On the geometry of quantum complexity

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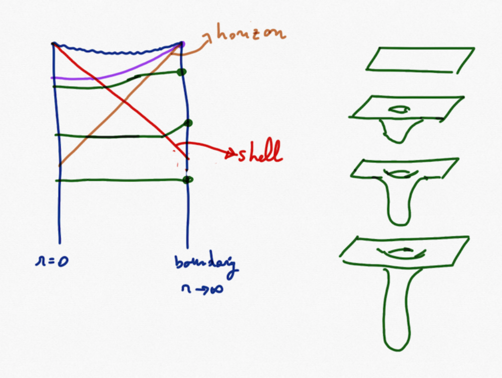
Intro and motivations

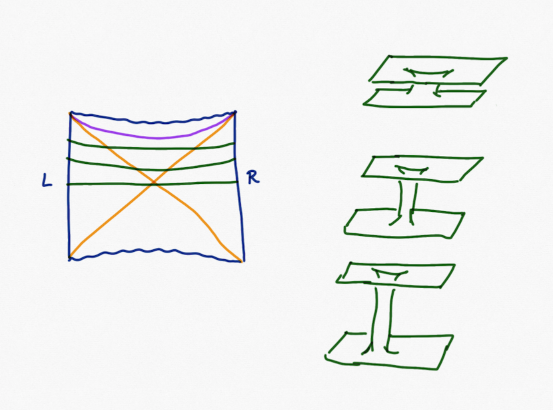
The growth of the bridge: gravitational collapse

The growth of Einstein-Rosen bridge continues far beyond the thermalisation time: Entanglement is not enough!



Susskind 1411.0690

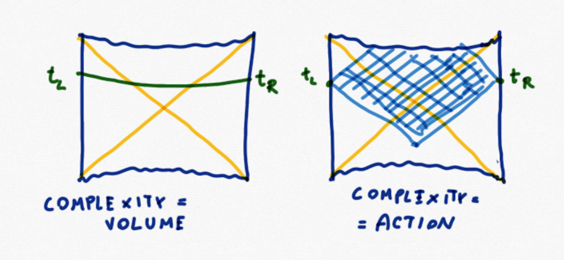
The growth of the bridge: eternal BH



Thermofield double state

$$|\Psi_{TFD}\rangle \propto \sum_n e^{-E_n\beta/2 - iE_n(t_L+t_R)}|E_n\rangle_R|E_n\rangle_L \,.$$

Holographic conjectures



$$C_V \sim \frac{\mathrm{Max}(V)}{G\,L}\,, \qquad C_A = \frac{\mathcal{S}}{\pi\hbar}$$

$$\frac{dC}{dt} \sim T\,S$$

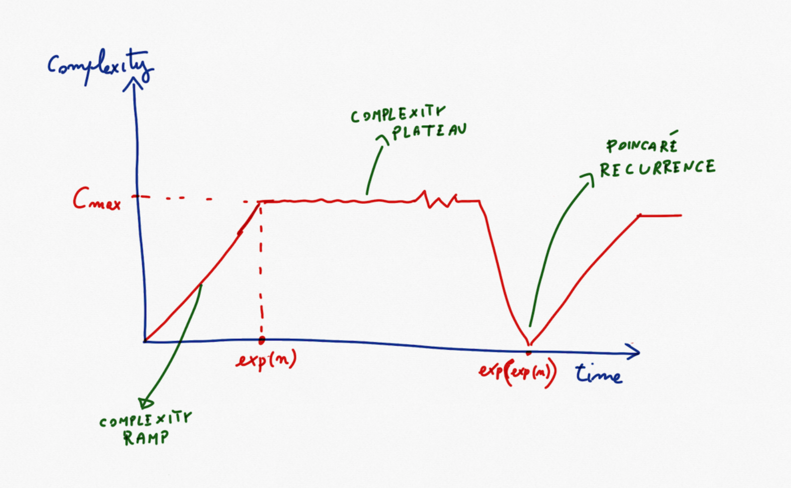
Quantum complexity

Complexity is heuristically defined as the minimum number of simple unitary operations required to reach a given state from a reference state

Example: a system of n qubits

- Simple state $|0\rangle = |00000\ldots\rangle$
- Generic state $|\psi\rangle = \sum_{i=1}^{2^n} \alpha_i |i\rangle$
- Simple operation: act on 2 qubits

Time evolution of complexity



Susskind, 1507.02287

Nielsen's complexity geometry

For n qubits, $U \in SU(2^n)$

$$\dot{U}(t) = -iH(t)U(t)\,, \qquad H = H^I \mathcal{O}_I$$

Cost function: $\qquad l = \int dt F(t)$

Which norm for complexity?

