I. Short Introduction to Cosmology

1. Historical notes:

The meaning of the word cosmology stems from antient grek:

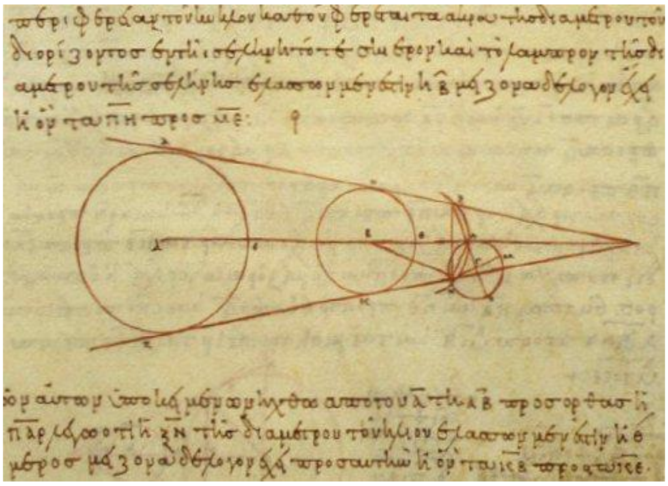
ὁ κόσμοσ	the universe
ὁ λόγοσ	the word

~. 600 BC: Anaximander and Thales von Milet

- Knowledge that earth is round. Prediction of solar eclipse 585 b.C.!

~. 310 – 230 BC.: Aristarch of Samos

- Heliocentrical world model
- Calculates size of moon and distance to sun!
- Rightly estimates distance to stars using parallax!

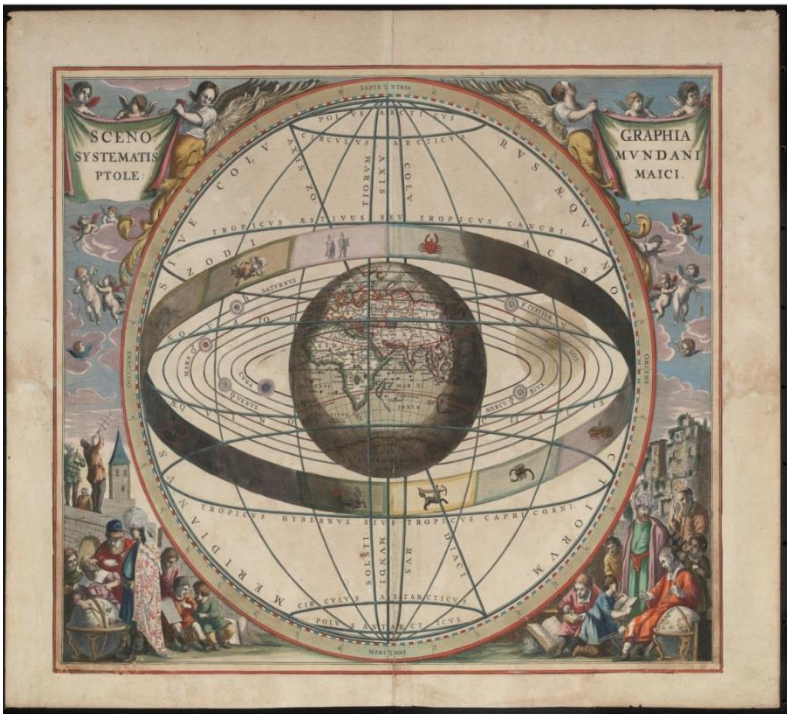


Aristarch of Samos
(3. century BC) :
Calculation of size of moon and distance to sun

83 – 161 AC: Claudius Ptolemäus

- Almagest: Astronomical textbook: Geocentrical world model
- Did explain all observations at its time!

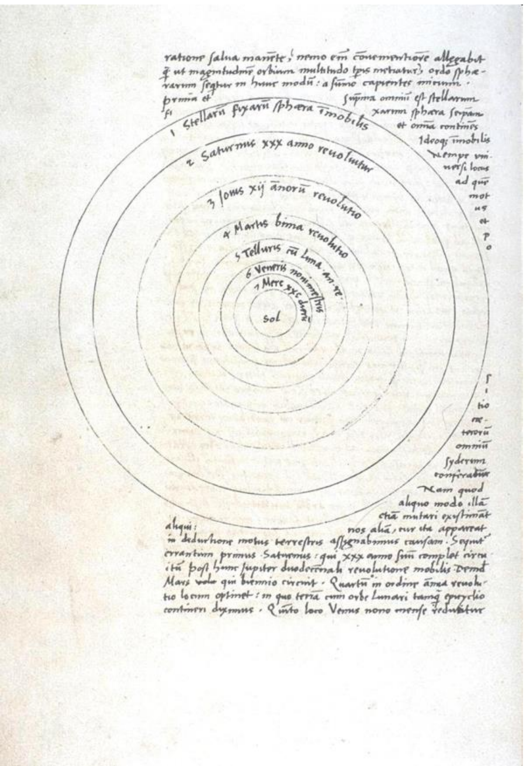
➔ Ptolemäic geo-centrical world model



1473 – 1543 Nicolaus Copernicus

- Helio-centric world model: „de revolutionibus orbium coelestium“ (Von den Umdrehungen der Himmelskörper)
- Perfection of circular movement
- Rightly estimates distance to stars using prallax

