ALMA MATER STUDIORUM · UNIVERSITY OF BOLOGNA
School of Science
Department of Physics and Astronomy
Master Degree in Physics

Backreaction from magnetogenesis during inflation

Sofia Corbà

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study of the inflationary evolution in the presence of backreaction effects due to the production of primordial magnetic fields

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Coupling:

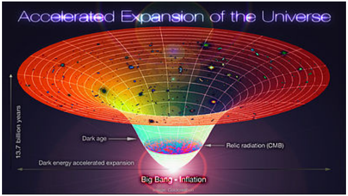
$$\mathcal{L} = -\int d^4x \ \frac{1}{4}\sqrt{-g} \ \ f^2(\phi) \ \ F_{\mu\nu}F^{\mu\nu}$$

Inflation

Inflation

Initial period of accelerated expansion governed by a scalar field

$$S = -\frac{M_{pl}^2}{2}\int d^4x\sqrt{-g}R + \int d^4x\sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right)$$



$$\ddot{\phi} + 3H\dot{\phi} + V_\phi = 0$$

$$H^2 = \frac{1}{3M_{pl}^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right)$$

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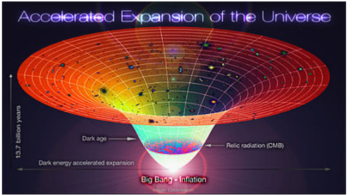

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- Horizon Problem
- Flatness Problem
- CMB Anisotropies
- Large-scale structure

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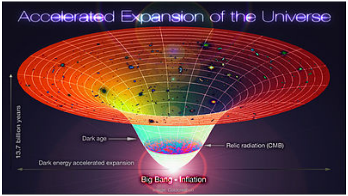

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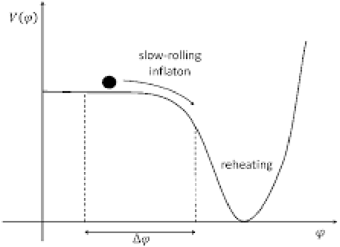

- Horizon Problem
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Reheating $\rightarrow$ Standard Model degrees of freedom

The action of the system is: $S = -\frac{M_{pl}^2}{2}\int d^4x\sqrt{-g}\,\mathcal{R} + S_M$

$$S_M = \int d^4x\sqrt{-g}\,\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) - \frac{\alpha}{4\Lambda}F_{\mu\nu}\tilde{F}^{\mu\nu} - \frac{f^2(\phi)}{4}F_{\mu\nu}F^{\mu\nu}\right)$$

Primordial Magnetogenesis

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The equations become:

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV(\phi)}{d\phi} = \boxed{\frac{1}{2}\frac{d(f^2(\phi))}{d\phi}(\mathbf{E}^2 - \mathbf{B}^2)}$$

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Primordial Magnetogenesis

The equations for the electromagnetic field are:

$$\nabla \cdot \mathbf{E} = 0 \qquad \partial_0(a^2\mathbf{E}) + \frac{1}{f^2(\phi)}\frac{d(f^2(\phi))}{d\phi}\dot{\phi}\, a^2\mathbf{E} - a\nabla \wedge \mathbf{B} = 0$$

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$$\frac{d^2\mathbf{A}}{d\eta^2} + \frac{1}{f^2(\phi)}\frac{d(f^2(\phi))}{d\phi}\frac{d\phi}{d\eta}\frac{d\mathbf{A}}{d\eta} - \nabla^2\mathbf{A} = 0$$

$$A_i(\eta, \mathbf{x}) = \frac{1}{f(\phi)}\sum_{\lambda=\pm}\int\frac{d\mathbf{k}}{(2\pi)^{3/2}}u_{\mathbf{k}}^{\lambda}(\eta)\epsilon_i^{\lambda}(\mathbf{k})e^{i\mathbf{kx}}$$

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$$\boxed{\frac{d^2 u_{\mathbf{k}}^{\lambda}(\eta)}{d\eta^2} + \left(k^2 - \frac{1}{f(\phi)}\frac{d^2 f(\phi)}{d\eta^2}\right)u_{\mathbf{k}}^{\lambda}(\eta) = 0}$$