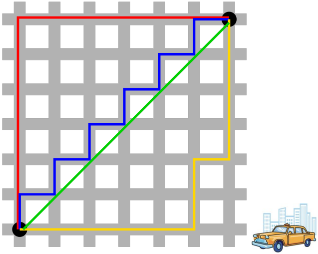
$$F_2 = \sqrt{\sum_I (H^I)^2}$$

Gives bi-invariant metric on U

$$d_{F_2} \leq \pi^2$$

Does not scale exponentially with n

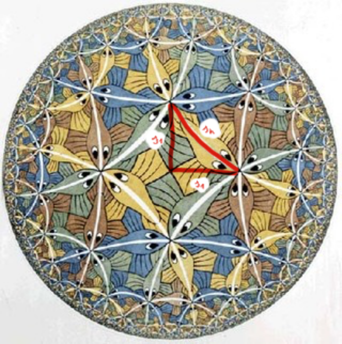
The taxicab metric



$$F_1 = \sum_I |H^I|$$

Better suited, but uncomfortable to deal with (Finsler geometry)

Hyperbolic space acts as a taxicab



$$\cosh s_h = \cosh^2(s_1)$$

Interpolates between $\mathcal{L}^2$ norm (small distances, $s_h = \sqrt{2} s_1$) and taxicab (large distances, $s_h = 2 s_1$)

Precursors and switchback

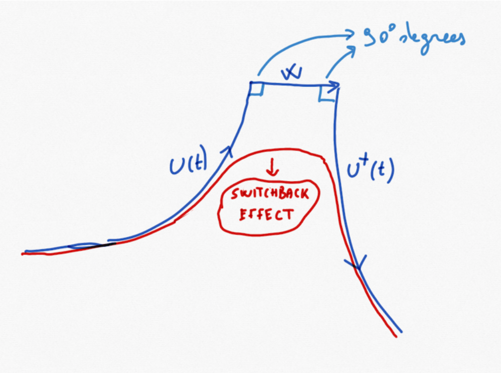
W is a simple operator with $O(1)$ complexity

$$W(t) = U(t)\, W\, U^\dagger(t)$$

Complexity of precursor

$$\mathcal{C}(W(t)) \approx 2K(t - t_*)$$

Precursors and negative curvature



Brown, Susskind, Zhao, 1608.02612

Motivations for negative curvature

1. To get exponential complexity in n, we should suppress shortcuts (and taxicab geometry does it!)
2. Switchback effect
3. Ergodicity of geodesics (Anosov)

Geometry of unitaries complexity

$\mathcal{L}^2$ norm with penalty factors

$$I = \int dt \langle H(t), H(t) \rangle^{1/2}$$

$$\langle H, K \rangle = \frac{\mathrm{Tr}\ (H\ \mathcal{G}(K))}{2^n}$$

For superoperator $\mathcal{G}$ not proportional to identity, it defines a right-invariant metric on the space of unitaries

We consider the case of n qubits, $U \in SU(2^n)$.
Generalised Pauli matrices σ

$$\langle \sigma, \tau \rangle = q_\sigma \delta_{\sigma\tau}\ , \qquad \text{penalty factor} \quad q_\sigma$$

Sectional curvatures

Right invariant metric on group manifold, (Arnold, Milnor)

Sectional curvatures vanish if $[\rho, \sigma] = 0$, otherwise:

$$K(\rho\,,\sigma) = \frac{1}{q_\rho\, q_\sigma}\left[-3\, q_{[\rho\,,\sigma]} + 2\,(q_\rho + q_\sigma) + \frac{(q_\rho - q_\sigma)^2}{q_{[\rho\,,\sigma]}}\right]$$