➔ Copernican helio-central world-model


ratione salua manente; nemo enī conuenientiore allegabit q̄ ut magnitudine orbium multitudo tpis metiatur; ordo sphærarum sequitur in hunc modū: a summo capientes initium · Prima et suprema omniū est stellarum fixarum sphæra seipsam et omnia continens Ideoq; imobilis Nempe uniuersi locus ad quē motus et positio ceterorū omniū syderum conferatur Nam quod aliquo modo illā etiā mutari existimant aliqui: nos aliā, cur ita appareat in deductione motus terrestris assignabimus causam. Sequitur errantium primus Saturnus: qui xxx anno suū complet circuitū post hunc Jupiter duodecennali reuolutione mobilis Deinde Mars ~~volo~~ qui biennio circuit · Quartū in ordine ānua reuolutio locum obtinet: in quo terra cum orbe Lunari tamq̄ epicyclo contineri diximus · Quinto loco Venus nono mense reducitur

Copernican world-model: Page from „De Revolutionibus Orbium Coelestium“

1572: Tycho Brahe observes Supernova without observing parallax

→ Sphere of stars is NOT eternal!

1643 – 1727 Sir Isaac Newton:

- Couples astronomical (cosmological) observations to earthly mechanics!
- Predicts clumping due to gravity → Predicts contraction of universe that is not observed
  → eternal, homogeneous Universe!

1750 Thomas Wright of Durham:

- Interpretation of sun as on one star of many in our galaxy (nebula)→ "Island universe"

1755 Immanuel Kant:

- Interpretation of nebulae as galaxies as our own

2. Fundamental Observations

Let's assume (most intuitive???)

- homogeneous
- eternal, endless
- euklidean
- static

Most intuitive ???

Univers → Olbers Paradoxon:

For above assupmtion (static & endless): Always find a star in line of sight.
→ As surface brightness is independent of distance, night sky should be bright as sun

→ At least one assumption wrong!

We observe:

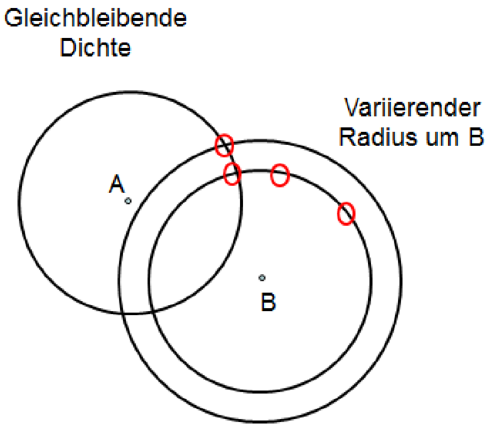
On "large scales" universe is homogeneous:

- Cosmic Microwave
  Distribution of Galaxies

No scientific reason why our position in universe should be special

→ Universe should be isotropic from all positions

Assumption of isotropic universe at two distinct points in the universe leads to conclusion of homogeneity of universe!



Due to isotropy at each equidistant point around A density ist he same. The same holds for equidistant points around B.

→Isotropy around A AND B can only be given if universe is homogeneous!

Cosmological Principle:

Our universe is

- homogeneous

- isotropic

Redshift of galaxies:

E. Hubble 1929: Publication of Hubbles law (verry sloppy!).
[wrong units; no uncertainties; removal of data without „good reason“, no statistical significance given]

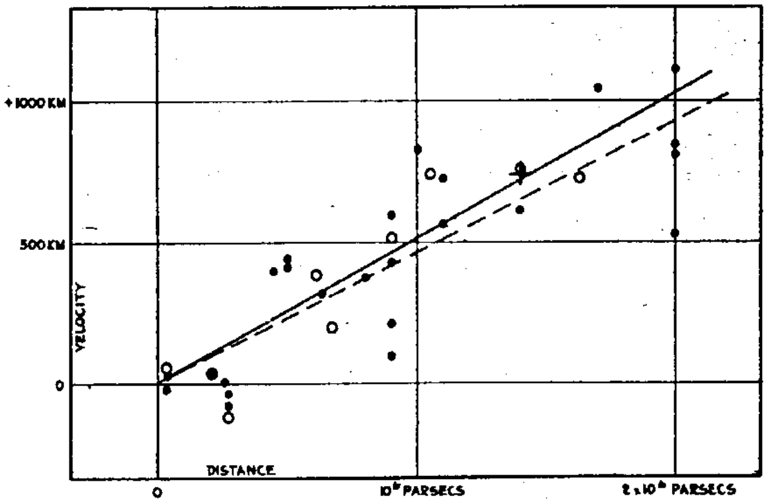


FIGURE 1

Hubble diagram in original Publication (Proc. NAS, Vol 15, 1929).

