Pole skipping away from maximal chaos

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Based on
1908.03574 with Mezei
2010.08558 with Choi and Mezei

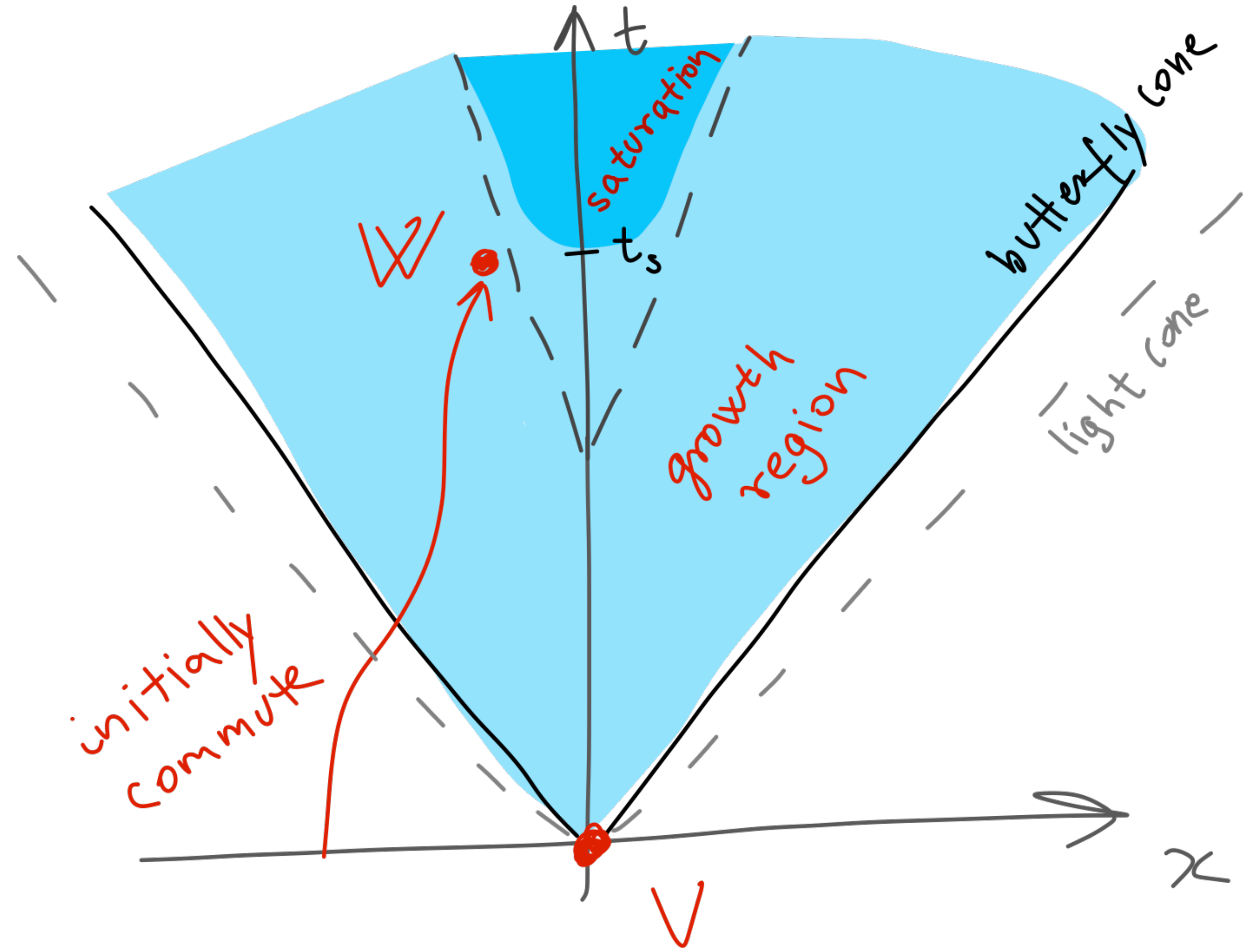
Plan

1. Introduction: quantum butterfly effect
2. Pole skipping
3. Large q SYK chain

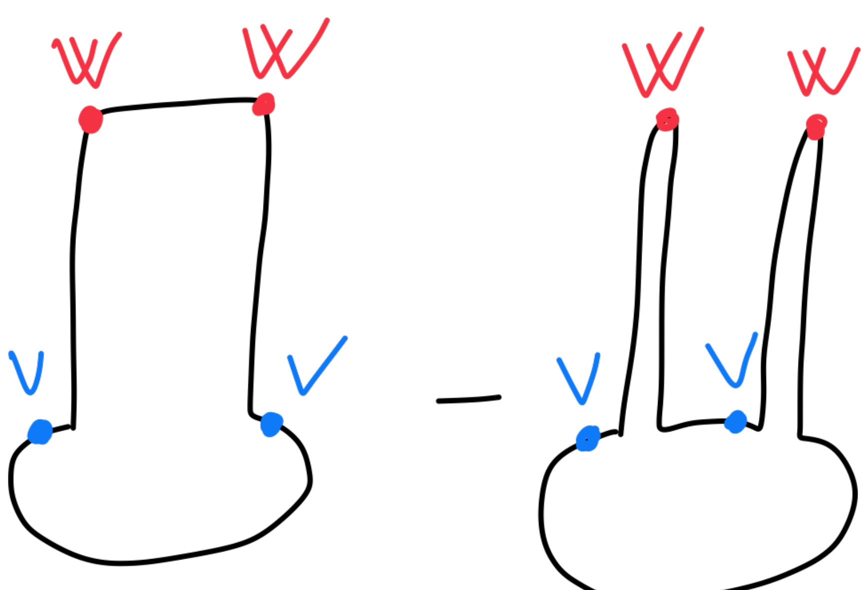
Butterfly effect

“Footprint” of a Heisenberg operator

$$C(t,x) = -\langle [W(t,x), V(0,0)]^2 \rangle_\beta$$



$C(t,x) \propto$



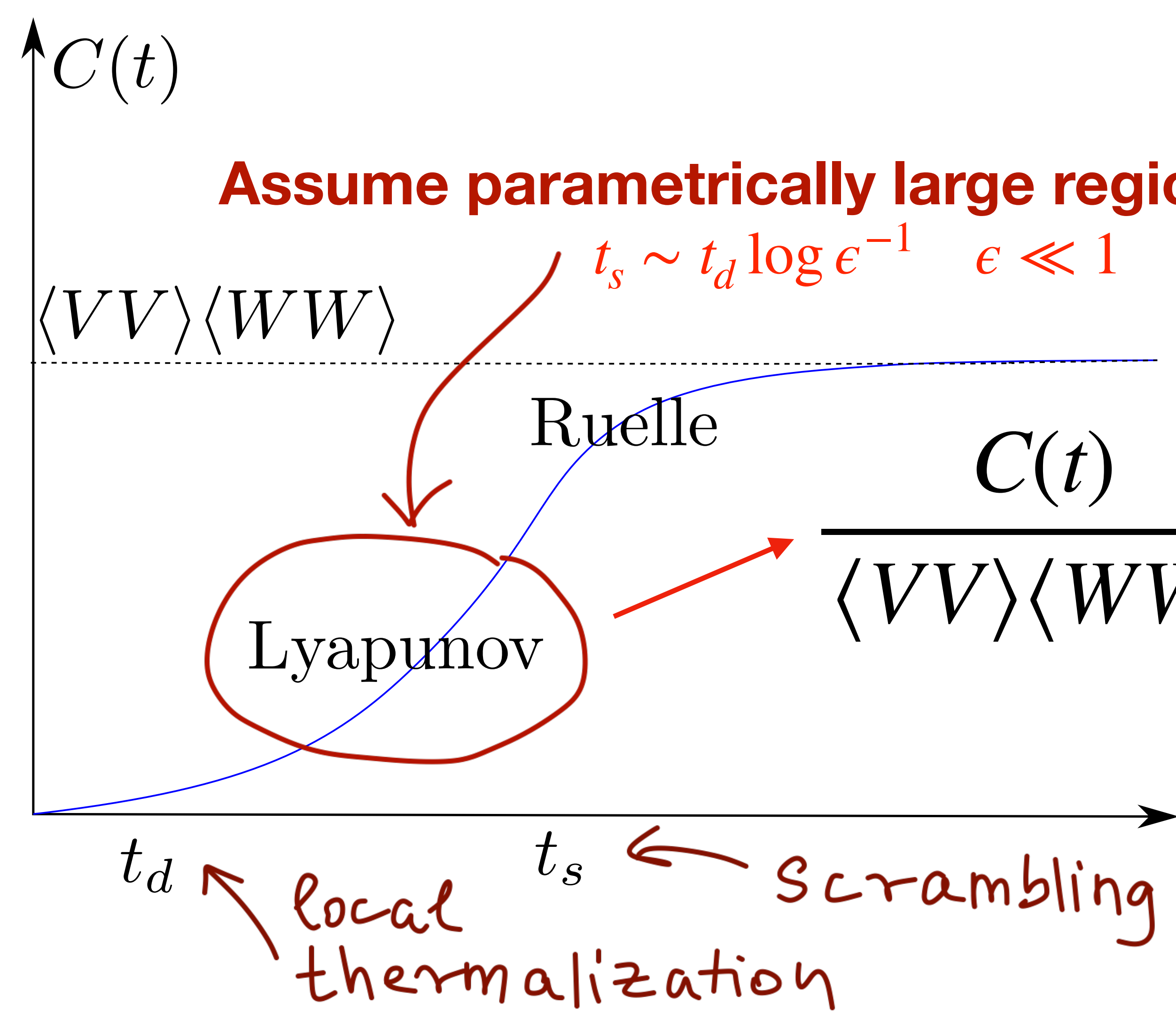
TOC OTOC

Sensitivity to initial data [Larkin-Ovchinnikov]