Primordial Magnetogenesis

$$\rho_E = \frac{H^2 f^2(\phi)}{4\pi^2 a^2} \int_0^\infty dk\ k^2 \left| \frac{1}{f(\phi)} \frac{du_{\mathbf{k}}(N)}{dN} - \frac{1}{f^2(\phi)} \frac{df(\phi)}{d\phi} \frac{d\phi}{dN} u_{\mathbf{k}}(N) \right|^2$$

$$\rho_E = \frac{H^2 f^2(\phi)}{4\pi^2 a^2} \int_0^\infty dk\ k^2 \left| \frac{d}{dN} \left(\frac{u_{\mathbf{k}}(N)}{f(\phi)} \right) \right|^2$$

$$\rho_B = \frac{f^2(\phi)}{4\pi^2 a^4} \int_0^\infty dk\ k^4 \left| \frac{u_{\mathbf{k}}(N)}{f(\phi)} \right|^2$$

$$N(t) = \ln \frac{a(t_{end})}{a(t)}$$

Backreaction from magnetogenesis in string inflation

Inflation:
ultraviolet sensitive phenomenon $\rightarrow$ String Theory

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Cosmological applications:

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$$N = 1, \quad \text{D} = 4 \quad \text{SUPERGRAVITY}$$

By expanding the 10D fields in harmonic forms

- Complex Structure Moduli U_α
- Axio-dilaton S
- Kähler Moduli $T_i = \tau_i + i\theta_i$

Moduli Stabilization

Complex Structure Moduli Axio-dilaton	$\rightarrow$	Stabilized at tree level

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$$W = W_0 + \sum_i A_i e^{-a_i T_i} \qquad K = -2 \ln \left(\mathcal{V} + \frac{\xi}{2} \right)$$

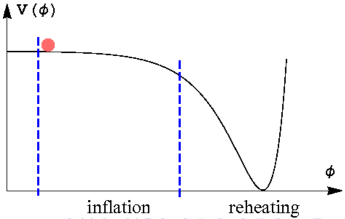
Moduli Stabilization

Complex Structure Moduli
Axio-dilaton $\rightarrow$ Stabilized at tree level

Kähler Moduli $\rightarrow$ Perturbative corrections to the Kähler Potential
Non perturbative corrections to the Superpotential

$$W = W_0 + \sum_i A_i e^{-a_i T_i} \qquad K = -2 \ln \left(\mathcal{V} + \frac{\xi}{2} \right)$$

$$V_F = e^K \left((K^{-1})^{i\bar{j}} D_i W D_{\bar{j}} \bar{W} - 3|W|^2 \right)$$

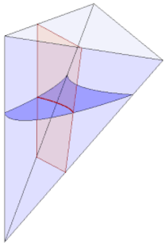


Why are we interested in Backreaction effects?

[*Cicoli*, *Ciupke*, *Mayrhofer*, *Shukla*]

$$\mathcal{V}(\tau_i) = \langle \mathcal{V} \rangle$$

$$\tau_s = \langle \tau_s \rangle$$



$$\boxed{\frac{\Delta\phi}{M_{pl}} \leq c \ln(\mathcal{V}) \quad \Rightarrow \quad \text{restriction on } N_e}$$

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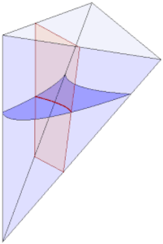

$$\boxed{\frac{\Delta\phi}{M_{pl}} \leq c \ln(\mathcal{V}) \quad \Rightarrow \quad \text{restriction on } N_e}$$

We need a mechanism which slows down the inflaton

$\Downarrow$

COUPLING OF THE INFLATON WITH ELECTROMAGNETIC FIELDS

String inflationary models

- Kähler Inflation

$$\mathcal{V} = \alpha\left(\tau_1^{3/2} - \sum_{i=2}^{n} \lambda_i \tau_i^{3/2}\right) \qquad f_K(\phi) = \sqrt{1 + \gamma\left(\frac{\phi^{4/3}}{\langle\phi\rangle^{4/3}} - 1\right)}$$

$$V(\tau_n) = \frac{W_0^2 \beta}{\mathcal{V}^3} + \frac{8aA\sqrt{\tau_n}e^{-2a\tau_n}}{3\mathcal{V}\lambda\alpha} - \frac{4aAW_0\tau_n e^{-a\tau_n}}{\mathcal{V}^2}$$