Observation that (apart from a few galaxies in local group) all galaxies move away from us, the faster the further away:

➔ Hubble law $V \cong H_0 d$

withe H_0 Hubble parameter and d distance.

H_0 correpsonds to present expansion rate of Universe.

$H_0 = h\ 100$ km/s Mpc^{-1}

What is Red shift?

→ Doppler Effect:

Moving source with (radial-) velocity v relative to observer → Wave-length changes

$$\Delta\lambda = \lambda \cdot \frac{v}{c}, \text{ or } \Delta\nu = \nu \cdot \frac{v}{c} \qquad (\nu = \frac{c}{\lambda})$$

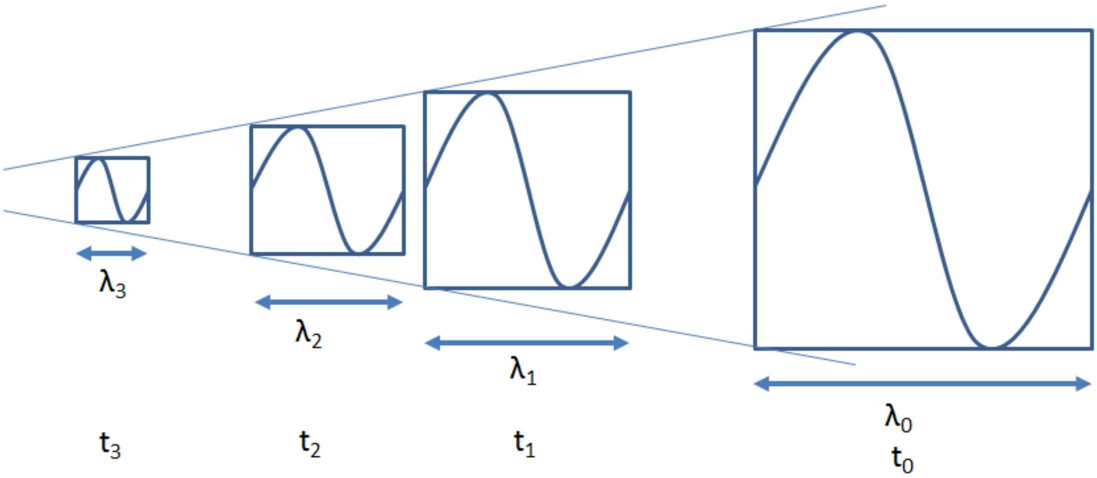
→ Looks like all galaxies are moving away from us.

This would mean: We are at the *"center of the Universe"*

Contradiction to Cosmological Principle!

Alternative interpretation:

Red-shift appears due to expansion of space since the time light was emitted



→ At every location of the universe: galaxies seem to move away isotropically from observer

→ Universe NOT static?

Consequence of Hubble expansion:

$H_0 = h\ 100\ \mathrm{km/s\ Mpc}^{-1}$

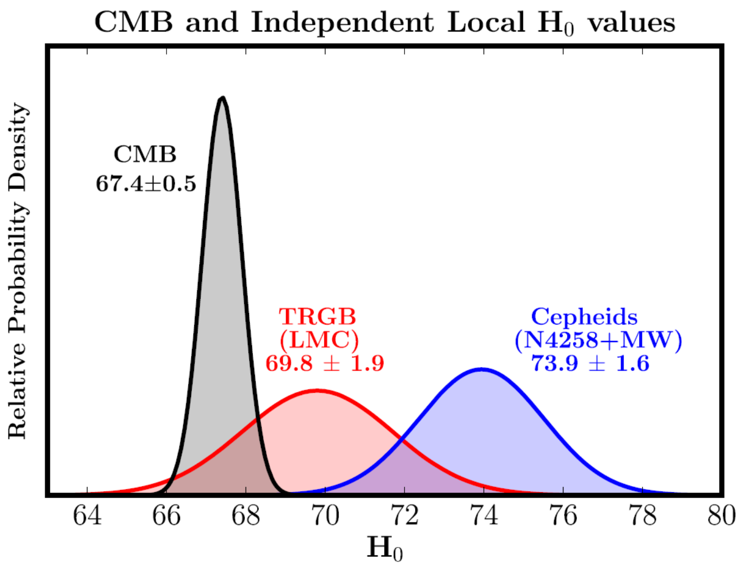
Currently: dispute about value: evidence for beyond standard model cosmology?
W. Freedman et al The Astrophysical Journal, 882:34 (29pp), 2019 September 1 1907.05922

Different measurements of h:

CMB:
h = (67.4±0.5)
Planck Collaboration arXiv:1807.06209

SNIa/RGB:
h = 69.8 ± 0.8 (stat) ±1.7 (sys)
W. Freedman et al The Astrophys. J., 882:34 (29pp), 2019 September 1 1907.05922

SNIa/Cepheids:
h=74.22 +/- 1.82, Riess et al.,The Astrophys. Journal, 876:85 (13pp), 2019



THE ASTROPHYSICAL JOURNAL, 882:34 (29pp), 2019 September 1

Figure 18. Completely independent calibrations of H_0. Shown in red is the probability density function based on our LMC CCHP TRGB calibration of CSP-I SNe Ia ; in blue is the Cepheid calibration of H_0 (Riess et al. 2016) using the Milky Way parallaxes and maser distance to NGC 4258 as anchors (excluding the LMC). The Planck value of H_0 is shown in black.