$$[q(t), p(0)] \propto \{q(t), p(0)\} = \frac{\partial q(t)}{\partial q(0)}$$

Butterfly effect

$$C(t) = -\langle [W(t), V(0)]^2 \rangle_{\text{thermal}}$$

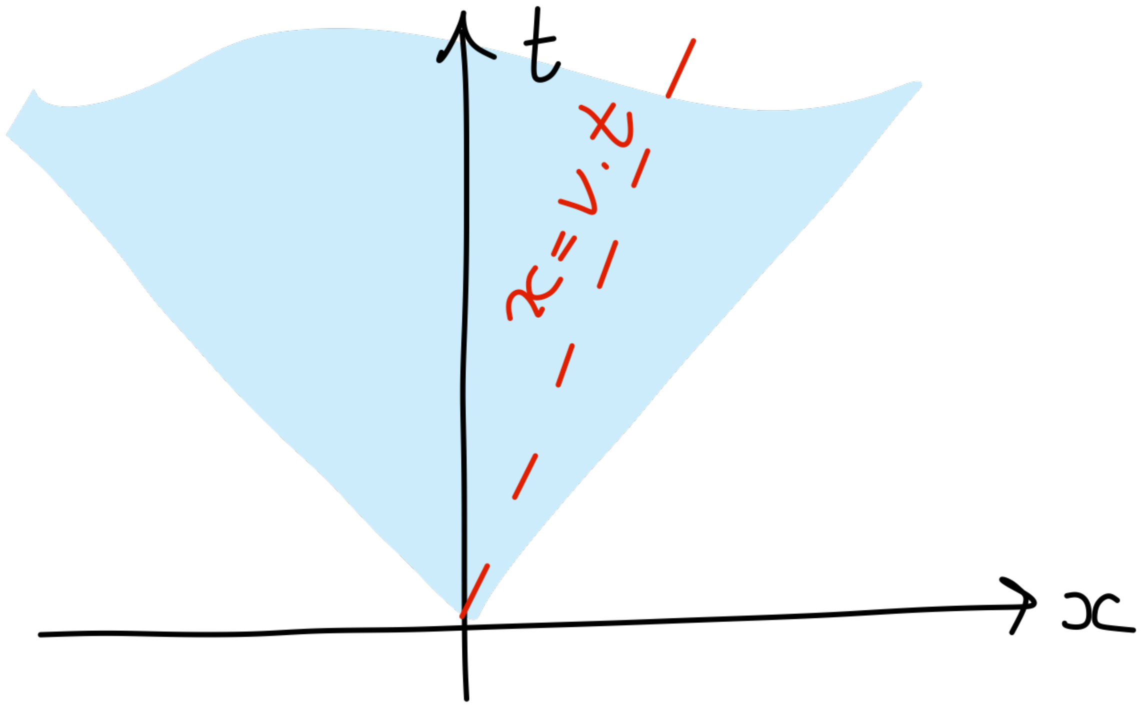


$$\frac{C(t)}{\langle VV \rangle \langle WW \rangle} \approx \epsilon e^{\lambda_L t} + O(\epsilon^2)$$

Butterfly effect with local interactions

Scaling ansatz [Kemani-Huse-Nahum,Xu-Swingle]

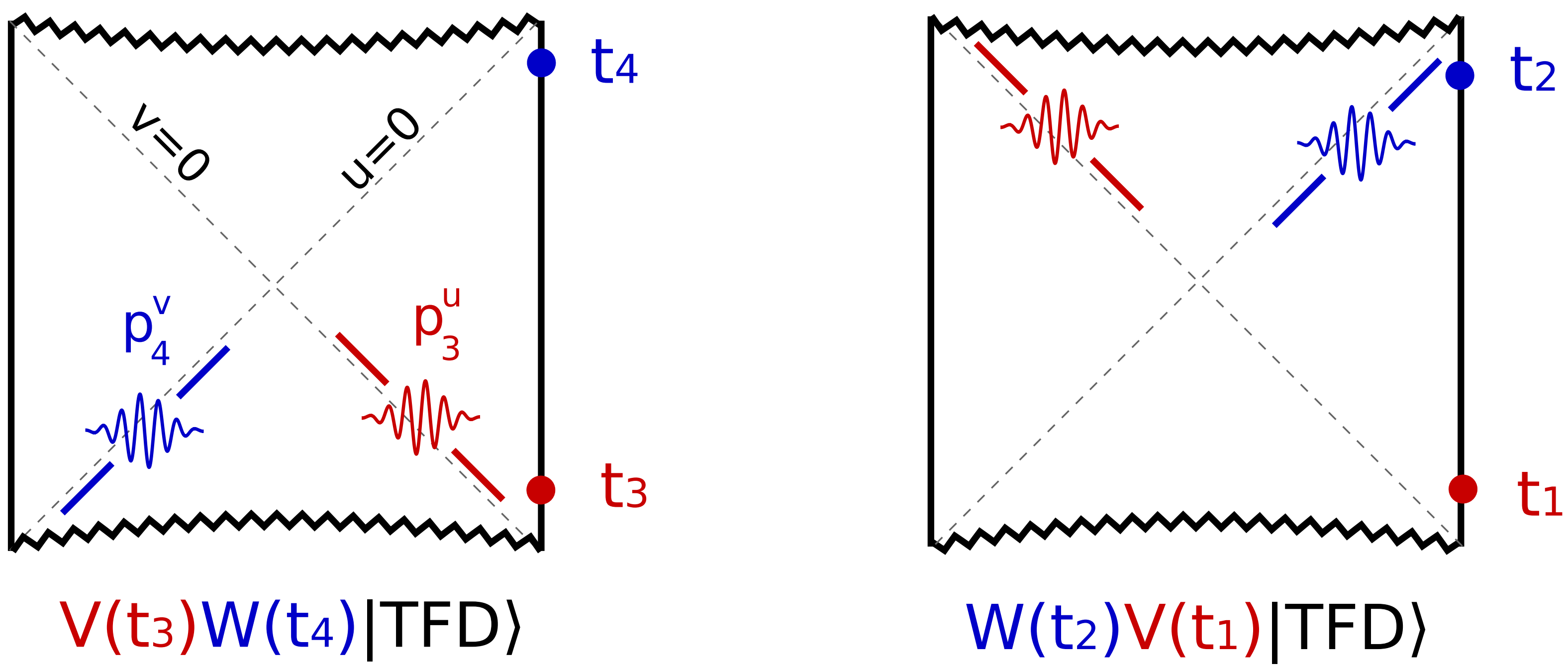
$$C(t,x) = -\langle [W(t,x), V(0,0)]^2 \rangle \sim \epsilon e^{\lambda(\frac{x}{t})t}$$



- $\lambda(\mathrm{v})$ **is a velocity dependent Lyapunov exponent**
- **For** $x \ll t$ **we recover the ordinary exponent** $\lambda_L = \lambda(0)$
- **The edge of the "butterfly cone" is defined by** $\lambda(\mathrm{v}_B) = 0$
  v_B **is the butterfly speed**

OTOC in AdS/CFT

AdS/CFT with Einstein gravity (near horizon scattering) [Shenker-Stanford]



$$\text{OTOC}(t,x) \approx 1 - \# G_N e^{\frac{2\pi}{\beta}(t-|x|/\mathrm{v}_B)} \quad \rightarrow \quad \lambda(\mathrm{v}) = \frac{2\pi}{\beta}\left(1 - \frac{\mathrm{v}}{\mathrm{v}_B}\right)$$

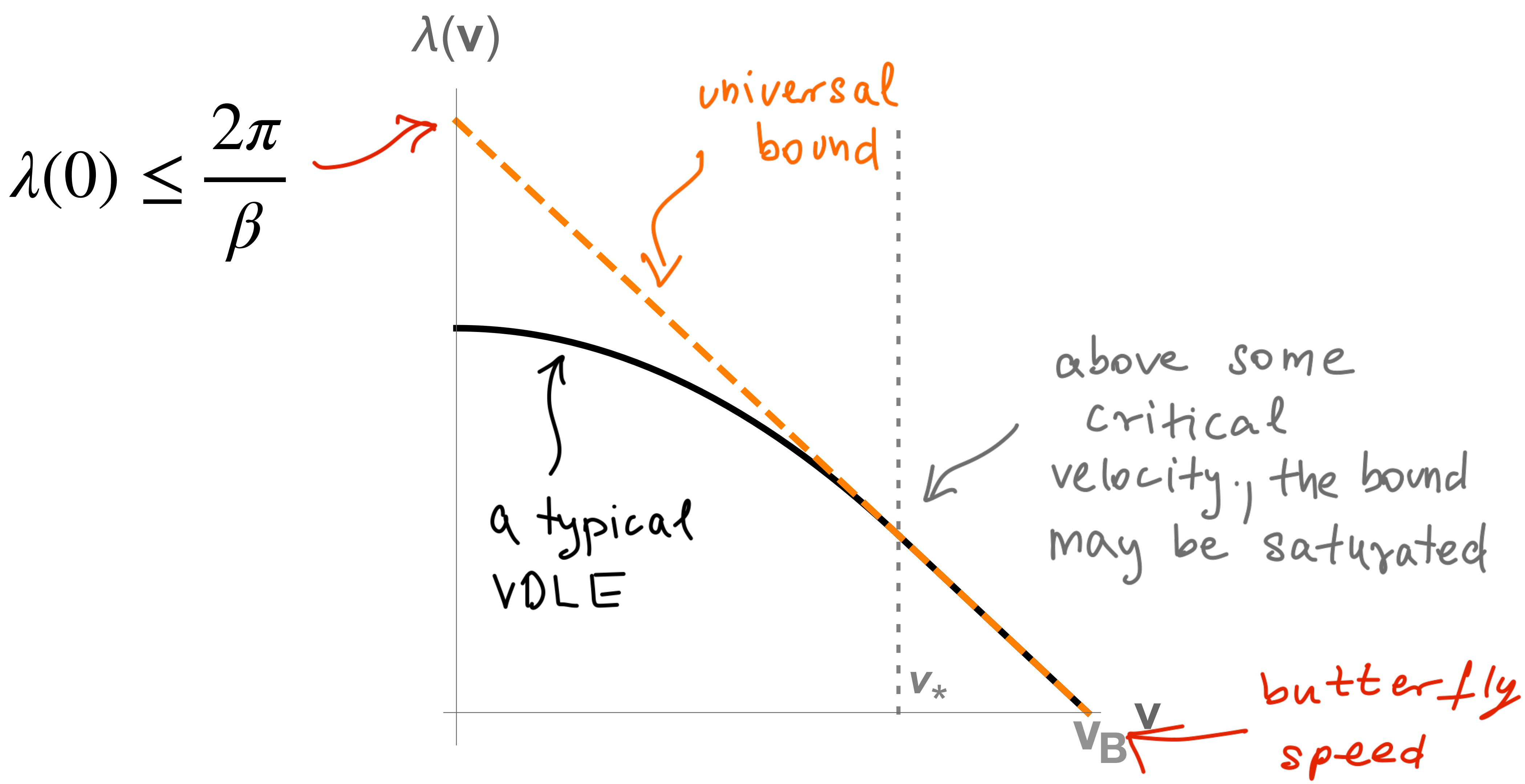
$$\lambda_L \equiv \lambda(0) = \frac{2\pi}{\beta}$$

