Complexity for CFTs in General Dimensions

Based on [hep-th/2103.06920]
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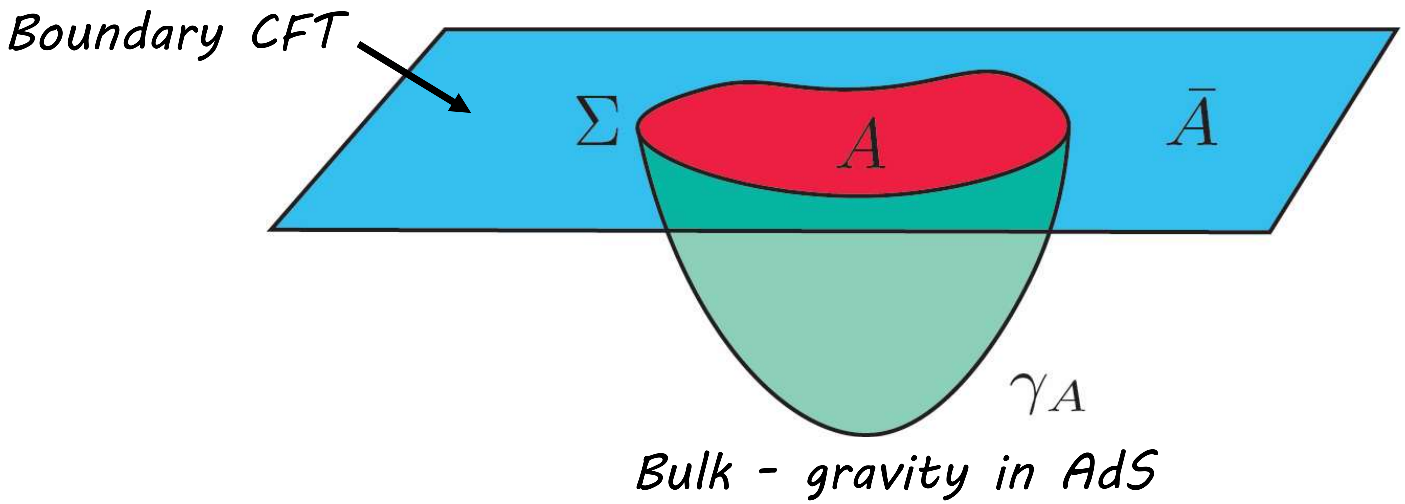
Holography and Quantum Information

Bekenstein-Hawking formula (1973') $S = \frac{A}{4G_N}$

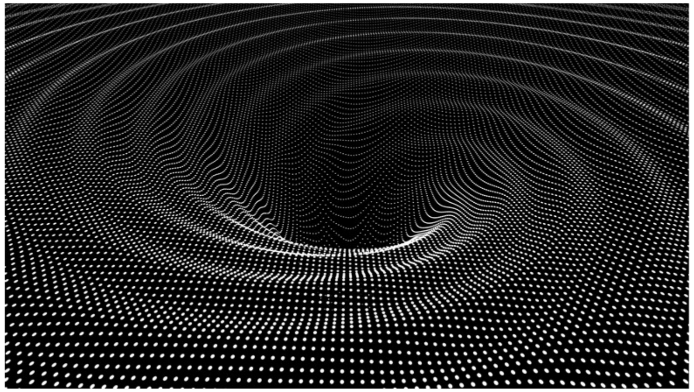
Ryu-Takayanagi formula (2006') for entanglement entropy

$$S_{EE} = \min_{\Sigma \sim \partial\gamma_A} \frac{Area(\gamma_A)}{4\, G_N}$$



Black holes grow forever

Even though, from the outside, they appears to stay constant in size, the interior volume grows bigger and bigger all the time as space stretches toward the center point.



Puzzle: How is this growth encoded in the boundary?

Quantum Computational Complexity

entropy

"Entanglement is not enough"

Susskind (2014')

Field Theory Complexity

- To understand this proposal we need a definition of complexity in QFT.

- Progress for Gaussian states of free QFTs
  [Jefferson, Myers; SC, Heller, Marrochio, Pastawski; ...]

- Progress for 2d CFTs
  [Caputa, Magan; Flory, Heller; Erdmenger, Gerbershagen, Weigel; ...]

How can we define complexity in CFTs in General dimensions?