→„Expansion of space, i.e. Universe: to Mpc per second 100 km space „are added“

Or:

$H_0 = h \cdot 3.24 \cdot 10^{-18}\ s^{-1}$ or $H_0 = h \cdot \frac{\text{å}}{m} yr^{-1}$

→ constant H_0 : Distance of 1m today was = at time Hubble time $T_{hubble} = 1/h \cdot 10^{-10}$ years!

→ characteristic time scale of our universe

Hubble time well agrees with age of oldest known object in the universe!

→ Total energy content of universe was gathered in much smaller space.

→ Energy density was very high → Universe was „very hot“

→“Big Bang“

Hubble-Distance: $D_{hubble} = c\ T_{hubble} = c/\ H_0$

$\cong 1/h$ 3000 Mpc $\cong 1/h\ 10^{10}$ light years

Size of biggest observed structures in galaxy surveys: ~10^9 light year

→Hubble volume $D_{hubble}{}^3$ fits ~10^3 of these structures

→Over Hubble-distance: Assumption of homogeneity seems justified!

3. Einstein´s Field Equations

The derivation of Einstein´s field equations is very technical and matter of a dedicated lecture. Let´s, however, look at the main basic idea leading to its derivation:

The equivalence principle:

Newton´s first law: $F = m_i\, a$

Law of gravitation: $F = m_g\, a$

where m_i is the inertial and m_g is the gravitating mass. The **weak equivalence principle** states that $m_i = m_g$. This a priori cannot be stated. However, measurements show that

$|1 - \frac{m_g}{m_i}| < 10^{-12}$

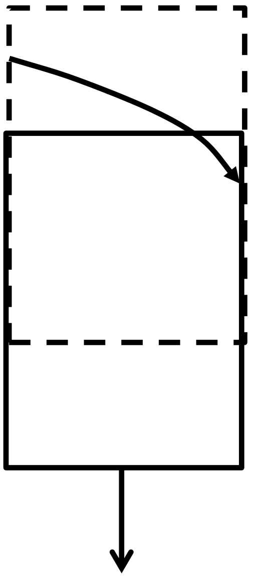
How can this be measured?

If this is generalized to inertial systems we obtain the **strong equivalence principle**:

From within an inertial system there is no way to tell the difference whether one is in free fall in a gravitational potential or whether on does not experience any gravitational force. I.e. there is no way to experimentally distinguish between “free fall in a potential” and “weightlessness”.

[Caution with non-homogeneity of gravitational fields: they cause tidal effects!]

→ Gedankenexperiment:



Consider an inertial system that is in free fall. A light ray propagating through vertically to the force through this inertial system seems straight to an observer in this system. An observer on earth would, however, see a bent light ray

→ Gravitating mass is curving space!

A short look at the relativistic description of space time:

$x^0 = c\,t$, $x^1 = xt$, $x^2 = y$, $x^3 = z$ or

$x^\mu = (c\,t, \vec{r}\,) = (x^0, x^1, x^2, x^3)$

Observation: Invariance of speed of light, i.e. of the line element

For flat (Euclidean) space this means the line element

$$ds^2 = c^2dt^2 - [(dx^1)^2 + (dx^2)^2 + (dx^3)^2]$$

$$= \begin{pmatrix} dx^0 \\ dx^1 \\ dx^2 \\ dx^3 \end{pmatrix}^T \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dx^0 \\ dx^1 \\ dx^2 \\ dx^3 \end{pmatrix}$$

$$= \sum_{\mu,\nu} \eta_{\mu\nu}\, dx^\mu dx^\nu$$

$$= \eta_{\mu\nu} dx^\mu dx^\nu \qquad \text{(Einstein summation rule)}$$

$$= 0$$

The metric is orthogonal → no curvature.

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is the Minkowski metric, i.e. the metric describing a flat space-time (Minkowski space). It is defined by the line element $ds^2 = 0$ in flat Euclidean space.

The development of our universe, i.e. of our space time structure, is determined by the initial conditions and the physical laws that govern its dynamics as a function of its content.