Chaos bound

Universal bounds:

$$\lambda_L \equiv \lambda(0) \leq \frac{2\pi}{\beta}$$ [Maldacena-Shenker-Stanford]

$$\lambda(\mathrm{v}) \leq \frac{2\pi}{\beta}\left(1 - \frac{\mathrm{v}}{\mathrm{v}_B}\right)$$ [Mezei-Sarosi]

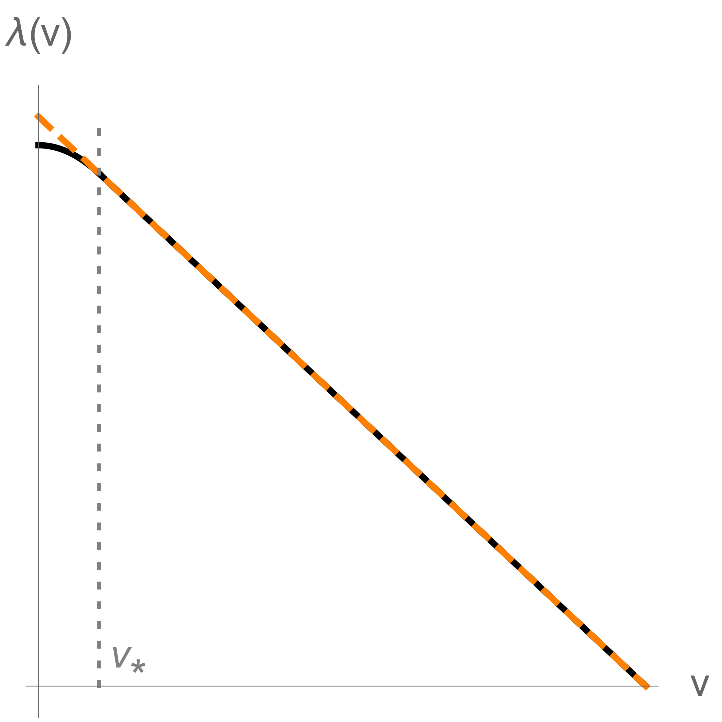


Example

AdS/CFT with stringy corrections [Shenker-Stanford]

$$\lambda(\mathrm{v}) = \begin{cases} \frac{\pi}{\beta}\left(2 - \frac{\mathrm{v}_*}{\mathrm{v}_B} - \frac{\mathrm{v}^2}{\mathrm{v}_*\mathrm{v}_B}\right), & \mathrm{v} < \mathrm{v}_* \\ \frac{2\pi}{\beta}\left(1 - \frac{\mathrm{v}}{\mathrm{v}_B}\right), & \mathrm{v} \geq \mathrm{v}_* \end{cases}.$$

$$\mathrm{v}_* = \frac{d^2}{4\mathrm{v}_\mathrm{B}}\left(\frac{\ell_{\mathrm{string}}}{\ell_{\mathrm{AdS}}}\right)^2$$



How special are maximally chaotic theories?

Original expectation: $\lambda_L = \frac{2\pi}{\beta}$ **implies a weakly coupled gravity dual**

Counter example: SYK/tensor models in 0+1d

Higher dimensions: $\lambda_L = \frac{2\pi}{\beta}$ **for local operators is much stronger**

Example: CFT on $S^1_{2\pi} \times \mathbb{H}_{d-1}$ **(conformal to Rindler)**

$$\lambda(\mathrm{v}) \leftrightarrow j(\Delta)$$

↑

spin of leading Regge trajectory

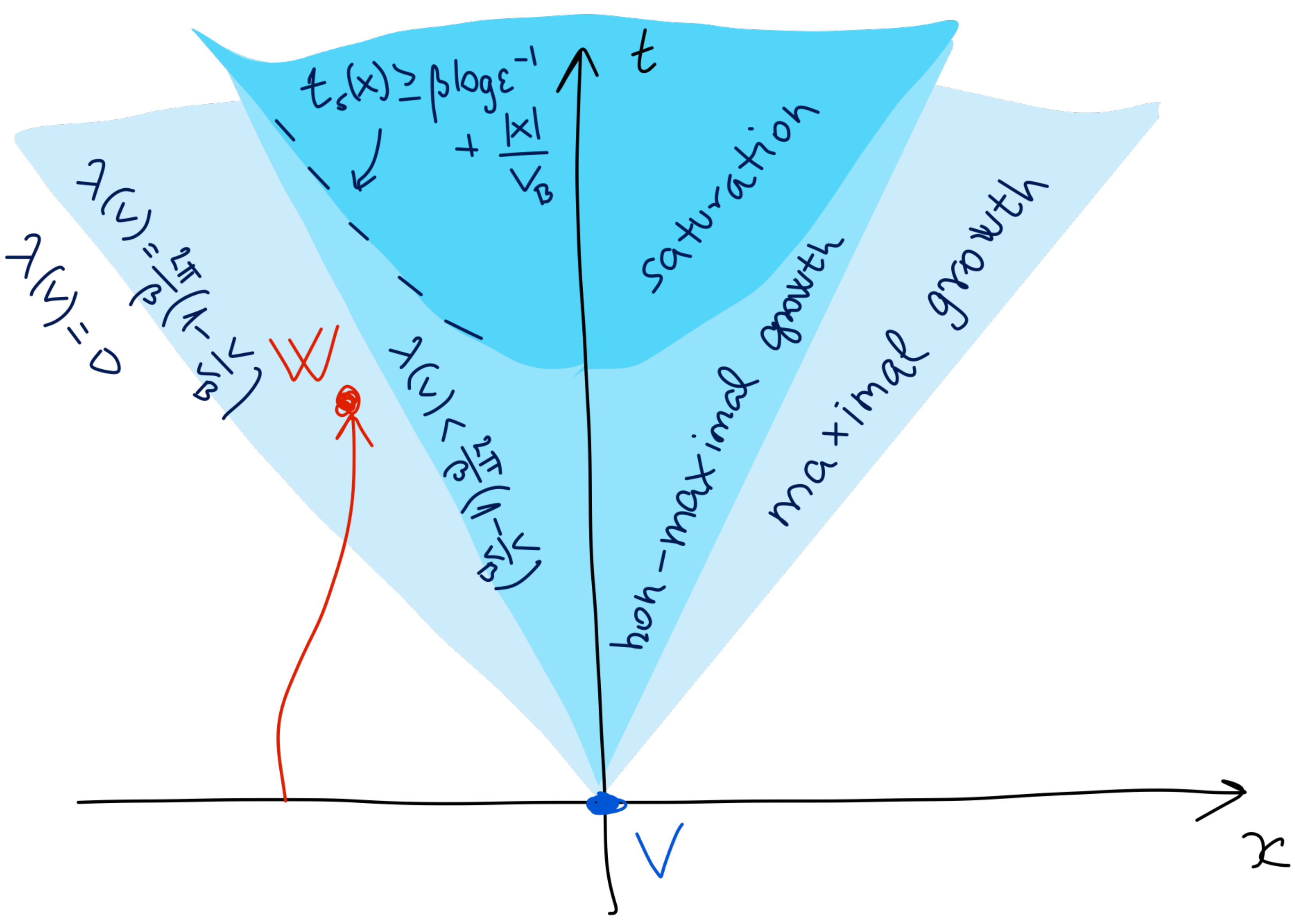
$$\lambda(\mathrm{v}) = \frac{2\pi}{\beta}\left(1 - \frac{\mathrm{v}}{\mathrm{v}_B}\right) \quad \rightarrow \quad j(\Delta) = 2$$

flat trajectory;
infinite gap for higher spin single traces

If gravity EFT is dual to an ensemble average, $\lambda(0) = \frac{2\pi}{\beta}$ **suggests that each element of the ensemble should have a weakly coupled bulk dual**

In a nutshell

$$C(t,x) = -\langle [W(t,x), V(0,0)]^2 \rangle \quad \sim \epsilon e^{\lambda(\frac{x}{t})t}$$



Pole skipping

Energy density retarded two point function $G^R_{\varepsilon\varepsilon}(\omega, p)$

has a family of hydrodynamic poles defined by

$$\omega_{\text{pole}}(p \to 0) = 0$$

for small p **the possibilities:**

$$\omega_{\text{pole}}(p) = \begin{cases} \pm c_s p + \dots & \textbf{(sound)} \\ -iDp^2 + \dots & \textbf{(energy diffusion)} \end{cases}$$

Prediction of AdS/CFT: Residue on this pole line vanishes at

$$(\omega, p)_{\textbf{p.s.}} = i\underset{\downarrow \; \frac{2\pi}{\beta}}{\lambda_L}\left(1, \frac{1}{v_B}\right)$$

[Grozdanov,Schalm,Scopelliti]