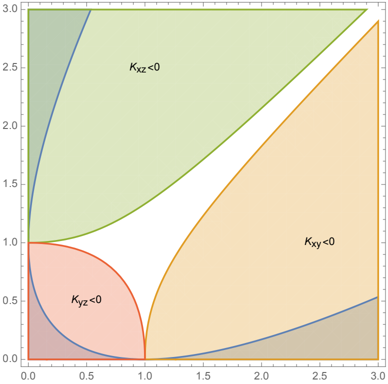
Negative curvature is associated to directions such that

$$[\mathrm{easy}\,,\mathrm{easy}] = \mathrm{hard}$$

Scalar curvature

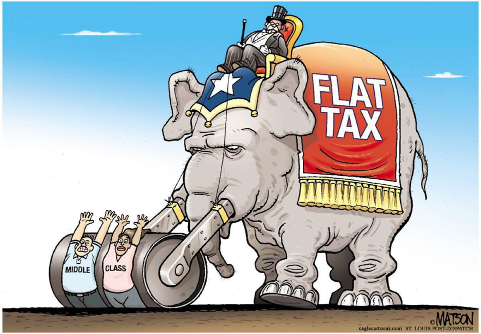
$$R = \sum_{\sigma,\,\rho} K(\rho\,,\sigma)$$

One qubit



$$q_{\sigma_x} = 1\,, \qquad q_{\sigma_y} = Q\,, \qquad q_{\sigma_z} = P\,,$$

Many qubits: Draconian



$$
\begin{aligned}
q(w) &= 1\,, & w &\le 2 & \mathcal{P} \\
q(w) &= q\,, & w &> 2 & \mathcal{Q}
\end{aligned}
$$

Dowling, Nielsen, quant-ph/0701004

Curvatures for Draconian Penalties

$$R \approx -55qn^3 + \frac{16^n}{2q}$$

In order to have negative R, $q \approx 4^n$

	$[\rho, \sigma] \in \mathcal{P}$	$[\rho, \sigma] \in \mathcal{Q}$
$\rho, \sigma \in \mathcal{P}$	$K = 1$	$K = 4 - 3q$
$\rho, \sigma \in \mathcal{Q}$	$K = \frac{4q-3}{q^2}$	$K = \frac{1}{q}$
$\rho \in \mathcal{P}$, $\sigma \in \mathcal{Q}$	$K = q$	$K = \frac{1}{q^2}$

Singular in negative curvature region!

Many qubits: Progressive

$$q(w) = \alpha^{w-1}\,,$$

At large α, sectional curvatures scale as

$$K = \mathrm{constant} + O(\alpha^{-1})\,.$$

Progressive penalties at large α

$$K = 1 \qquad \mathcal{N}_+ \approx 12\, 7^{n-1} n$$

$$K = -3 \qquad \mathcal{N}_- = \frac{\mathcal{N}_+}{2} - 3n$$

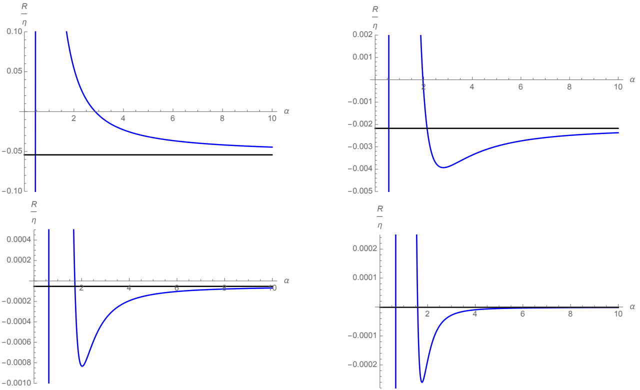
$$R = 3n \left(4^n - 2\, 7^{n-1}\right) .$$

We can have negative R with non singular sectional curvatures !

Number of sectional curvatures scales as $\eta \approx 16^n$
Average sectional curvature is small at large n and α:

$$\bar{K} = \frac{R}{\eta} \approx -\frac{6}{7} n \left(\frac{7}{16}\right)^n + \frac{1}{\alpha} \frac{9}{4^n} \frac{n(n-1)}{2} .$$

Progressive penalties, generic α



The exact value of R/η plotted as a function of α in the case of progressive penalties, for $n = 5, 10, 15, 20$. The minimum in the picture appears for $n \geq 8$.