→ **Einstein´s field equation:**

$$\boxed{\overbrace{\boldsymbol{R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu}}}^{\text{Geometrical properties of space-time}} = \overbrace{\boldsymbol{8\, \pi\, G/c^4 T_{\mu\nu}}}^{\text{Source of “field”}}}$$

$R_{\mu\nu}$: Ricci Tensor, $g_{\mu\nu}$: metric tensor (contains time dependent scaling factor), R: Ricci scaler, Λ: Cosmological constant; G. Newton´s gravitational constant,
$T_{\mu\nu}$: Energy momentum tensor

Rough meaning of the equation: The space time structure and its dynamic are determined by its content.

The Einstein tensor: $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ describes the structure of space time. It quantifies the interconnections between the space time dimensions. (see Riemann Tensor, Christoffel symbols).

$8\,\pi\, G/c^4 T_{\mu\nu}$ defines the gravitating energy content of space time.

Λ is a constant that can be added without loss of generality. It corresponds to a "vacuum energy" of space time.

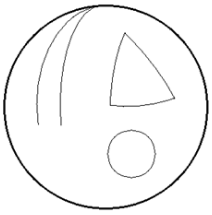
Einstein´s field equations are differential equations. I.e. they describe how the metric, describing space time, evolves driven by the energy content of space time.

4. The Robertson-Walker metric

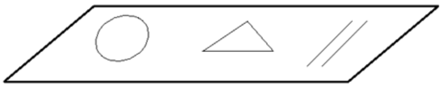
For a homogeneous and isotropic space time, i.e. for a universe according to the cosmological principle it can be shown that the simplest metric can be described by the line element:

$$\boxed{ds^2 = dt^2 - a^2(t)(\frac{dr^2}{1-kr^2} + r^2 d\vartheta^2 + r^2 sin^2\vartheta d\varphi^2),}$$

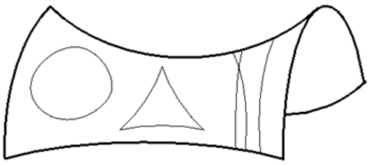
the Robertson-Walker metric.
Here $a(t)$ is a time dependent, dimensionless scaling factor,
r, ϑ, φ are the spherical co-moving coordinates
$k = -1, 0\,, 1$ is the curvature constant.
In a 2 dimensional representation the different geometries as a function of curvature constants can be displayed as a saddle for $k = -1$, a Euclidean plane for $k = 0$ or a sphere for $k = 1$.



$k = +1$



$k = 0$

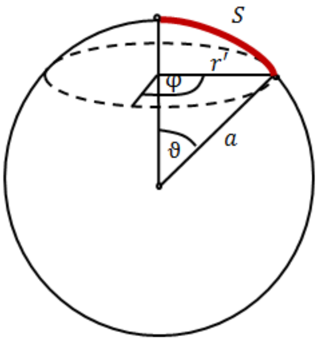


$k = -1$

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{-a^2}{1-kr^2} & 0 & 0 \\ 0 & 0 & -a^2r^2 & 0 \\ 0 & 0 & 0 & -a^2r^2sin^2\vartheta \end{pmatrix}$$

is the metric tensor.

The metric can be derive by demanding that a circle around the pole of a sphere has a circumference (in the metric) of $c = 2\,\pi\, r'$.



$r = \frac{r'}{a} = sin\vartheta$

→ ds^2=$a^2(\frac{dr^2}{1-kr^2}$+$r^2\ d\varphi^2)$

The curvature can be described by

$K = \frac{3}{\pi}\lim_{S\to 0}\frac{2\pi S-c}{S^3}$, where

S = the measured radius of a circle

c = the measured circumference

For two dimensions this is easy to imagine, as it is easy to embed the auxiliary dimension (radius of the sphere, a into euclidean space.

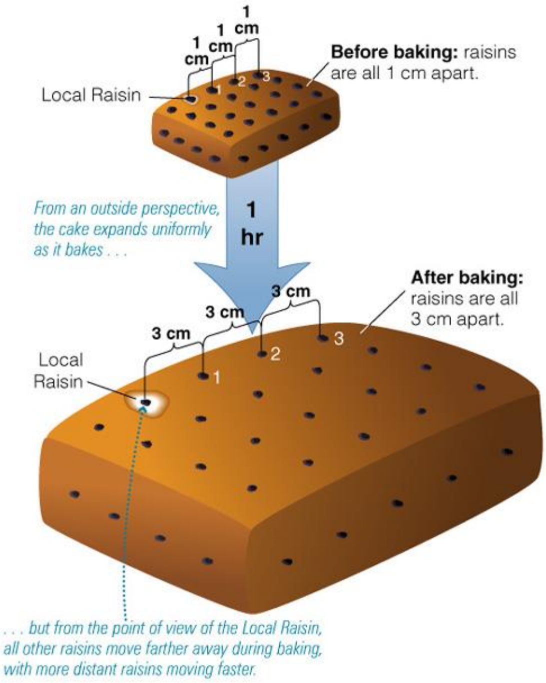
This is mathematically not even needed, as a is dimensionless
→ Same derivation in three dimensional space.

For $k = -1$ even in two dimensions the derivation is as intuitive, the basic idea is, however, the same as for $k = 1$ and $k = 0$.