Pole skipping

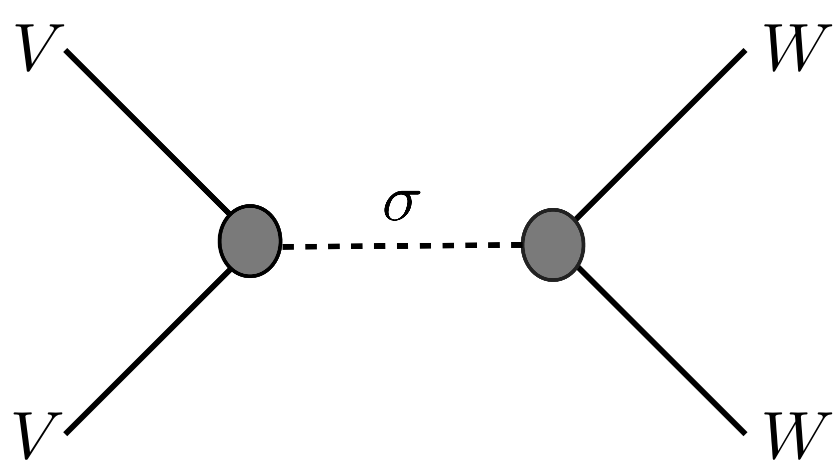
Prediction of AdS/CFT: **Residue on this pole line vanishes at**

$$(\omega, p)_{\textbf{p.s.}} = i\lambda_L \left(1, \frac{1}{v_B}\right)$$

[Grozdanov,Schalm,Scopelliti]

Proposed explanation [Blake-Lee-Liu]

Lyapunov growth coming from exchange of hydrodynamic “fluid field”



$$\propto 1 - \epsilon e^{\lambda_L (t - |x|/v_B)}$$

$\Rightarrow \langle \sigma\sigma \rangle(\omega, p)$ **should have a pole at** $(\omega, p) = i\lambda_L \left(1, \frac{1}{v_B}\right)$

But $\langle T^{00}[\sigma] T^{00}[\sigma] \rangle(\omega, p)$ **should not have an exponential growth**

$\Rightarrow$ **in** $\langle T^{00} T^{00} \rangle(\omega, p)$ **the pole should be absent**

Pole skipping

Example: 2d CFTs (maximal chaos in 2d) [Haehl-Rozali]

$$S_{\text{hydro}} \propto \int dt[\partial_{x^-}\sigma_L(\partial_t^3 - \partial_t)\sigma_L + \partial_{x^+}\sigma_R(\partial_t^3 - \partial_t)\sigma_R]$$

$$\langle \sigma_L \sigma_L \rangle(\omega, p) \propto \frac{1}{\omega(\omega^2+1)(\omega - p)}$$

Exp growth from pole at $\omega = -i, p = i$

in real space giving e^{t-x}

But energy density is a local functional of the hydro field:

$$T_L \propto \partial_t^3 \sigma_L - \partial_t \sigma_L$$

$$G^R_{T_L T_L} \propto \frac{\omega^2(\omega^2+1)^2}{\omega(\omega^2+1)(\omega - p)} = \frac{\omega(\omega^2+1)}{\omega - p}$$

No growth, pole is skipped.

Pole skipping

$$(\omega, p)_{\textbf{p.s.}} = i\lambda_L \left(1, \frac{1}{\mathrm{v}_B}\right)$$

Most known examples have maximal chaos $\lambda_L = \dfrac{2\pi}{\beta}$

Is there a pole skipping phenomenon away from maximal chaos?
If yes, what's the generalization?

Pole skipping

Known examples with non-maximal chaos:

- **2d CFT** $\langle T^{00}T^{00}\rangle$ **is universal, displays pole skipping at** $(\omega, p) = i\dfrac{2\pi}{\beta}(1,1)$ [Haehl-Rozali]

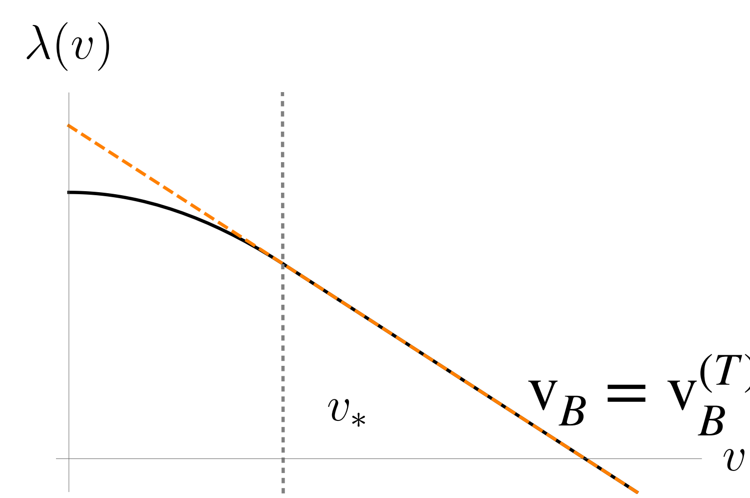
  All 2d CFTs have $v_B = 1$ [Mezei-Sarosi]

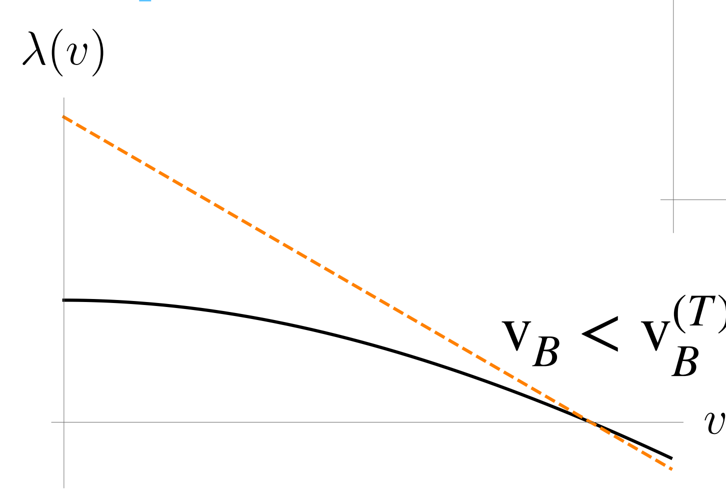
  Not all 2d CFTs have $\lambda_L = \dfrac{2\pi}{\beta}$

- $d > 2$ **CFT on Rindler space** $\langle T^{00}T^{00}\rangle$ **is universal, displays pole skipping at** $(\omega, p) = i\left(1, d-1\right)$ [Haehl-Rozali]

  On Rindler space $v_B^{(T)} = (d-1)^{-1}$ [Perlmutter]

  In general $v_B \leq v_B^{(T)}$ [Mezei-Sarosi]





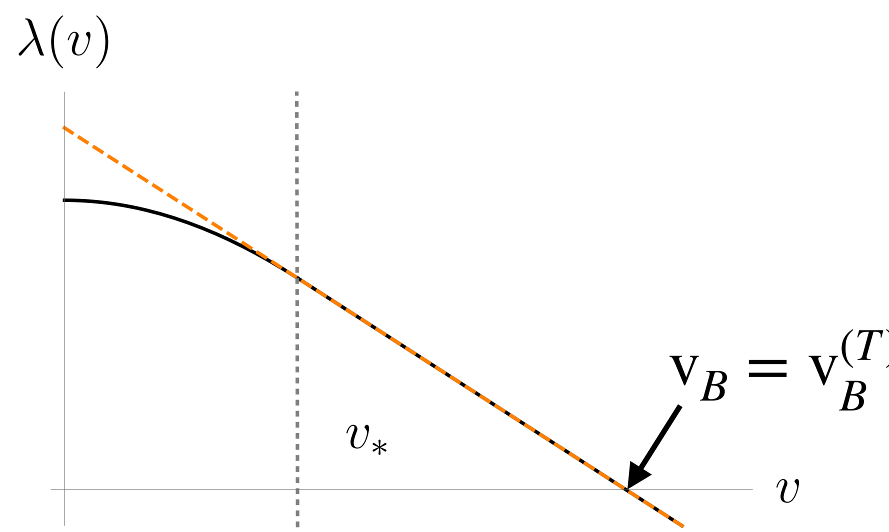
- **Pole skipping with higher derivatives** [Grozdanov]

Pole skipping

Our conjecture:

$$(\omega, p)_{\textbf{p.s.}} = i\lambda_L^{(T)}\left(1, \frac{1}{\mathrm{v}_B^{(T)}}\right)$$

$\lambda_L^{(T)} \equiv \frac{2\pi}{\beta}$, **and** $\mathrm{v}_B^{(T)}$ **are the stress tensor contributions**

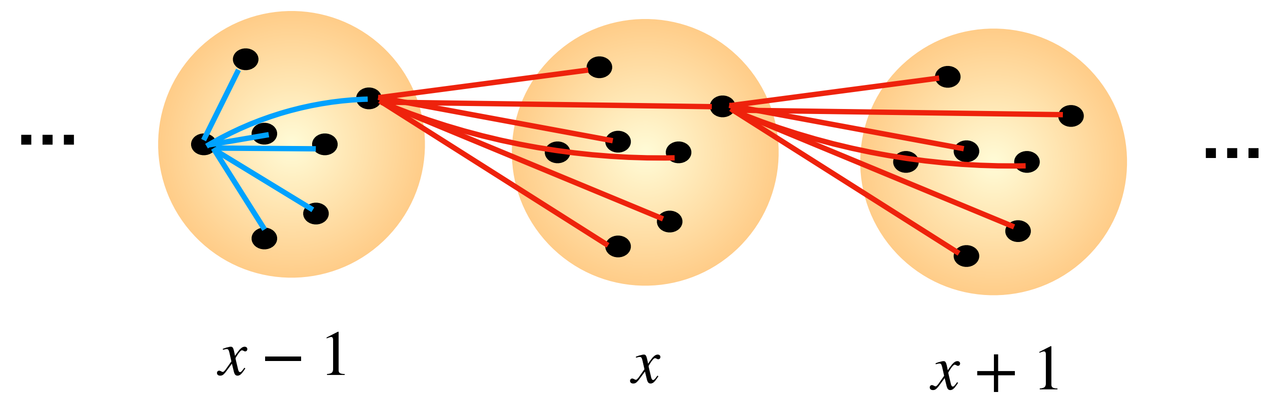


Anticlimatic in that λ_L **cannot be read from stress tensor 2pt func**

Strong in that $\mathrm{v}_B^{(T)} = \mathrm{v}_B$ **in many non-maximally chaotic theories**