Outline

Part I - Overview of Complexity

Part II - Continuous state complexity
- Circuits, distance functions

Part III - Complexity in CFT
- Higher dimensional case
- Comparison with existing results in 2d

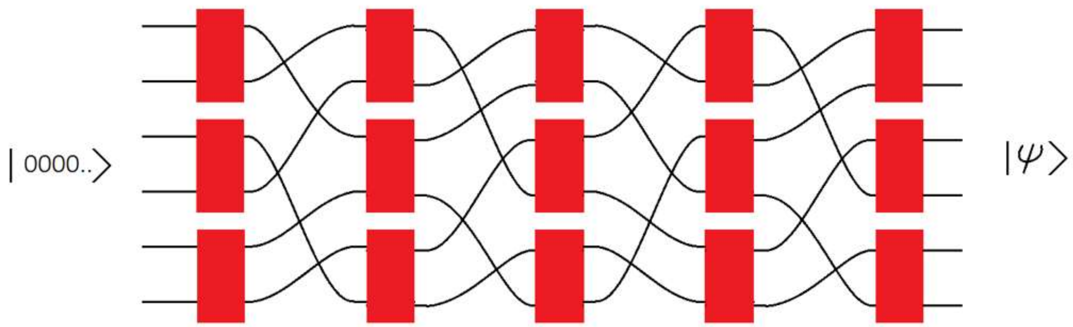
Part IV - Relation to Coadjoint Orbits

Future Directions and Summary

Part 1 - Overview of Complexity

Quantum Computational Complexity

Complexity of a quantum state is defined as the minimal number of simple unitary operations one has to put together starting from a simple (unentangled) reference state in order to obtain a given target state.



Circuit complexity of $|\psi\rangle$ is the minimum number of gates needed to go from $|0000..\rangle$ to $|\psi\rangle$.

Example - Spin/Qubit Chain

- Start with simple unentangled reference state

$$|\psi_R\rangle = |\uparrow\uparrow \cdots \uparrow\rangle$$

- Apply simple universal gates to construct a Quantum circuit

Act on a small number of qubits

Approximate any state

Gates

Phase shifter gates

$$R(\theta) = \begin{array}{cc} \begin{bmatrix} 1 & 0 \\ 0 & e^{2\pi i\theta} \end{bmatrix} & \begin{array}{c} |0\rangle \\ |1\rangle \end{array} \end{array}$$

Hadamard gate

$$H = \frac{1}{\sqrt{2}} \begin{array}{cc} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} & \begin{array}{c} |0\rangle \\ |1\rangle \end{array} \end{array}$$

controlled-NOT gate

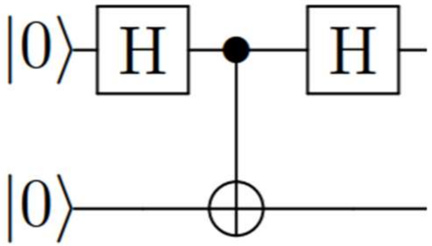
$$\mathrm{CNOT} = \begin{array}{cc} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} & \begin{array}{c} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array} \end{array}$$

- Approximate target state with unitary operations built from those gates $|\psi_T\rangle = U|\psi_R\rangle = g_n \ldots g_2\, g_1|\psi_R\rangle$.
- With tolerance $\langle\psi_T|\psi_T^{(\epsilon)}\rangle \geq 1 - \epsilon$.

Example - Spin/Qubit Chain

Complexity = minimal number of gates.

Circuit



$$\frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$$

Gates

Phase shifter gates

$$R(\theta) = \begin{array}{cc} \begin{bmatrix} 1 & 0 \\ 0 & e^{2\pi i\theta} \end{bmatrix} & \begin{array}{c} |0\rangle \\ |1\rangle \end{array} \end{array}$$

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controlled-NOT gate

$$\mathrm{CNOT} = \begin{array}{cc} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} & \begin{array}{c} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array} \end{array}$$

- Depends on the various choices! (but some properties are universal).
- Important in quantum computation which uses the quantum superposition to perform a computation in parallel on many inputs.