The concept of co-moving coordinates:

There is a frame of reference in which coordinates r, ϑ, φ are constant while scaling factor $a(t)$ is changing.

For $k = 0$ this can be interpreted as "raisins moving away from each other in a cake in the oven".



For $k = 1$ in a 2 dimensional view, using an *auxiliary dimension* this can be interpreted as an address on the surface of a balloon blown up to radius a. Regardless of the radius, the "address" in terms of the coordinates r, ϑ, φ stays the same.

For $k = -1$: How to interpret a swelling saddle? Let your fantasy play!

It is possible to introduce conformal time: $d\eta = \frac{dt}{a(t)}$ to treat the time coordinate the same way.

→ $ds^2 = a^2(\eta)(d\eta^2 - \frac{dr^2}{1-kr^2} - r^2 d\Omega^2)$ where $d\Omega^2 = d\vartheta^2 + sin^2\vartheta d\varphi^2$

Meaning of the scale factor:

The distance between two points is proportional to the scale factor.

→ For increasing $a(t)$: space is expanding!

Compare to Hubble´s law → $H = \frac{\dot{a}}{a}$

If we now assume that $\dot{a} > 0$ for all t

→ Universe looks the same for all observers at time t_0 regardless of their coordinates.

How to synchronize time of observation t_0?

Use cosmological "clocks" like cosmic background radiation (CMB): There is a well-defined relation between observation time t_0 and the temperature of the CMB
→ There is a non-moving reference system of commoving coordinates.

In reality: There is a dipole moment in the CMB of order 10^{-3}. This can be interpreted as the movement of sun relative to co-moving reference frame. Indeed, there should be movement of: our sun around the galactic center, the movement of our galaxy within the local cluster (Virgo cluster) and the movement of the Virgo cluster towards the "great attractor".

But how does $a(t)$ change with time?

All the dynamics are written in Einstein´s field equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \Lambda g_{\mu\nu} = 8\,\pi\; G/c^4 T_{\mu\nu}$$

The Ricci tensor $R_{\mu\nu}$ contains derivatives of the metric tensors to all coordinates, including time.

The metric tensor $g_{\mu\nu}$ contains the time dependent scaling factor. Solutions need to be found that satisfy the equation.

The energy momentum tensor $T_{\mu\nu}$ describes the energy content of the "cosmic fluid".

The cosmological constant Λ can be added. Only with this constant it is possible to construct a model of a static universe.

To derive the standard model of cosmology the "energy content" of the universe is treated as a "fluid".

An observer in the universe is situated in an inertial system that is in free fall in the "cosmic fluid". It is taken to be a perfect fluid, i.e. non viscous. The strong equivalence principle states that we cannot distinguish between free fall and the lack of a gravitational potential (locally in the free fall frame: $g_{\mu\nu} = \eta_{\mu\nu}$).

In general: $T^{ik} = \left(\rho + \frac{p}{c^2}\right)u^i u^k - \frac{p}{c^2}\eta^{ik}$
where u^i is the four velocity of the cosmic fluid and η^{ik} is the local metric

In the rest frame of the cosmic fluid: $u^i = (1{,}0{,}0{,}0)$

The equivalence principle tells us that space is curved →the metric tensor needs to be used:

$$T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right)u^\mu u^\nu - p g^{\mu\nu}$$

In the rest frame of the fluid this reduces to:

$$T^{\mu\nu} = \frac{1}{c^2}\begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

→ System of differential equations!

In this inertial system the universe appears isotropic!

The De Sitter model:

What would a universe with vanishing Energy-momentum tensor look like?

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \Lambda g_{\mu\nu} = 0$$

→ With the simplest general metric that adheres to the cosmological principle, $g_{\mu\nu}$, and the cosmological constant Λ we can find for the 00 and 11 component (calculate "metric connections"- Christoffel symbols, i.e. derivatives of metric tensor $g_{\mu\nu}$) to get:

$$\underbrace{3\left(\frac{\dot{a}}{a}\right)^2 + \frac{3k}{a^2} = \Lambda}_{\text{From 00 compnonets}} \qquad \text{and} \qquad \underbrace{2\left(\frac{\ddot{a}}{a}\right) + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \Lambda}_{\text{from 11,22 and 33 components}}$$

→ for $k = 0$ & $\Lambda > 0$

$$H^2(t) = \frac{1}{3}\Lambda = \left(\frac{\dot{a}}{a}\right)^2 \qquad \rightarrow \qquad a = \sqrt{\frac{3}{\Lambda}}\dot{a}$$

→ $a(t) = e^{Ht}$ with $H = \sqrt{\frac{\Lambda}{3}}$ → Inflation!

