<\Delta\alpha} \mathrm{d}\Phi_{1\to 3} = \frac{M_Z^2}{64(2\pi)^5} \int \mathrm{d}\Omega_{\hat{p}} \int_0^{1-2\Delta E/M_Z} \mathrm{d}z\, z(1-z) \int_0^{2\pi} \mathrm{d}\beta \int_0^{\Delta\alpha} \mathrm{d}\alpha_1\, \sin\alpha_1$$

## Integration over collinear regime:

Collecting results yields

$$\Gamma_{Z\to f\bar{f}}\Big|_{\mathrm{coll},f} = \frac{1}{2M_Z}\int_{\alpha_1<\Delta\alpha} \mathrm{d}\Phi_{1\to 3}\,\langle|\mathcal{M}_\gamma|^2\rangle$$

$$= \underbrace{\frac{1}{2M_Z}\,\frac{M_Z^2}{64(2\pi)^5}\int \mathrm{d}\Omega_{\hat{p}}\,\langle|\mathcal{M}_0(\hat{p})|^2\rangle}_{=\frac{M_Z^2}{8(2\pi)^3}\Gamma_{Z\to f\bar{f},0}} \int_0^{1-2\Delta E/M_Z} \mathrm{d}z\, z(1-z)\underbrace{\int \mathrm{d}\beta}_{\to 2\pi}$$

$$\times \int_0^{\Delta\alpha}\mathrm{d}\alpha_1\,\sin\alpha_1\,\frac{Q_f^2\,e^2}{qp}\left[\frac{1+z^2}{1-z}-\frac{m_f^2}{qp}\right]$$

$$\equiv \Gamma_{Z\to f\bar{f},0}\,\delta_{\mathrm{coll},f}$$

with

$$\delta_{\mathrm{coll},f} = \frac{Q_f^2\,\alpha}{8\pi}\int_0^{1-2\Delta E/M_Z}\mathrm{d}z\,z(1-z)\int_0^{\Delta\alpha}\mathrm{d}\alpha_1\,\sin\alpha_1\,\frac{M_Z^2}{qp}\left[\frac{1+z^2}{1-z}-\frac{m_f^2}{qp}\right]$$

$$\equiv \frac{Q_f^2\,\alpha}{8\pi}\int_0^{1-2\Delta E/M_Z}\mathrm{d}z\,z(1-z)\left[\frac{1+z^2}{1-z}K_1-K_2\right]$$

## Integration over collinear regime (continued):

Asymptotic behaviour of $qp_1$: $\alpha_1 < \Delta\alpha \ll 1, \quad m_f \ll M_Z$

$$qp_1 = q^0(p_1^0 - |\mathbf{p}_1|\cos\alpha_1) \sim q^0 p_1^0 \left(1 - \cos\alpha_1 + \frac{m_f^2}{2(p_1^0)^2}\right)$$

$$\sim \tfrac{1}{4} M_Z^2\, z(1-z)\left(1 - \cos\alpha_1 + \frac{2m_f^2}{z^2 M_Z^2}\right)$$

But: note hierarchy $\frac{m_f}{M_Z} \ll \Delta\alpha \ll 1$

Angular integration:

$$K_1 = \int_0^{\Delta\alpha} \mathrm{d}\alpha_1\, \sin\alpha_1\, \frac{M_Z^2}{qp_1}$$

## Integration over collinear regime (continued):

Asymptotic behaviour of $qp_1$: $\alpha_1 < \Delta\alpha \ll 1, \quad m_f \ll M_Z$

$$qp_1 = q^0(p_1^0 - |\mathbf{p}_1|\cos\alpha_1) \sim q^0 p_1^0 \left(1 - \cos\alpha_1 + \frac{m_f^2}{2(p_1^0)^2}\right)$$

$$\sim \tfrac{1}{4} M_Z^2\, z(1-z)\left(1 - \cos\alpha_1 + \frac{2m_f^2}{z^2 M_Z^2}\right)$$