Main example: SYK chain

$$\{\chi_{i,x}, \chi_{j,x}\} = \delta_{xy}\delta_{ij}$$



$$H = i^{q/2}\sum_{x=0}^{M-1}\Big(\sum_{1<i_1<\ldots<i_q<N} J_{i_1\ldots i_q,x}\chi_{i_1,x}\cdots\chi_{i_q,x} + \sum_{\substack{1<i_1<\ldots<i_{q/2}<N\\ 1<j_1<\ldots<j_{q/2}<N}} J'_{i_1\ldots i_{q/2}j_1\ldots j_{q/2},x}\chi_{i_1,x}\cdots\chi_{i_{q/2},x}\chi_{j_1,x+1}\cdots\chi_{j_{q/2},x+1}\Big)$$

[Gu-Qi-Stanford]

The J and J' are random variables, their variances are the couplings

$$\frac{\pi w}{\cos\frac{\pi w}{2}} \sim \beta\sqrt{\overline{J^2}+\overline{J'^2}} \qquad \gamma \sim \sqrt{\overline{J'^2}/(\overline{J^2}+\overline{J'^2})}$$

Solvable large N limit [Kitaev]

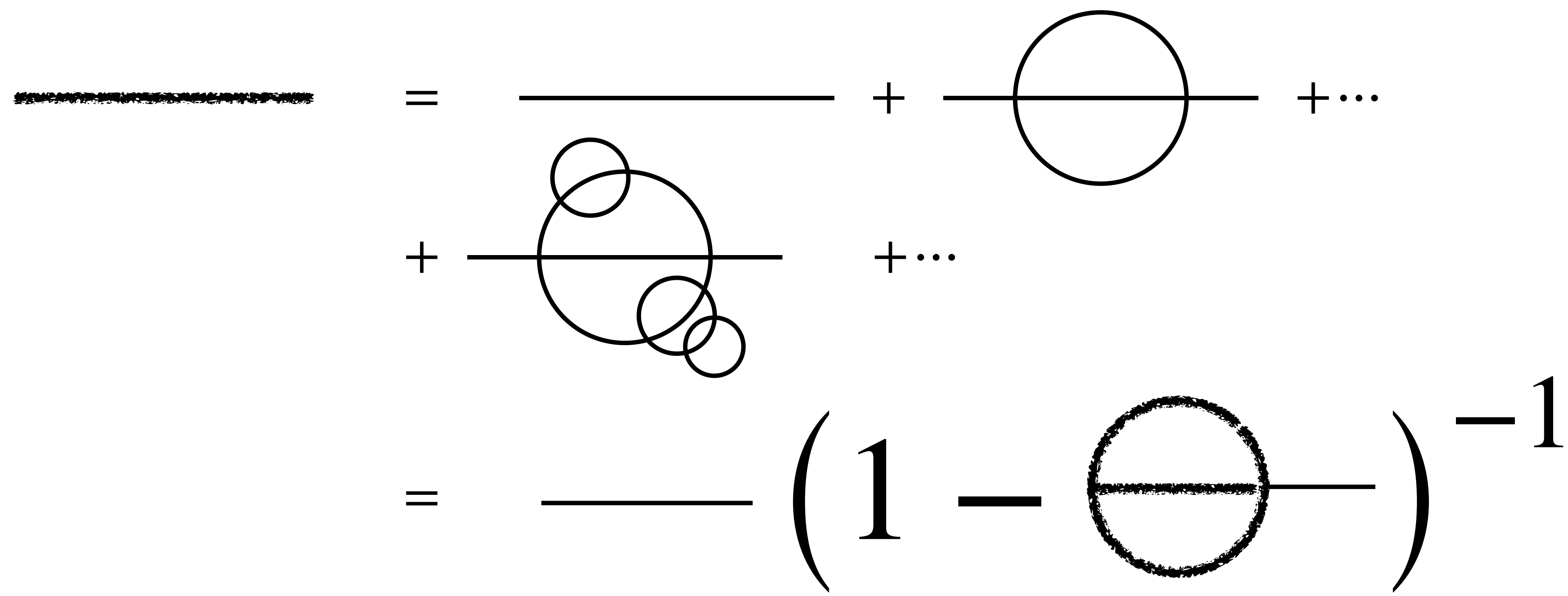
Even more solvable if also large q, $q/N \to 0$ [Maldacena-Stanford]

Four point function in SYK models

Large N is dominated by summable diagramms

Two point function:

$$\sum_i \langle \psi_i \psi_i \rangle =$$



Still complicated to solve for

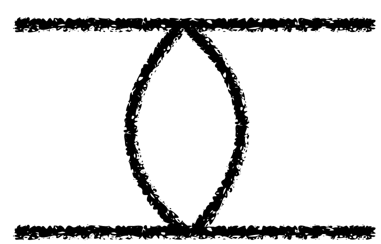
but much “cheaper” problem than direct diagonalization

Four point function in SYK models

Large N is dominated by summable diagramms

Four point function:

$$\sum_{i,j}\langle\psi_i\psi_i\psi_j\psi_j\rangle_{\text{conn}} =$$



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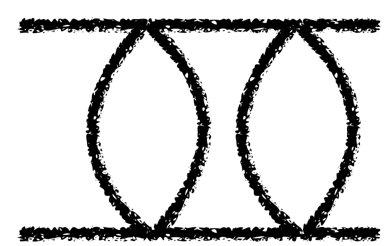
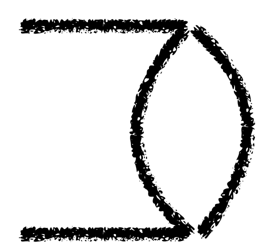
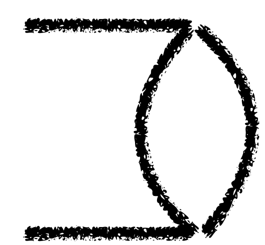


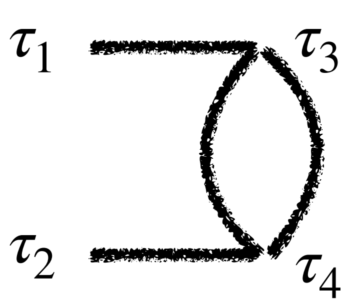
$+\cdots$

$$= \Big(1 -$$



$$\Big)^{-1}$$



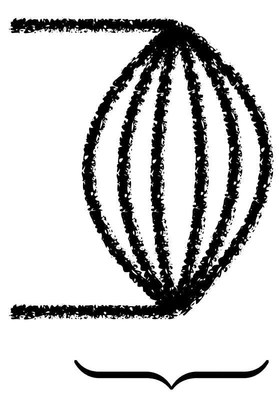


$$= K(\tau_1, \tau_2 \,|\, \tau_3, \tau_4)$$

“Ladder kernel”

Four point function in SYK models

The Schwinger-Dyson equations simplify drastically in the large q limit



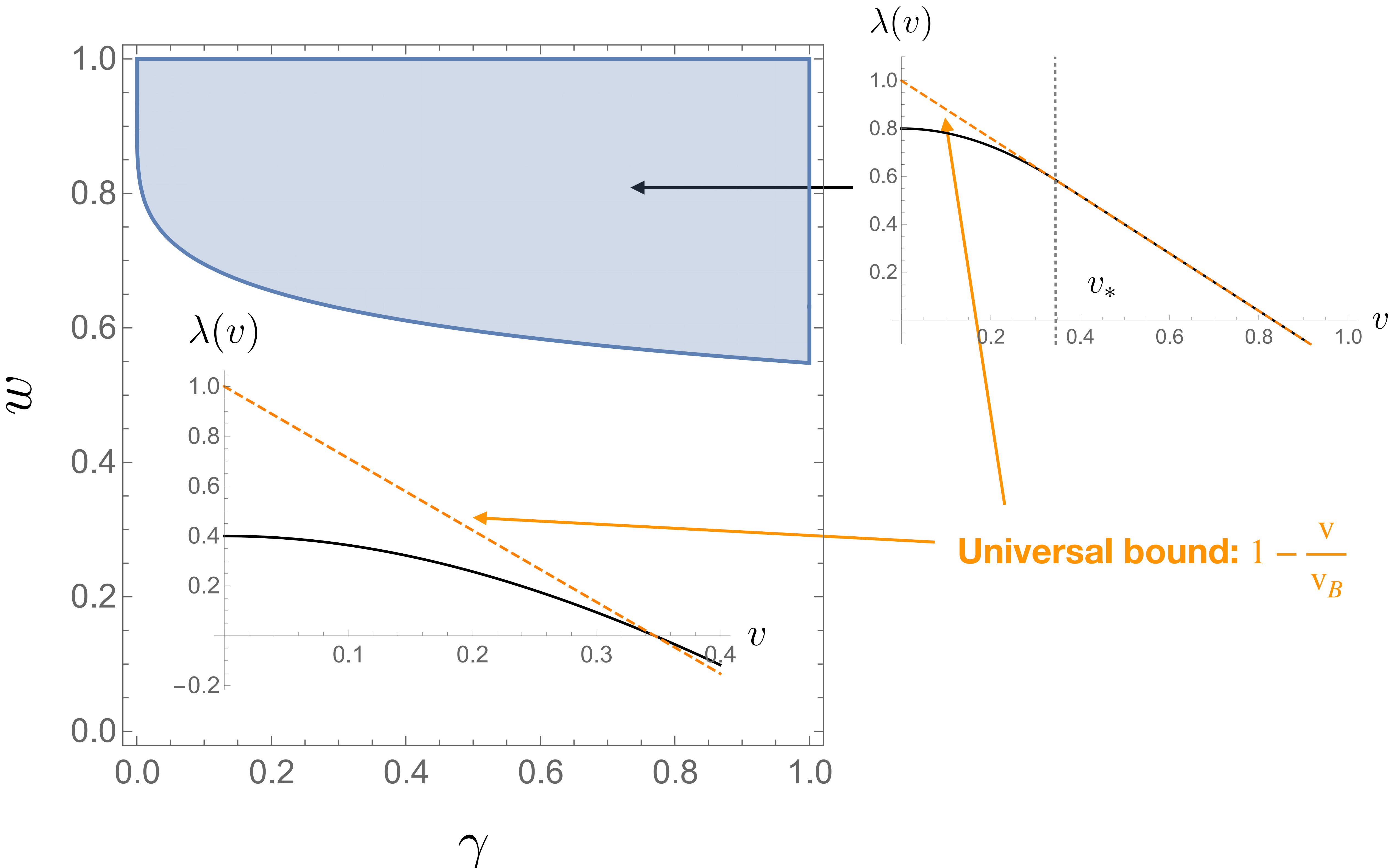
$q-2$

$$= K(\tau_1, \tau_2 \,|\, \tau_3, \tau_4) \approx \left[\delta(\tau_1 - \tau_3)\delta(\tau_2 - \tau_4) + \cdots\right]\left(\partial_{\tau_3}\partial_{\tau_4} + V(\tau_3 - \tau_4)\right)$$

The problem reduces to solving partial differential equations!