- Fibre Inflation

$$\mathcal{V} = \alpha\left(\sqrt{\tau_1}\tau_2 - \gamma\tau_3^{3/2}\right) \qquad f_F(\phi) = \sqrt{1 + \gamma\left(\frac{e^{k\phi}}{e^{k\langle\phi\rangle}} - 1\right)}$$

$$V(\hat{\phi}) = 3 - 4e^{-\frac{\hat{\phi}}{\sqrt{3}}} + e^{-\frac{4\hat{\phi}}{\sqrt{3}}}$$

Equations

- $$\frac{d^2\phi}{dN^2} + \left(3 - \frac{1}{2}\left(\frac{d\phi}{dN}\right)^2\right)\left(\frac{d\phi}{dN} + \frac{V'(\phi)}{V(\phi)}\right) = 0$$

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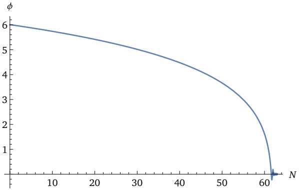
Backreaction

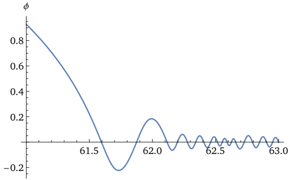
- $$\frac{d^2\phi}{dN^2}+\left(3-\frac{1}{2}\left(\frac{d\phi}{dN}\right)^2\right)\left(\frac{1}{6}\frac{d\rho_E}{dN}\frac{d\phi}{dN}+\frac{d\phi}{dN}(V(\phi)+\rho_E)+\frac{1}{3}\left(\frac{d\phi}{dN}\right)^2\rho_E\frac{f'(\phi)}{f(\phi)} + V'(\phi) - 2\frac{f'(\phi)}{f(\phi)}\rho_E\right)\frac{1}{V(\phi)+\rho_E} = 0$$

- $$\frac{d\rho_E}{dN} + 4\rho_E + 2\frac{df(\phi)}{dN}\frac{1}{f(\phi)}\rho_E = \left(V(\phi)+\rho_E\right)^2\frac{1}{4\pi^2}\left(1+\left(\frac{df(\phi)}{dN}\frac{1}{f(\phi)}\right)^2\right)\left(3-\frac{1}{2}\left(\frac{d\phi}{dN}\right)^2\right)^{-2}$$

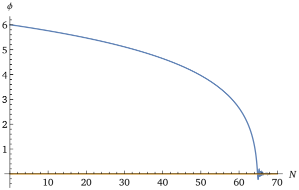
Numerical results: Fibre Inflation

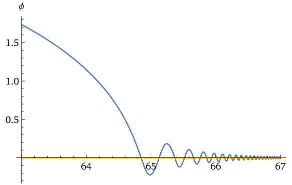
In absence of Backreaction



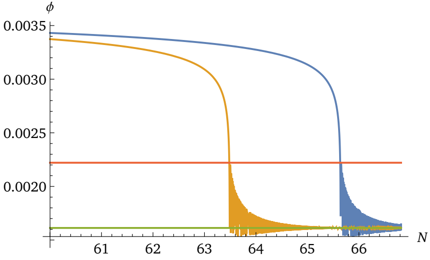


In presence of Backreaction





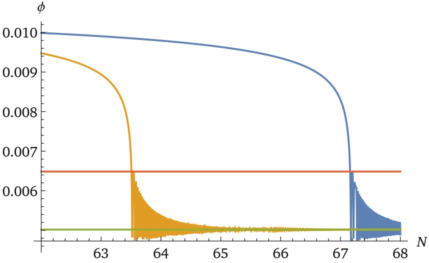
Numerical results: Kähler Inflation



$W_0 = 400 \quad \mathcal{V} = 10^4$

$\alpha = 1/100 \quad \lambda = 1$

$A = 1 \quad a = 2\pi$



$W_0 = 500 \quad \mathcal{V} = 10^4$

$\alpha = 1/100 \quad \lambda = 10$

$A = 10 \quad a = 2\pi$

Conclusions

Backreaction due to the coupling of the inflaton with the electromagnetic field

$\Downarrow$

slowdown of the inflationary evolution

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slowdown of the inflationary evolution

- compatibility of string inflationary models with phenomenology;
- production of primordial magnetic fields.