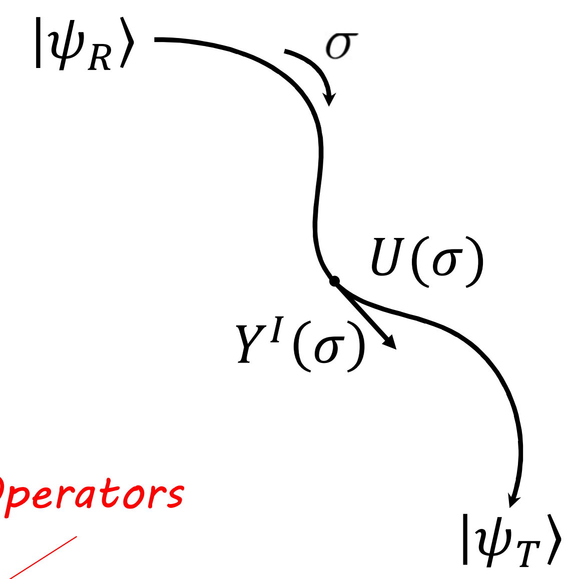
Part II - Continuous State Complexity

Continuous Circuits

A continuous circuit approach

$$|\psi(\sigma)\rangle = U(\sigma)|\psi_R\rangle \qquad \sigma \in [0,1]$$

$$|\psi_T\rangle = U(1)|\psi_R\rangle$$



Build unitaries from closed algebra of operators

Control Functions

Operators

$$U(\sigma) = \overleftarrow{\mathcal{P}} e^{-i\int_0^{\sigma} H(\sigma')d\sigma'} \qquad H(\sigma) = \sum_I Y^I(\sigma)\mathcal{O}_I$$

Instantaneous Hamiltonian

For unitary complexity: Nielson et al. (2006)

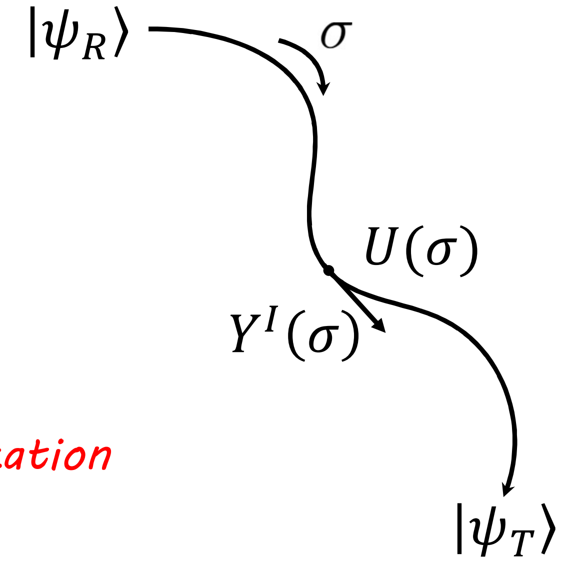
Cost functions

Length of a given circuit

$$\mathcal{D}(U) = \int_0^1 d\sigma\, \mathcal{F}(\underbrace{U(\sigma)}_{\text{position}}, \underbrace{Y^I(\sigma)}_{\text{velocity}})$$

Cost function

(on space of unitaries)



Unitary Complexity: $\mathcal{C} = \min_{U(1)=U_T} \mathcal{D}(U)$

State complexity: $\mathcal{C} = \min_{U(1)|\psi_R\rangle=|\psi_T\rangle} \mathcal{D}(U)$

Additional minimization

Cost functions induce a geometry on the space of unitaries/states $\Longrightarrow$ Find geodesics

Different choices of cost function give different definitions for complexity

For unitary complexity: Nielson et al. (2006)

Cost functions

$\mathcal{F}_1$ cost-function: $|\langle\psi|\partial_\sigma\psi\rangle|$

$\sim \left|\sum Y^I\right| \neq \sum\left|Y^I\right|$

Berry phase

$$\Rightarrow \boxed{\mathcal{F}_1(\sigma) = \left|\langle\psi_R\middle|U^\dagger\partial_\sigma U\middle|\psi_R\rangle\right|}$$

Fubini-Study norm:

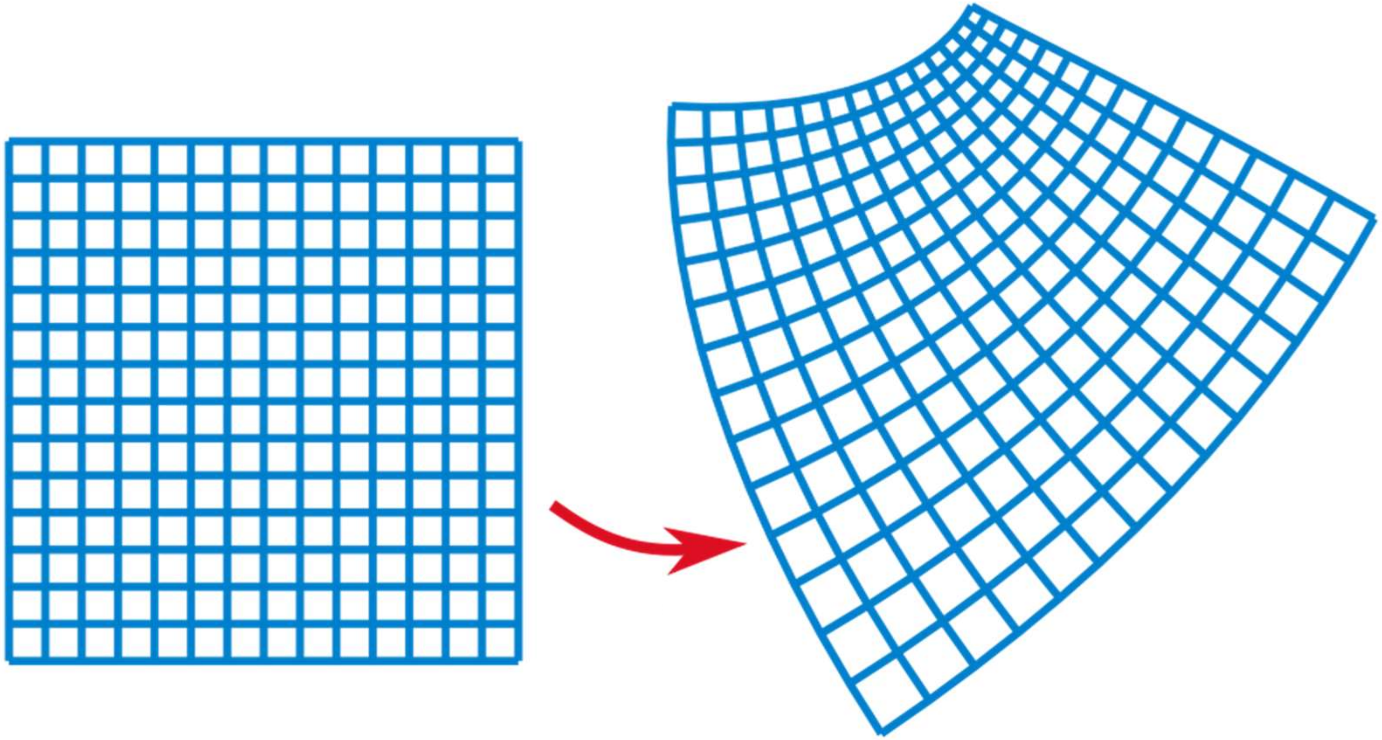
Variance: $\sim \langle H^2\rangle - \langle H\rangle^2$

Removes overall phases

$$\langle\partial_\sigma\psi|\partial_\sigma\psi\rangle - |\langle\psi|\partial_\sigma\psi\rangle|^2$$

$$\Rightarrow \boxed{\frac{ds_{FS}^2}{d\sigma^2} = \langle\psi_R\middle|\partial_\sigma U^\dagger\partial_\sigma U\middle|\psi_R\rangle - \left|\langle\psi_R\middle|U^\dagger\partial_\sigma U\middle|\psi_R\rangle\right|^2}$$

$$\mathcal{D}_{FS} = \int ds_{FS}$$



Part III - Complexity in CFT

CFTs in $\mathrm{d} > 2$

Euclidean conformal algebra in higher dimensions

P_μ - translations

D- dilatations

$L_{\mu\nu}$ - rotations

K_μ - special conformal

Work in radial quantization

$$(P_\mu)^\dagger = K_\mu \qquad (L_{\mu\nu})^\dagger = -L_{\mu\nu} \qquad (D)^\dagger = D$$

As the reference state we will consider a spinless primary

$$|\psi_R\rangle = |\Delta\rangle \qquad D|\Delta\rangle = \Delta|\Delta\rangle; \qquad K_\mu|\Delta\rangle = L_{\mu\nu}|\Delta\rangle = 0;$$

CFTs in $d > 2$

The primary and its descendants form a reflection-positive representation of the Euclidean conformal algebra

Under a Wick rotation, this maps to a unitary representation of the Lorentzian conformal algebra

We compute the complexity for unitary circuits of the Lorentzian conformal algebra

State $|\Delta\rangle$ unmodified (up to an overall phase) by exponentiating the Hermitian generators $D, iL_{\mu\nu}$ - $SO(d) \times SO(2)$ stabilizer subgroup of $SO(d,2)$.