Velocity dependent Lyapunov exponent

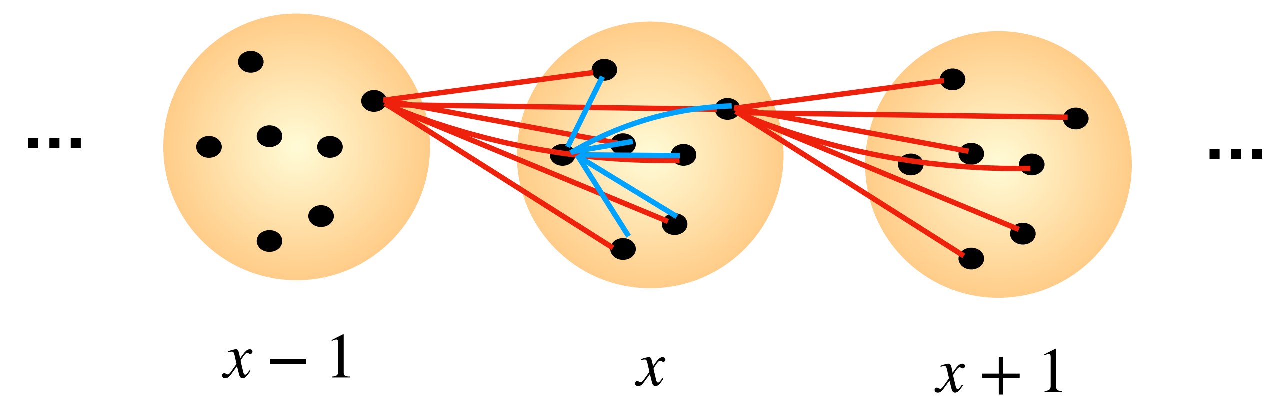
[Choi,Mezei,Sarosi]



w : "total" coupling, γ : inter-site coupling

Energy correlations in SYK chain

$$H = \sum_{x=1}^{M} \varepsilon_x(0)$$



$$\varepsilon_x(0) = i^{q/2}\Big(\sum_{i_1<\ldots<i_q} J_{i_1\ldots i_q,x}\chi_{i_1,x}\cdots\chi_{i_q,x}$$

$$+\frac{1}{2}\sum_{\substack{i_1<\ldots<i_{q/2}\\ j_1<\ldots<j_{q/2}}}\Big[J'_{i_1\ldots i_{q/2}j_1\ldots j_{q/2},x}\chi_{i_1,x}\cdots\chi_{i_{q/2},x}\chi_{j_1,x+1}\cdots\chi_{j_{q/2},x+1} + J'_{i_1\ldots i_{q/2}j_1\ldots j_{q/2},x-1}\chi_{i_1,x-1}\cdots\chi_{i_{q/2},x-1}\chi_{j_1,x}\cdots\chi_{j_{q/2},x}\Big]\Big)$$

Aim: calculate

$$G_{\varepsilon\varepsilon}(\tau,x) = \langle T\varepsilon_{x+y}(\tau)\varepsilon_y(0)\rangle_{\text{conn}} = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} e^{in\tau}\int_{-\pi}^{\pi}\frac{dp}{2\pi}\, e^{ipx} G^M_{\varepsilon\varepsilon}(n,p)$$

Energy correlations in SYK chain

$$G_{\varepsilon\varepsilon}(\tau, x) = \langle T \varepsilon_{x+y}(\tau) \varepsilon_y(0) \rangle_{\rm conn}$$

Idea: extract from fermion four point function [Choi,Mezei,Sarosi]

$$H = \sum_{x=1}^{M} \varepsilon_x(0), \quad \varepsilon_x(0) \propto \chi^q \qquad \textbf{and} \qquad \{\chi_{i,x}, \chi_{j,x}\} = \delta_{xy}\delta_{ij}$$

↓

$$\sum_i \chi_{i,x}(0)[\chi_{i,x}(0), H] = q\, \varepsilon_x(0)$$

$$\lim_{\substack{\tau_1 \to \tau_2 \\ \tau_3 \to \tau_4}} \partial_{\tau_1}\partial_{\tau_3}\left(\textbf{Tr}\left[e^{-\beta H} \chi_{i,x}(\tau_1)\chi_{i,x}(\tau_2)\chi_{j,y}(\tau_3)\chi_{j,y}(\tau_4)\right] - \textbf{disconnected} \right) \propto G_{\varepsilon\varepsilon}(\tau_2 - \tau_4, x - y)$$

↑

Reminder: $\left(1 -\right.$  $\left.\right)^{-1}$ 

Energy correlations in SYK chain

Turning the crank, we obtain an expression [Choi,Mezei,Sarosi]

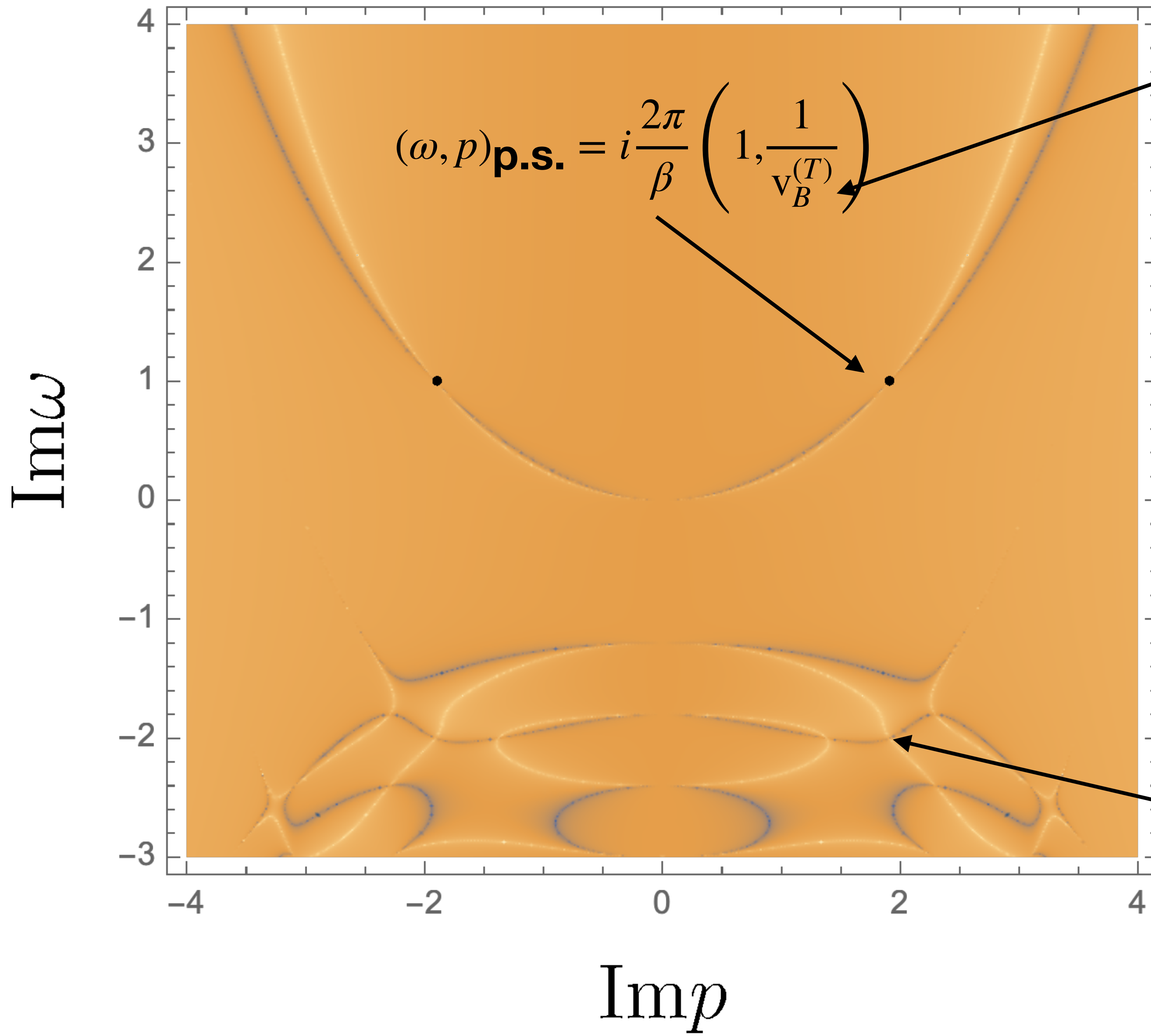
$$G^R_{\varepsilon\varepsilon}(\omega, p) = -\frac{Nw}{2q^2}\left(\partial_\theta \log \psi_n(\theta_w) + \tan\frac{\pi w}{2}\right)\Big|_{n\to -i\omega+\epsilon}$$