This corresponds to negative pressure in energy momentum tensor in co-moving coordinates:

$$T_{\mu\nu}^{\Lambda} = \frac{1}{c^2}\begin{pmatrix} \rho_\Lambda & 0 & 0 & 0 \\ 0 & -\rho_\Lambda & 0 & 0 \\ 0 & 0 & -\rho_\Lambda & 0 \\ 0 & 0 & 0 & -\rho_\Lambda \end{pmatrix} \qquad \text{where} \qquad \rho_\Lambda = \frac{\Lambda}{8\pi G}$$

The Standard Model of Cosmology

In reality we have to account for the energy content of the universe

→ Include energy-momentum tensor for "cosmic fluid"!

$$T_{\mu\nu} = \frac{1}{c^2}\begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

→ Add term $8\pi G\rho$ to the 00 component. Here ρ describes the total energy density of mass ρ_m and radiation ρ_{rad}. If we define ρ_Λ=$\rho_{vac} = \frac{\Lambda}{8\pi G}$ we obtain the

Friedman equation

$$\boxed{(\frac{\dot{a}}{a})^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho_{tot}}$$

with $\rho_{tot} = \rho_m + \rho_{rad} + \rho_\Lambda$

From equations of 11 & 22 & 33 components we obtain:

$$2(\frac{\ddot{a}}{a}) + (\frac{\dot{a}}{a})^2 + \frac{k}{a^2} = -\,8\pi G p$$

Further we can derive using some arithmetics involving Chistoffel symbols and free-fall frameboundary conditions:

$$\dot{p}a^3 = \frac{d}{dt}(a^3[\rho + p]) \qquad \text{or} \qquad \boxed{\frac{d}{dt}(\rho a^3) = -p\frac{d}{dt}a^3}$$

Compare to first law of thermodynamics:

$$\underset{\text{Inner energy}}{dU} = -\underset{\text{Pressure}}{P}\,dV \longleftarrow \text{Volume change}$$

Equation of state: $p = \alpha\,\rho$

$$\rightarrow \boxed{\rho = const\ a^{-3(1+\alpha)}}$$

We know: $E = mc^2$

→ There are two different energy densities:

- (Relativistic) Radiation: $v \sim c$
  → α=1/3, as pressure is averaged over all directions
- Non relativistic matter: $v \ll c$
  → $\alpha = 0$, as matter does not exert pressure

The energy densities of the two components are denoted ρ_{rad} and ρ_m

$\rho_m \gg \rho_{rad}$ → $\rho \propto a^{-3}$

$\rho_m \ll \rho_{rad}$ → $\rho \propto a^{-4}$

This is straight forward to interpret:
In the first case: the energy density in a co-moving volume element does not change as the number of particles and their mass are constant.
In the second case: the same applies for radiation, i.e. number of photons is not changing, however, their wavelength, does: additional factor a due to red shift.

It is easy to show: $a(t) \propto t^{\frac{2}{3(1+\alpha)}}$

$a_{rad} \propto \sqrt{t}$ for a radiation dominated universe

$a_m \propto t^{\frac{2}{3}}$ for a matter dominated universe

6. The Expansion of the Universe

We have seen (Hubble diagram) that all galaxies seem to be moving away from us and interpreted this as expansion of space.
The apparent velocity with which objects are moving away from us:
$v = \frac{\dot{a}}{a} d = H\, d$ Hubble's law, with $H = \frac{\dot{a(t)}}{a(t)}$ is often called Hubble's constant for t_0
From Friedman equation and equation for 11, 22 and 33 components we get

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)}$$

The present epoch seems matter dominated
→ $p \sim 0$ → $\ddot{a} < 0$, i.e. expansion should be decelerating (if we neglect Λ).

We define the critical density:
$\rho_{crit} = \frac{3H^2}{8\pi G}$

Today we have: $\rho_{crit}(t_0)$~ 9·$10^{-30} \frac{g}{cm^3}$ ~ 6 protons per m^3.

Using this we can re-write the Friedman equation:

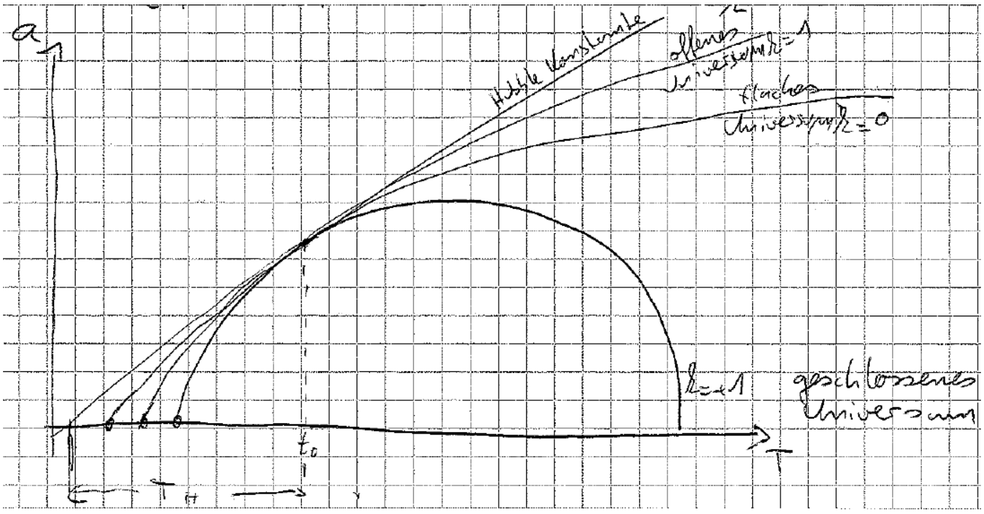
$$(\frac{\dot{a}}{a})^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho_{tot}$$