But: note hierarchy $\frac{m_f}{M_Z} \ll \Delta\alpha \ll 1$

Angular integration:

$$K_1 = \int_0^{\Delta\alpha} \mathrm{d}\alpha_1\, \sin\alpha_1\, \frac{M_Z^2}{qp_1}$$

$$= \frac{4}{z(1-z)} \int_0^{\Delta\alpha} \mathrm{d}\alpha_1\, \frac{\sin\alpha_1}{1 - \cos\alpha_1 + \frac{2m_f^2}{z^2 M_Z^2}} + \ldots$$

## Integration over collinear regime (continued):

### Asymptotic behaviour of $qp_1$: $\alpha_1 < \Delta\alpha \ll 1, \quad m_f \ll M_Z$

$$
\begin{aligned}
qp_1 &= q^0(p_1^0 - |\mathbf{p}_1|\cos\alpha_1) \sim q^0 p_1^0 \left(1 - \cos\alpha_1 + \frac{m_f^2}{2(p_1^0)^2}\right) \\
&\sim \tfrac{1}{4} M_Z^2\, z(1-z) \left(1 - \cos\alpha_1 + \frac{2m_f^2}{z^2 M_Z^2}\right)
\end{aligned}
$$

But: note hierarchy $\dfrac{m_f}{M_Z} \ll \Delta\alpha \ll 1$

### Angular integration:

$$
\begin{aligned}
K_1 &= \int_0^{\Delta\alpha} \mathrm{d}\alpha_1\, \sin\alpha_1\, \frac{M_Z^2}{qp_1} \\
&= \frac{4}{z(1-z)} \int_0^{\Delta\alpha} \mathrm{d}\alpha_1\, \frac{\sin\alpha_1}{1 - \cos\alpha_1 + \frac{2m_f^2}{z^2 M_Z^2}} + \ldots \\
&= \frac{4}{z(1-z)} \left[ \ln\left(1 - \cos\alpha_1 + \frac{2m_f^2}{z^2 M_Z^2}\right) \right]_0^{\Delta\alpha} + \ldots
\end{aligned}
$$

## Integration over collinear regime (continued):

Asymptotic behaviour of $qp_1$: $\alpha_1 < \Delta\alpha \ll 1, \quad m_f \ll M_Z$

$$qp_1 = q^0(p_1^0 - |\mathbf{p}_1|\cos\alpha_1) \sim q^0 p_1^0 \left(1 - \cos\alpha_1 + \frac{m_f^2}{2(p_1^0)^2}\right)$$

$$\sim \tfrac{1}{4} M_Z^2\, z(1-z) \left(1 - \cos\alpha_1 + \frac{2m_f^2}{z^2 M_Z^2}\right)$$

But: note hierarchy $\quad \dfrac{m_f}{M_Z} \ll \Delta\alpha \ll 1$

Angular integration:

$$K_1 = \int_0^{\Delta\alpha} d\alpha_1 \,\sin\alpha_1 \frac{M_Z^2}{qp_1}$$

$$= \frac{4}{z(1-z)} \int_0^{\Delta\alpha} d\alpha_1 \frac{\sin\alpha_1}{1 - \cos\alpha_1 + \frac{2m_f^2}{z^2 M_Z^2}} + \ldots$$

$$= \frac{4}{z(1-z)} \left[ \ln\left(1 - \cos\alpha_1 + \frac{2m_f^2}{z^2 M_Z^2}\right)\right]_0^{\Delta\alpha} + \ldots$$

$$= \frac{4}{z(1-z)} \left[ \ln\left(\frac{\Delta\alpha^2}{2}\right) - \ln\left(\frac{2m_f^2}{z^2 M_Z^2}\right)\right] + \mathcal{O}(\Delta\alpha) + \mathcal{O}\left(\frac{m_f}{M_Z}\right)$$

## Integration over collinear regime (continued):

$$K_2 \;=\; \int_0^{\Delta\alpha} \mathrm{d}\alpha_1 \, \sin\alpha_1 \, \frac{m_f^2 \, M_Z^2}{(qp_1)^2}$$