$$\begin{aligned}
\psi_n(\theta) &= c_o\psi_n^o(\theta) + c_e\psi_n^e(\theta)\,, \\
c_o &= \frac{\Gamma\left(1-\frac{h}{2}-\frac{n}{2w}\right)\sin\left(\frac{\pi h}{2}+\frac{\pi n}{2w}\right)\sin\left(\frac{n\pi}{2}\right)}{\Gamma\left(\frac{1}{2}-\frac{h}{2}+\frac{n}{2w}\right)}\,, \\
c_e &= \frac{\Gamma\left(\frac{1}{2}-\frac{h}{2}-\frac{n}{2w}\right)\cos\left(\frac{\pi h}{2}+\frac{\pi n}{2w}\right)\cos\left(\frac{n\pi}{2}\right)}{2\Gamma\left(1-\frac{h}{2}+\frac{n}{2w}\right)}\,, \\
\psi_n^e &= \sin(\theta)^h\,{}_2F_1\left(\frac{h-n/w}{2}, \frac{h+n/w}{2}, \frac{1}{2}, \cos^2\theta\right)\,, \\
\psi_n^o &= \cos(\theta)\sin(\theta)^h\,{}_2F_1\left(\frac{1+h-n/w}{2}, \frac{1+h+n/w}{2}, \frac{3}{2}, \cos^2\theta\right)\,, \\
h &= \frac{1}{2}\left(1+\sqrt{9+4\gamma(\cos(p)-1)}\right)\,, \\
\theta_w &= \frac{\pi}{2}(1-w)\,,
\end{aligned}$$

The only known non-perturbative thermal correlator that is not fixed by symmetry

Energy correlations in SYK chain

Pole skipping



Reminder:

$v_B^{(T)}$ **non-trivial function of the couplings**

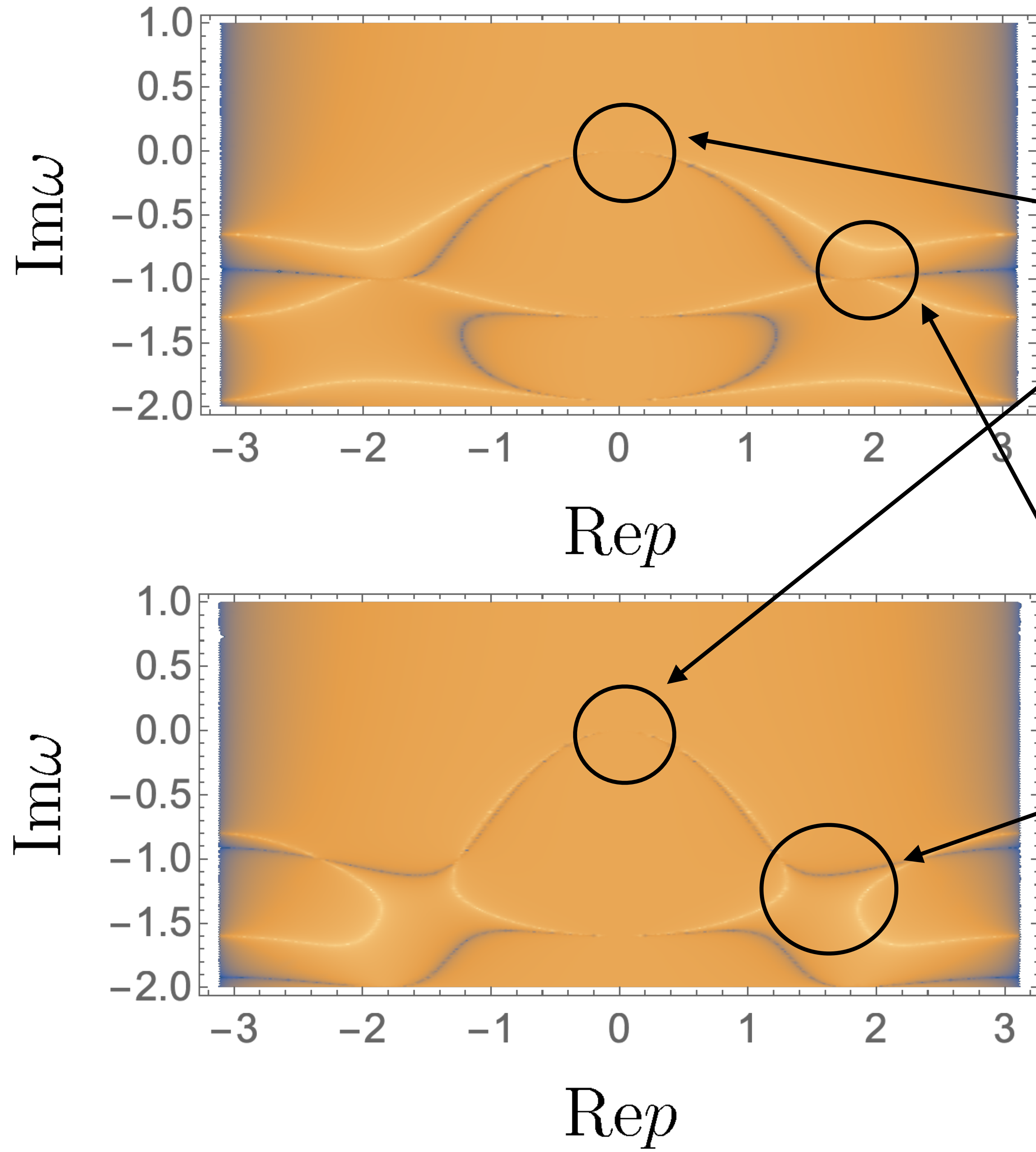
Many lower half plane pole skipping points, not related to chaos

Energy correlations in SYK chain

Diffusive dispersion relations

$w = 0.65$

$w = 0.8$



Hydro regime
$i\omega \approx Dp^2$

Diffusion pole rejoining with first gapped mode, outside of region of convergence for hydro

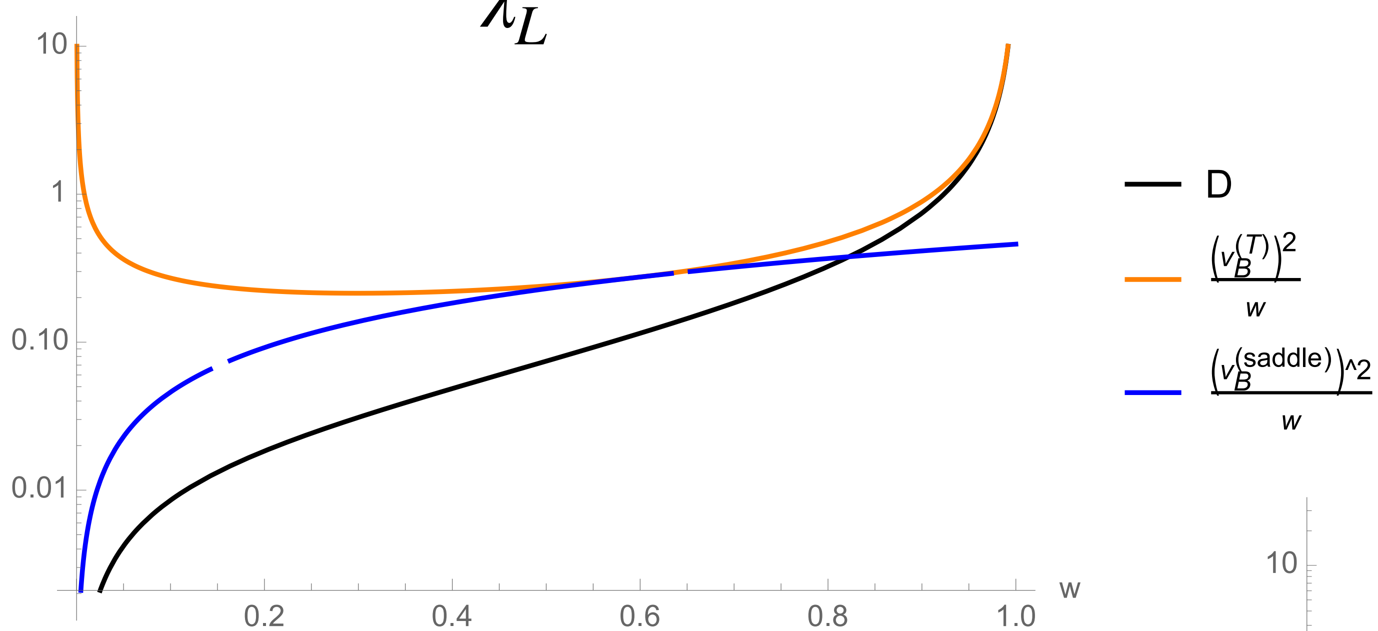
Diffusion

One may extract the diffusion constant:

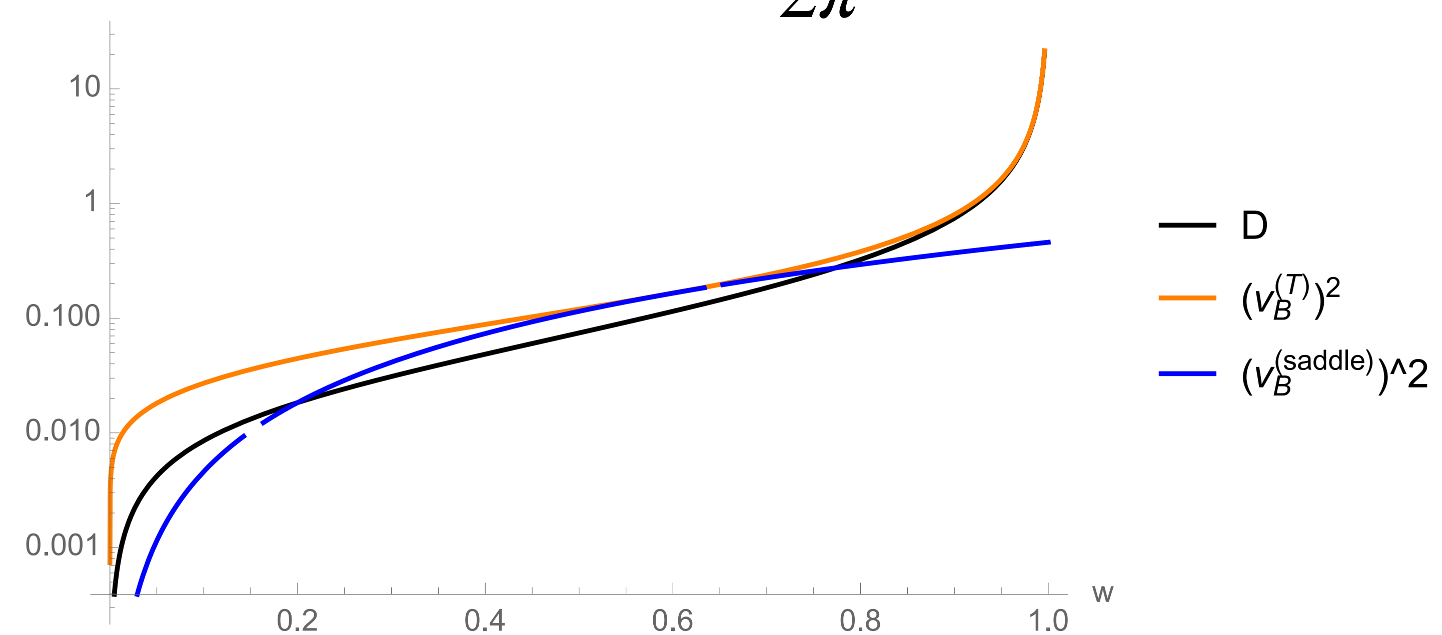
$$D = \frac{1}{12}\gamma w \left(\pi w \tan\left(\frac{\pi w}{2}\right) + 2 \right)$$

Conjectured bounds:

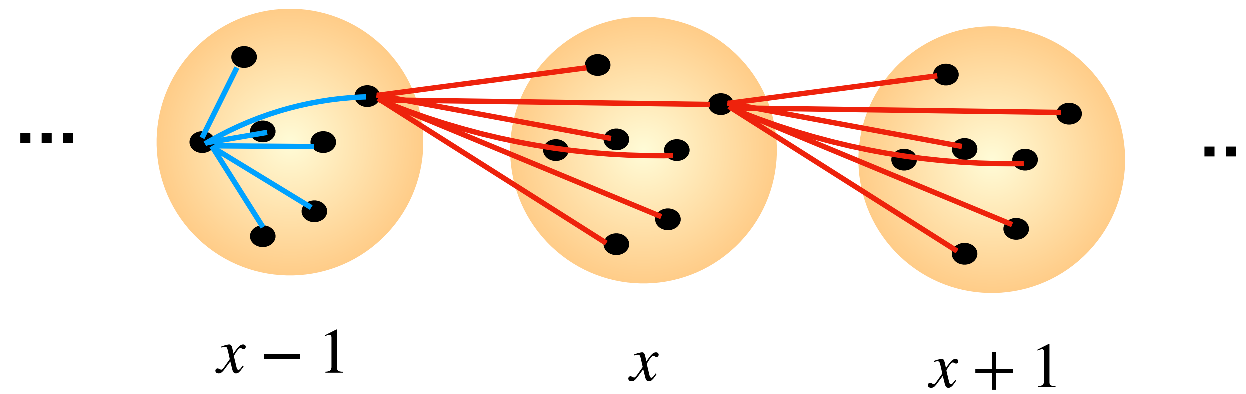
$$D \leq \frac{v_B^2}{\lambda_L}$$ [Hartman-Hartnoll-Mahajan]



$$D \leq \beta \frac{(v_B^{(T)})^2}{2\pi}$$ [Grozdanov]



In a nutshell

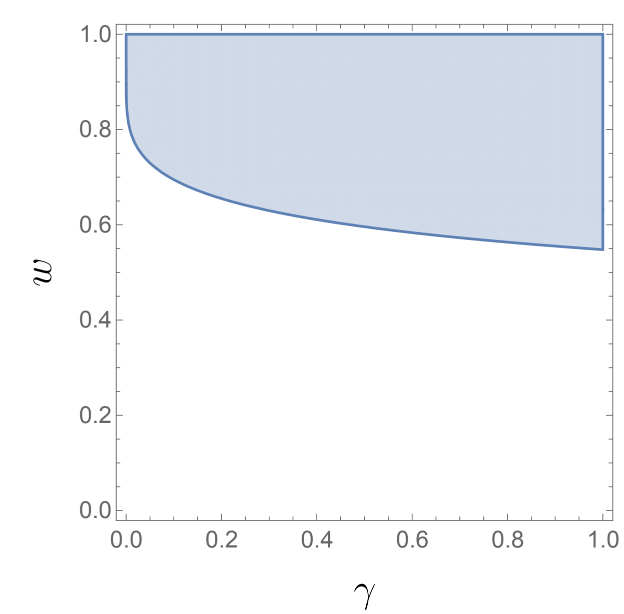


- **SYK chain has analytically solvable limit (large N and large q)**

interpolating between weakly coupled ($w=0$) and maximally chaotic ($w=1$) physics

- **We can calculate $\lambda(\mathrm{v})$ exactly as a function of the coupling**

There is a phase where chaos is maximal above a critical velocity, and there is a phase where it isn't



In a nutshell

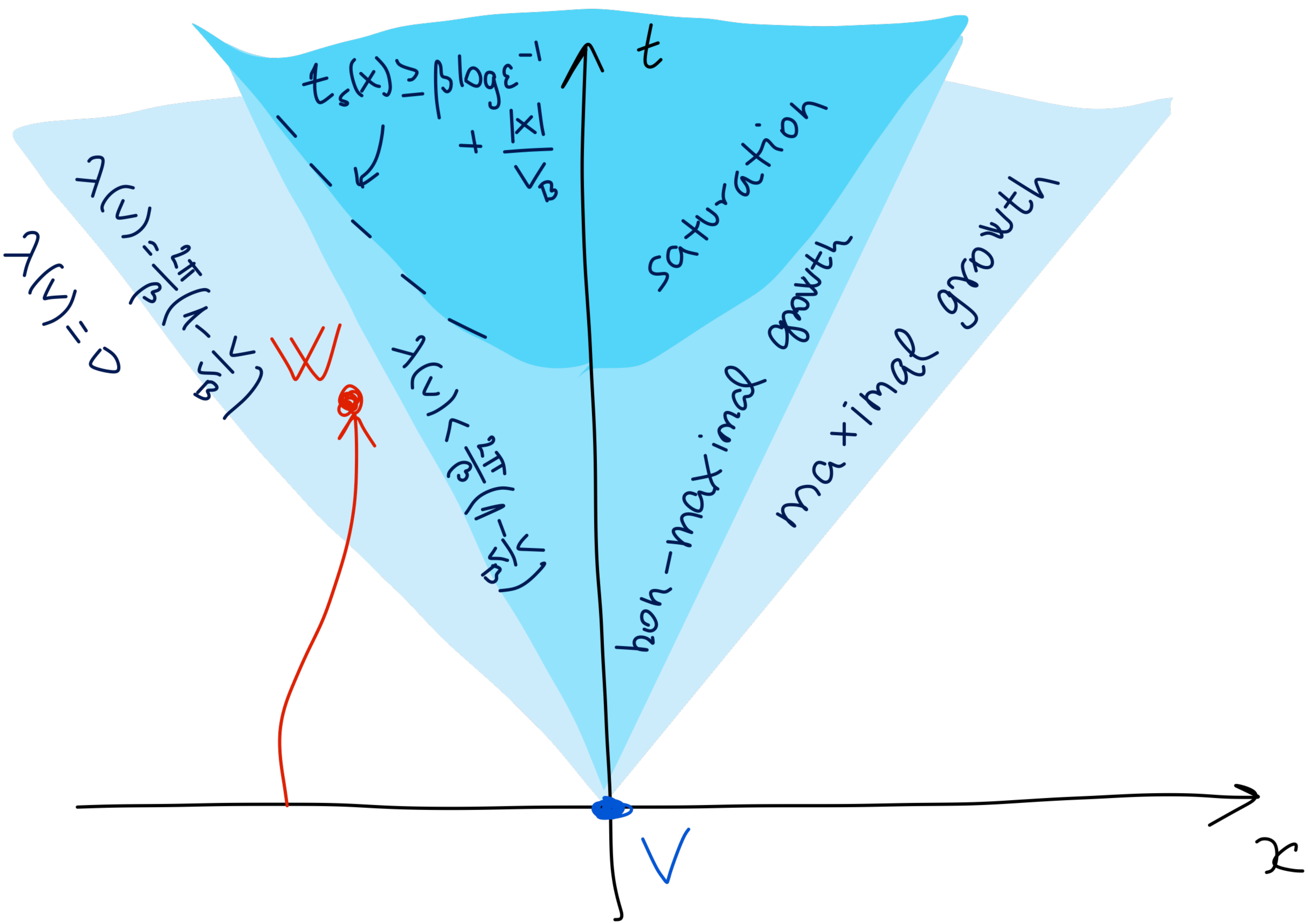
- $G^R_{\varepsilon\varepsilon}(\omega, p)$ **can be calculated exactly as a function of the couplings**

- **This is the only such known thermal correlator, has interesting analytic properties**

- **Confirms the modified pole-skipping conjecture**

$$(\omega, p)_{\textbf{p.s.}} = i\lambda_L^{(T)}\left(1, \frac{1}{\mathrm{v}_B^{(T)}}\right)$$

Summary



t
$t_s(x) \geq \beta \log \varepsilon^{-1} + \frac{|x|}{v_B}$
saturation
non-maximal growth
maximal growth
$\lambda(v) = \frac{2\pi}{\beta}\left(1 - \frac{v}{v_B}\right)$
$\lambda(v) = 0$
$\lambda(v) < \frac{2\pi}{\beta}\left(1 - \frac{v}{v_B}\right)$
W
V
x