Thanks for your attention

The Standard Cosmological Model

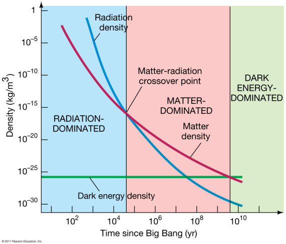
Friedmann-Robertson-Walker metric:

$$dS^2 = dt^2 - a^2(t) \left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta \, d\phi^2) \right)$$

Einstein Equations: $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu - p g_{\mu\nu} \qquad p = w\rho$$

Friedmann Equations: $\dot{a}^2 + k = \frac{8\pi G}{3}\rho a^2 \qquad \ddot{a} = -\frac{4\pi G}{3}(\rho + 3p)a$



$$\begin{cases} w = 0 & \text{matter} \\ w = 1/3 & \text{radiation} \\ w = -1 & \text{cosmological constant} \end{cases}$$

$$\rho(t) = \rho_0 \left(\frac{a_0}{a(t)} \right)^{3(1+w)}$$

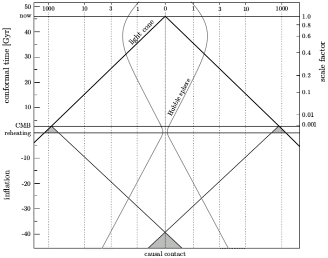
Problems in the Standard Cosmological Model

- Horizon Problem

The high degree of homogeneity of the CMB

$\Downarrow$

thermal equilibrium at decoupling



- Flatness Problem

$$\Omega = \frac{\rho}{\rho_c} \qquad \frac{d\Omega}{d\log a} = \Omega(\Omega - 1)(1 + 3w)$$

$\Omega = 1$: unnatural fine tuning of the initial conditions

Solution $\rightarrow$ Inflation

Inflation: initial period of accelerated expansion governed by a scalar field that violates the strong energy condition $w > -1/3$

- Large scale structure
- CMB anisotropies

Inflation

Scalar field with negative pressure: $T_{\mu\nu} = \text{diag}(\rho, -p, -p, -p)$

$$S = -\frac{M_{pl}^2}{2}\int d^4x\sqrt{-g}R + \underbrace{\int d^4x\sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right)}_{S_M}$$

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} \rightarrow \begin{cases} \rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) \\ p = \frac{1}{2}\dot{\phi}^2 - V(\phi) \end{cases}$$

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \qquad H^2 = \frac{1}{3M_{pl}^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right)$$

SLOW ROLL: $3H\dot{\phi} \simeq -V'(\phi) \qquad H^2 \simeq \frac{1}{3M_{pl}^2}V(\phi)$

Inflation

Slow Roll parameters:

$$\epsilon = \frac{M_{pl}^2}{2}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 \qquad \eta = M_{pl}^2 \frac{V''(\phi)}{V(\phi)} \qquad \epsilon, |\eta| \ll 1$$

Number of e-foldings:

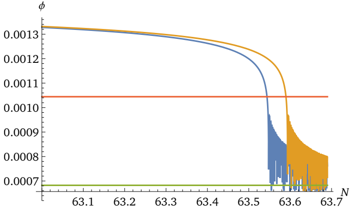
$$N_e = \ln \frac{a(t_{end})}{a(t)} \simeq \frac{1}{M_{pl}^2} \int_{\phi_{end}}^{\phi} \frac{V}{V'} d\phi$$

End of inflation: $\epsilon \sim 1$ $\qquad |\eta| \sim 1$ $\qquad N_e \simeq 50 - 60$

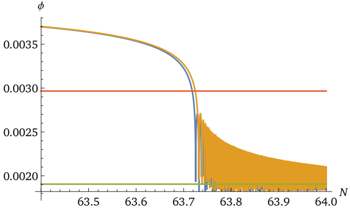
Reheating $\rightarrow$ Standard Model degrees of freedom

Numerical results: Kähler Inflation

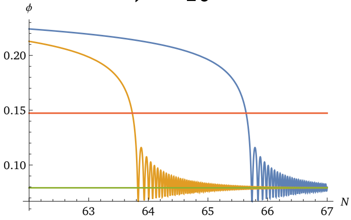
$W_0 = 57 \quad \alpha = 1 \quad \lambda = 1 \quad A = 1.87 \quad a = 2\pi$