## Integration over collinear regime (continued):

$$
\begin{aligned}
K_2 &= \int_0^{\Delta\alpha} \mathrm{d}\alpha_1 \, \sin\alpha_1 \, \frac{m_f^2 \, M_Z^2}{(qp_1)^2} \\
&= \frac{16 m_f^2}{M_Z^2 \, z^2 (1-z)^2} \int_0^{\Delta\alpha} \mathrm{d}\alpha_1 \, \frac{\sin\alpha_1}{\left(1 - \cos\alpha_1 + \frac{2m_f^2}{z^2 M_Z^2}\right)^2} + \ldots
\end{aligned}
$$

## Integration over collinear regime (continued):

$$K_2 = \int_0^{\Delta\alpha} \mathrm{d}\alpha_1 \, \sin\alpha_1 \, \frac{m_f^2 \, M_Z^2}{(qp_1)^2}$$

$$= \frac{16 m_f^2}{M_Z^2 \, z^2 (1-z)^2} \int_0^{\Delta\alpha} \mathrm{d}\alpha_1 \, \frac{\sin\alpha_1}{\left(1 - \cos\alpha_1 + \frac{2m_f^2}{z^2 M_Z^2}\right)^2} + \ldots$$

$$= \frac{16 m_f^2}{M_Z^2 \, z^2 (1-z)^2} \left[ \frac{-1}{1 - \cos\alpha_1 + \frac{2m_f^2}{z^2 M_Z^2}} \right]_0^{\Delta\alpha} + \ldots$$

## Integration over collinear regime (continued):

$$
\begin{aligned}
K_2 &= \int_0^{\Delta\alpha} \mathrm{d}\alpha_1 \, \sin\alpha_1 \, \frac{m_f^2 \, M_Z^2}{(qp_1)^2} \\
&= \frac{16 m_f^2}{M_Z^2 \, z^2 (1-z)^2} \int_0^{\Delta\alpha} \mathrm{d}\alpha_1 \, \frac{\sin\alpha_1}{\left(1 - \cos\alpha_1 + \frac{2m_f^2}{z^2 M_Z^2}\right)^2} + \dots \\
&= \frac{16 m_f^2}{M_Z^2 \, z^2 (1-z)^2} \left[ \frac{-1}{1 - \cos\alpha_1 + \frac{2m_f^2}{z^2 M_Z^2}} \right]_0^{\Delta\alpha} + \dots \\
&= \frac{16 \, m_f^2}{M_Z^2 \, z^2 (1-z)^2} \, \frac{z^2 M_Z^2}{2 m_f^2} + \dots \\
&= \frac{8}{(1-z)^2} + \mathcal{O}(\Delta\alpha) + \mathcal{O}\left(\frac{m_f}{M_Z}\right)
\end{aligned}
$$

Note:

- non-commutativity of limit $m_f \to 0$ and integration obvious
- result independent of $\Delta\alpha$ !

## Integration over collinear regime (continued):

Insertion of $K_1$ and $K_2$ yields

$$\begin{aligned}
\delta_{\mathrm{coll},f} &= \frac{Q_f^2\,\alpha}{8\pi} \int_0^{1-2\Delta E/M_\mathrm{Z}} \mathrm{d}z\, z(1-z)\left[\frac{1+z^2}{1-z}K_1 - K_2\right] \\
&= \frac{Q_f^2\,\alpha}{\pi} \int_0^{1-2\Delta E/M_\mathrm{Z}} \mathrm{d}z \left[\frac{(1+z^2)}{1-z}\ln\left(\frac{zM_\mathrm{Z}\Delta\alpha}{2m_f}\right) - \frac{z}{1-z}\right] \\
&\;\vdots \\
&= \frac{Q_f^2\,\alpha}{\pi}\left[2\ln\left(\frac{M_\mathrm{Z}}{2\Delta E}\right)\ln\left(\frac{\Delta\alpha M_\mathrm{Z}}{2m_f}\right) + \ln\left(\frac{2\Delta E}{M_\mathrm{Z}}\right) - \frac{3}{2}\ln\left(\frac{\Delta\alpha M_\mathrm{Z}}{2m_f}\right) + \frac{9}{4} - \frac{\pi^2}{3}\right] \\
&= \delta_{\mathrm{coll},\bar{f}} \qquad \text{because of symmetry}
\end{aligned}$$