States complexity

Unitary vs state complexity

So far we discussed, following Nielsen's idea, complexity defined for operators. The complexity metric was a right-invariant metric on the Lie group

$$M = SU(2^n)$$

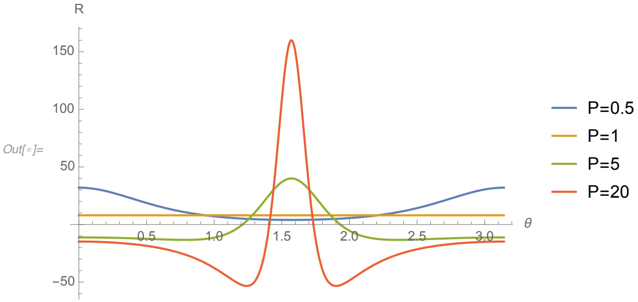
For holographic applications we would like to define complexity for states

$$B = \mathbb{CP}^{2^n-1} = \frac{SU(2^n)}{SU(2^n-1) \times U(1)} \,.$$

How are these two notions of complexity related ?

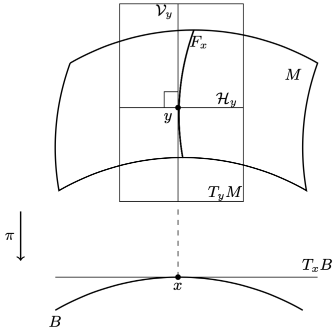
Single qubit: Bloch sphere $\mathbb{CP}^1$

$$q_{\sigma_x} = q_{\sigma_y} = 1\,, \qquad q_{\sigma_z} = P\,,$$



Brown and Susskind, 1903.12621

Submersions (O'Neill)



$$\pi : M \to B \text{ with:}$$

1. maximal rank
2. π_* preserves length of horizontal vectors (horizontal vectors are the ones orthogonal to the fiber $\pi^{-1}(x)$)

Submersion, curvatures and geodesics

O'Neill formula

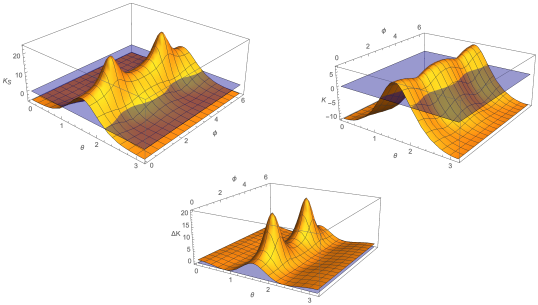
$$K_S(\tilde{h}_1, \tilde{h}_2) = K(h_1, h_2) + \frac{3}{4} \frac{|\mathcal{V}([h_1, h_2])|^2}{|h_1|^2 |h_2|^2 - \langle h_1, h_2 \rangle^2} ,$$

Relates sectional curvatures for states to the ones of unitaries

The extra contribution is always positive-definite (so curvature for states is always less negative compared to unitaries)

Geodesics in the state space B can be found projecting horizontal geodesics from the unitaries space M (result by O'Neill, 1967)

O'Neill formula for one qubit example



Comparison of $K_S(\tilde{h}_1, \tilde{h}_2)$, $K(h_1, h_2)$ and ΔK as a function of (θ, ϕ). $P = 6$, $Q = 3$.

States metric

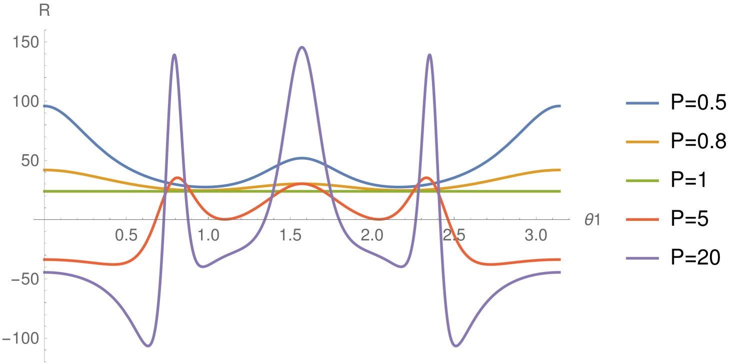
Closed-form expression for states metric

$$ds_S^2 = (\tilde{M}_{ij} - \tilde{M}_{ic}\tilde{M}_{ca}^{-1}\tilde{M}_{aj})u_i u_j \, .$$

$$\begin{aligned} \tilde{M}_{lm} &= M_{rs}(\mathrm{Ad}_{U_\theta^\dagger})_{rl}(\mathrm{Ad}_{U_\theta^\dagger})_{sm} \, , \\ u_s &= -i \, \mathrm{Tr}\left\{ U_\theta^\dagger dU_\theta \omega_s \right\} \end{aligned}$$

A bit cumbersome... we applied to one qutrit case

One qutrit



R for one qutrit state space $\mathbb{CP}^2$, with penalty factors P applied to all the generators of the unbroken subgroup.