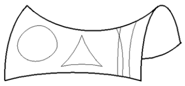
$H^2 + \frac{k}{a^2} = H^2 \frac{\rho_{tot}}{\rho_{krit}}$ or $\frac{\rho_{tot}}{\rho_{krit}} = \Omega = 1 + \frac{k}{H^2 a^2}$

→ $\Omega - 1 = \frac{k}{H^2 a^2}$ where $\Omega = \Omega_m + \Omega_{rad} + \Omega_\Lambda$

This is a differential equation! It states that the form, i.e. geometry of the universe is determined by the total energy content: mass plus radiation plus cosmological constant.

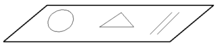
Sei zunächst $\Lambda = 0 \rightarrow \Omega_\Lambda = 0$

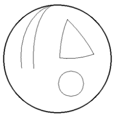




For $\Omega < 1$ → $k = -1$ → open universe, expands forever

For $\Omega = 1$ → $k = 0$ → open Euclidean universe, expansions stops



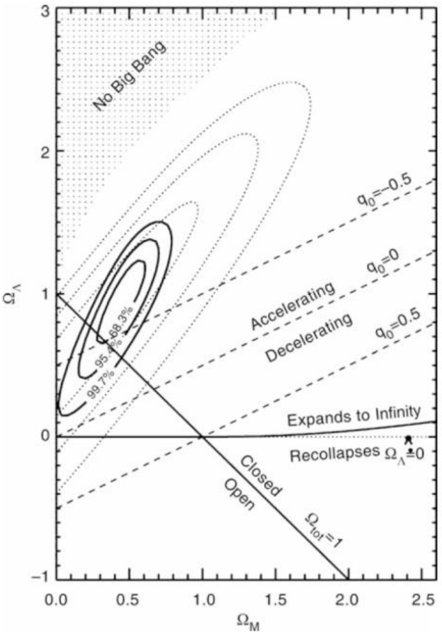


For $\Omega > 1$ → $k = 1$ → closed universe, re-collapses

Including the cosmological constant $\Lambda \neq 0$ complicates the situation.

But let us remember: for universe that is dominated by cosmological constant, i.e.

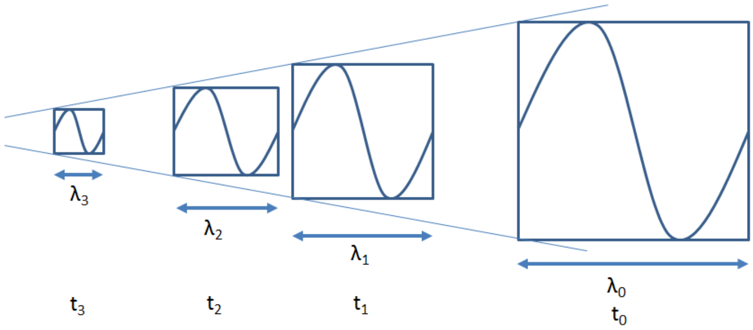
$\Omega_\Lambda \gg \Omega_m, \Omega_{rad}$ we found $a(t) \propto e^{\sqrt{\frac{\Lambda}{3}}t}$



For all parameter combinations except for some with $\Omega_\Lambda > 1$ – which are excluded by observations – there was a time in the past with $a = 0$, i.e. a singularity of space time

→ Big Bang model!

Red Shift revisited cosmologically



We had interpreted the redshift of galaxies as being due to expansion of space while light is propagating from the source galaxy to the observer.
The photon wavelength is growing as the co-moving volume element, ie is scaling with the scaling factor a.

$$\rightarrow \quad \frac{\lambda_{obs}}{\lambda_{emit}} = \frac{a_{obs}}{a_{emit}}$$

Astronomically redshift is defined by $1 + z = \frac{\lambda_{obs}}{\lambda_{emit}} = \frac{a_{obs}}{a_{emit}}$.
The redshift of astronomical objects is quantified by z.

Measurement of cosmological redshift usually happens by identifying the 21cm line of galactic hydrogen clouds or the Lyman-α line at 1216 å.

For example: galaxy identified in Nature 467(210)940
detection of the Lyman-α at 11615.6±2.4 → $z = \frac{11615.6}{1216}$-1 = 8.55

A co-moving line element of 1m today had an extension of $a_{emit} = \frac{a_{obs}}{(1+z)}$ = 10.5 cm when the light observed today was emitted.