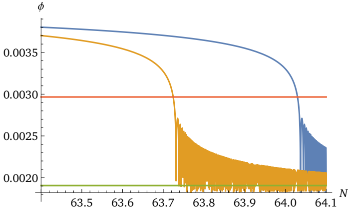
$\mathcal{V} = 10^7$



$\mathcal{V} = 10^6$



$\mathcal{V} = 10^2$

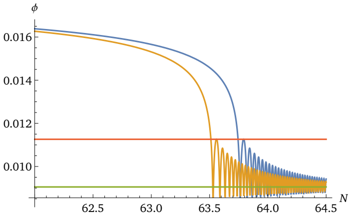


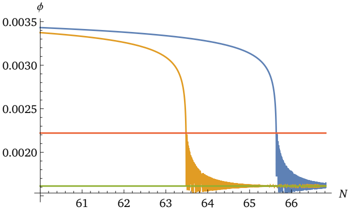
$\mathcal{V} = 10^6, \quad \rho(0) = 10^{-22}$

$W_0 = 130 \qquad \mathcal{V} = 10^4$

$\alpha = 1/100 \qquad \lambda = 1$

$A = 10 \qquad a = 2\pi/10$





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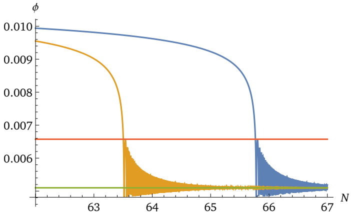
$W_0 = 400 \qquad \mathcal{V} = 10^4$

$\alpha = 1/100 \qquad \lambda = 10$

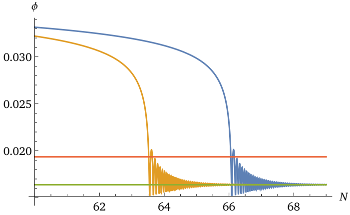
$A = 10 \qquad a = 2\pi$

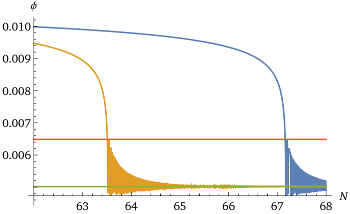


$W_0 = 400 \qquad \mathcal{V} = 10^4$

$\alpha = 1/100 \qquad \lambda = 1$

$A = 10 \qquad a = 2\pi/30$





$W_0 = 500 \qquad \mathcal{V} = 10^4$

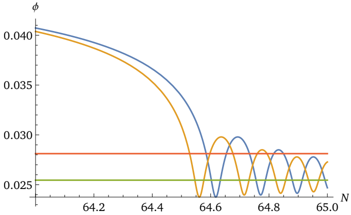
$\alpha = 1/100 \qquad \lambda = 10$

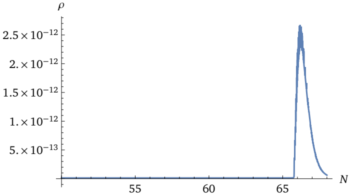
$A = 10 \qquad a = 2\pi$

$W_0 = 500 \qquad \mathcal{V} = 10^5$

$\alpha = 1/100 \qquad \lambda = 10$

$A = 50 \qquad a = 2\pi/50$





Electric energy density

$$\frac{d^2\phi}{dN^2} = -\left(3-\frac{1}{2}\left(\frac{d\phi}{dN}\right)^2\right)\left(\frac{1}{6}\frac{d\rho_E}{dN}\frac{d\phi}{dN}+\frac{d\phi}{dN}(V(\phi)+\rho_E)+\frac{1}{3}\left(\frac{d\phi}{dN}\right)^2\rho_E\frac{f'(\phi)}{f(\phi)}\right.$$

$$\left.+V'(\phi) - 2\frac{f'(\phi)}{f(\phi)}\rho_E\right)\frac{1}{V(\phi)+\rho_E}$$

$$\frac{d\rho_E}{dN} + 4\rho_E + 2\frac{df(\phi)}{dN}\frac{1}{f(\phi)}\rho_E =$$

$$\left(V(\phi)+\rho_E\right)^2\frac{1}{4\pi^2}\left(1+\left(\frac{df(\phi)}{dN}\frac{1}{f(\phi)}\right)^2\right)\left(3-\frac{1}{2}\left(\frac{d\phi}{dN}\right)^2\right)^{-2}$$