Towards exponential complexity

Towards exponential complexity

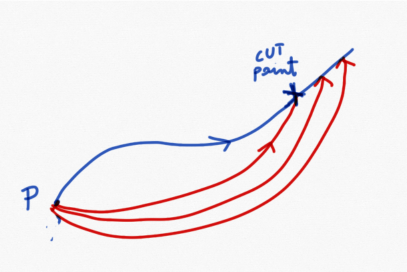
Both in unitaries and in state space, holographic expectations requires that maximal complexity scales exponentially with the number of states

This is not true for uniform penalties $q_\sigma = 1$:

- Bi-invariant metric for unitaries ($SU(2^n)$ for n qubits)
- Fubini-Study metric for states ($\mathbb{CP}^{2^n-1}$ for n qubits)

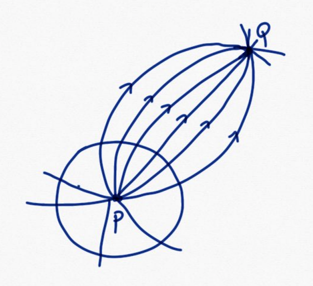
Can we prove that complexity scales exponentially with number of qubits, at least for opportune choice of penalties?

Cut point



Cut point: consider a geodesic starting from P. For short enough distances, the geodesic is the minimal path. A cut point is defined as a point after which the geodesic is no longer the minimal path.

Conjugate points

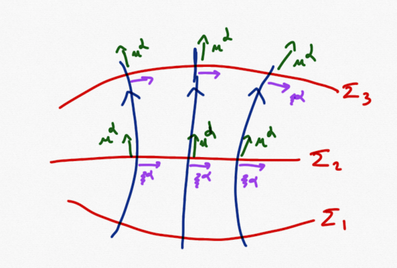


Conjugate points: these are two points P and Q along a geodesic for which it exists a continuous 1-parameter family of geodesics connecting them.

A geodesic fails to be minimising after its first conjugate point. But it might well fail to be minimising before it...

Congruence of geodesics

Congruence of geodesics, orthogonal to hypersurfaces Σ



$$u^{\alpha} u_{\alpha} = 1\,, \qquad \xi^{\alpha} u_{\alpha} = 0\,, \qquad g_{\alpha\beta} = h_{\alpha\beta} + u_{\alpha} u_{\beta}$$

Conjugate points and Raychaudhury eq

$$B_{\alpha\beta} = D_\beta u_\alpha = \frac{1}{d-1}\Theta h_{\alpha\beta} + \sigma_{\alpha\beta}$$

Expansion scalar $\Theta \to -\infty$ detects conjugate points

$$\Theta = \frac{1}{\Delta V}\frac{d\Delta V}{d\lambda}$$

Raychaudhury eq:

$$\frac{d\Theta}{d\lambda} = -\frac{1}{d-1}\Theta^2 - \sigma^{\alpha\beta}\sigma_{\alpha\beta} - R_{\alpha\beta}u^\alpha u^\beta \,,$$

Exponential geodesics

From Euler-Arnold eq

$$\dot{X} + i\mathcal{G}^{-1}\left([X, \mathcal{G}(X)]\right) = 0\,.$$

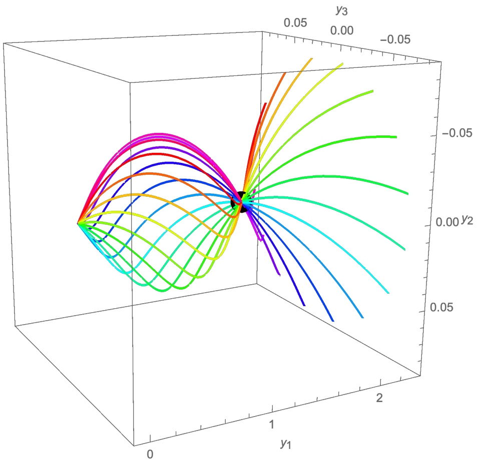
we find that the exponential of an eigenvector of penalty matrix $\mathcal{G}$ gives a simple class of geodesics (for unitaries)

Some of these geodesics are horizontal, and give a geodesic also for states

$$\lambda_c \leq \lambda_0 = \frac{\pi\sqrt{d-1}}{\sqrt{R_{\alpha\beta}u^{\alpha}u^{b}}}\,.$$