### Comment:

Result is generally valid for collinear final-state radiation

off (anti-)fermions with momentum $\hat{p}$ after substituting $\frac{1}{2}M_\mathrm{Z} \to \hat{p}^0$

## 3.4 The final result

Complete real photon correction factor for $\mathrm{Z} \to f\bar{f}\gamma$:

$$
\begin{aligned}
\delta_{\mathrm{real}} &= \delta_{\mathrm{hard}} + \delta_{\mathrm{soft}} + \delta_{\mathrm{coll},f} + \delta_{\mathrm{coll},\bar{f}} \\
&= \frac{Q_f^2\,\alpha}{\pi}\Bigg[4\ln\left(\frac{m_\gamma}{M_\mathrm{Z}}\right)\ln\left(\frac{m_f}{M_\mathrm{Z}}\right) - 2\ln^2\left(\frac{m_f}{M_\mathrm{Z}}\right) \\
&\qquad + 2\ln\left(\frac{m_\gamma}{M_\mathrm{Z}}\right) + \ln\left(\frac{m_f}{M_\mathrm{Z}}\right) + \frac{11}{4} - \frac{2\pi^2}{3}\Bigg]
\end{aligned}
$$

Comments:

- Dependence on auxiliary cutoff parameters $\Delta E$ and $\Delta\alpha$ drop out !

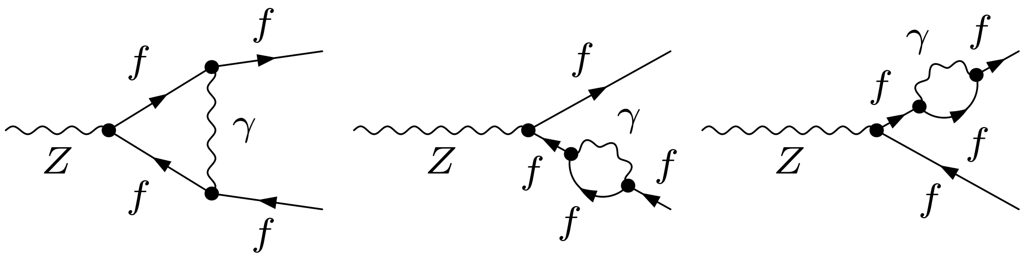
  But: cancellation of cutoff dependence achieved only numerically if non-singular integration is performed numerically

  $\hookrightarrow$ worsens numerical accuracy

  Alternative to slicing: subtraction techniques (no cutoff parameters)

- Soft singularity shows up as (unphysical) $\ln m_\gamma$ terms
- Collinear singularities show up as $\ln m_f$ terms
- Result is correct up to mass-suppressed terms of $\mathcal{O}(m_f)$

## Anticipation of virtual photonic correction: (see part II of exercises)




$$\delta_{\mathrm{virt,phot}} = \frac{Q_f^2\,\alpha}{\pi}\Bigg[-4\ln\left(\frac{m_\gamma}{M_Z}\right)\ln\left(\frac{m_f}{M_Z}\right)+2\ln^2\left(\frac{m_f}{M_Z}\right) -2\ln\left(\frac{m_\gamma}{M_Z}\right)-\ln\left(\frac{m_f}{M_Z}\right)-2+\frac{2\pi^2}{3}\Bigg]$$

## Complete photonic $\mathcal{O}(\alpha)$ correction for $\mathrm{Z}\to f\bar{f}(\gamma)$:

$$\delta_{\mathrm{phot}} = \delta_{\mathrm{real}} + \delta_{\mathrm{virt,phot}} = \frac{3Q_f^2\,\alpha}{4\pi}$$

## Comments:

- Soft-singular terms $\propto \ln m_\gamma$ cancel (Bloch–Nordsieck theorem)
- Collinear-singular terms $\propto \ln m_f$ also drop out !

  Reason: inclusiveness of initial and final state w.r.t. (one-)photon emission

  $\hookrightarrow$ Kinoshita–Lee–Nauenberg theorem guarantees cancellation of singularities