Using Raychaudhury eq, we can estimate conjugate points

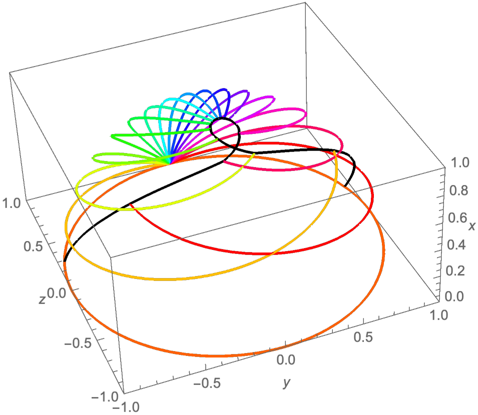
1 qubit case: unitaries



Example of an exact conjugate point (the black spot) of geodesics for $P = Q = 0.4$ in stereographic projection.

1 qubit case: states

$\lambda = 2.5,\ Q = 10,\ P = 10.$



The maximal complexity region lies just before the conjugate point

Large n: bi-invariant metric

In every direction in unitary space, there is a conjugate point before

$$\lambda_0 = \theta_0 \approx \pi\sqrt{2}\,.$$

Exponential of eigenvector of penalties remain geodesic also with arbitrary values of penalties

We call θ the length of the corresponding path measured with the bi-invariant metric

Large n: Draconian penalties

$$q \approx 4^n$$

Conjugate points of exponential geodesic, as a function of weight

$$\begin{aligned}
w &= 1\,, & \lambda_0 &\approx \frac{\pi 2^n}{\sqrt{6n}}\,, & \theta_0 &= \lambda_0\,, \\
w &= 3\,, & \lambda_0 &\approx \frac{\pi 2^n}{\sqrt{12q}} \approx \frac{\pi}{\sqrt{12}}\,, & \theta_0 &= \frac{\lambda_0}{\sqrt{q}} \approx \frac{1}{2^n}\frac{\pi}{\sqrt{12}}\,, \\
w &\geq 4\,, & \lambda_0 &\approx \pi\sqrt{2q} \approx \sqrt{2}\pi\, 2^n\,, & \theta_0 &= \frac{\lambda_0}{\sqrt{q}} \approx \sqrt{2}\pi\,,
\end{aligned}$$

Large n: Progressive penalties

Conjugate points for $w = 1$ and for $w > \frac{2}{3}(n+1)$

For $w = n$

$$\lambda_0 = \frac{\pi\, 2^{n/2}}{\sqrt{n}}\,, \qquad \theta_0 = \frac{\lambda_0}{\alpha^{n/2}} = \frac{\pi\, 2^{n/2}}{\sqrt{n}\,\alpha^{n/2}}\,.$$

For $w \approx \frac{2}{3}n$

$$\lambda_0 = \frac{\pi\, 2^{2n/3}}{\sqrt{n}}\,, \qquad \theta_0 = \frac{\lambda_0}{\alpha^{n/3}} = \frac{\pi\, 2^{2n/3}}{\sqrt{n}\,\alpha^{n/3}}\,.$$

Exponential complexity for progressive penalties

At largish $\alpha > 4$ the conjugate points are at small θ

A conjugate point at small θ is likely also a cut point (shortcuts usually are "non-perturbative", it is difficult to imagine a non-trivial shortcut at small θ)

$$\lambda_{\max} = \frac{\pi\, 2^{2n/3}}{\sqrt{n}}\,, \tag{1}$$

This is a strong indication of an exponential maximal complexity

Using the trick of projecting $w = n$ geodesic to states (it is horizontal!), this applies also to state complexity

Summary

We studied a few aspects of Nielsen's complexity geometry in quantum mechanics:

- Curvatures for large number of qubits for progressive penalties. Negative average sectional curvature can be achieved in a non-singular way.
- Relation between Unitary and States geometry, using the tool of Riemannian Submersions
- Evidence that maximal complexity scales exponentially with the number of qubits, by studying conjugate points of a simple class of geodesics

Many open questions

- How does maximal complexity scales with number of qubits, given some choice of penalties? How is this related to negative curvature?
- Precursor operators and switchback effect
- Mixed states complexity
- Complexity in SYK model ?
- Complexity in QFT ?

Thank you!