Complexity in CFTs in $d > 2$

$$U(\sigma) \equiv e^{i\alpha(\sigma)\cdot P} e^{i\gamma_D(\sigma) D} \left(\prod_{\mu<\nu} e^{i\lambda_{\mu\nu}(\sigma) L_{\mu\nu}} \right) e^{i\beta(\sigma)\cdot K}$$

Unitarity:

$$\gamma_D^I = -\frac{1}{2}\log(1 - 2\,\alpha\cdot\alpha^* + \alpha^2\alpha^{*2})$$

$$2\,\alpha\cdot\alpha^* - \alpha^2\alpha^{*2} < 1$$

Fubini-Study metric:

$$\boxed{\mathrm{d}s_{FS}^2 = 2\Delta \left[\frac{d\alpha\cdot \mathrm{d}\alpha^* - 2|d\alpha\cdot\alpha|^2}{1 - 2\,\alpha\cdot\alpha^* + \alpha^2\alpha^{*2}} + 2\frac{\left|d\alpha\cdot\alpha^* - {\alpha^*}^2\alpha\cdot d\alpha\right|^2}{\left(1 - 2\,\alpha\cdot\alpha^* + \alpha^2\alpha^{*2}\right)^2} \right]}$$

Metric on the coset space: $\dfrac{SO(d,2)}{SO(d)\times SO(2)}$

Gibbons (1999) - metric on the space of timelike geodesics in AdS

Complexity in CFTs in $d > 2$

$$ds_{FS}^2 = 2\Delta\left[\frac{d\alpha\cdot d\alpha^* - 2|d\alpha\cdot\alpha|^2}{1-2\,\alpha\cdot\alpha^* + \alpha^2\alpha^{*2}} + 2\frac{\left|d\alpha\cdot\alpha^* - \alpha^{*2}\alpha\cdot d\alpha\right|^2}{\left(1-2\,\alpha\cdot\alpha^* + \alpha^2\alpha^{*2}\right)^2}\right]$$

- Metric is Einstein-Kahler

$$R_{AB} = -\frac{2d}{\Delta}g_{AB}$$

$$ds_{FS}^2 = -\Delta\ \partial_\alpha\partial_{\alpha^*}\log\left(1-2\,\alpha\cdot\alpha^* + \alpha^2\alpha^{*2}\right)d\alpha\, d\alpha^*$$

- All sectional curvatures are negative
- Geodesics passing through $|\psi_R\rangle$ are

$$|\psi(\sigma)\rangle = \exp[\mathrm{i}\sigma\left(\tilde{\alpha}P_\mu + \tilde{\alpha}^*K_\mu\right)]|\psi_R\rangle$$

Straight line geodesic

Complexity in CFTs in $d > 2$

$$U(\sigma) \equiv e^{i\alpha(\sigma)\cdot P} e^{i\gamma_D(\sigma) D} \left(\prod_{\mu<\nu} e^{i\lambda_{\mu\nu}(\sigma) L_{\mu\nu}} \right) e^{i\beta(\sigma)\cdot K}$$

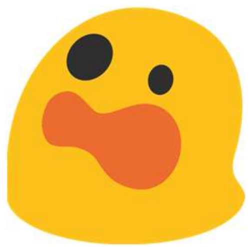
$\mathcal{F}_1$ cost-function

$$\mathcal{F}_1 = \Delta \left| \frac{d\alpha \cdot \alpha^* - d\alpha^* \cdot \alpha + \alpha^2 (d\alpha^* \cdot \alpha^*) - \alpha^{*2} (d\alpha \cdot \alpha)}{1 - 2\, \alpha \cdot \alpha^* + \alpha^2 \alpha^{*2}} + i\, d\gamma_D^R \right|$$

Dependence on the overall phase

2d-1 null directions (e.g., $\alpha \propto \alpha^*$)

$\Longrightarrow$ Not a norm



Complexity in CFTs in $d = 2$

Global conformal group in d=2: $SO(2,2) = SO(2,1) \times SO(2,1)$

Holomorphic Anti-holomorphic

Holomorphic generators: $L_0, L_1, L_{-1} \in SO(2,1)$ (and barred versions)

Radial quantization: $(L_1)^\dagger = L_{-1}$ $(L_0)^\dagger = L_0$

Reference state: $|h, \bar{h}\rangle = |h\rangle \otimes |\bar{h}\rangle$ primary (highest weight) state

$$L_0|h\rangle = h|h\rangle; \qquad L_1|h\rangle = 0$$

State $|\mathrm{h}\rangle$ unmodified (up to an overall phase) by exponentiating the Hermitian generator L_0 - $SO(2)$ stabilizer subgroup of $SO(1,2)$.

Complexity in CFTs in $d = 2$

$$U(\sigma) \equiv e^{i\zeta(\sigma)L_{-1}} e^{i\gamma(\sigma)L_0} e^{i\zeta_1(\sigma)L_1}$$
$$|\zeta(\sigma)\rangle \equiv U(\sigma)|h\rangle = N(\sigma) e^{i\zeta(\sigma)L_{-1}}|h\rangle$$

Fubini-Study metric:

$$\boxed{ds^2_{FS} = 2h\frac{d\zeta d\zeta^*}{(1-|\zeta|^2)^2}}$$ (+ barred)

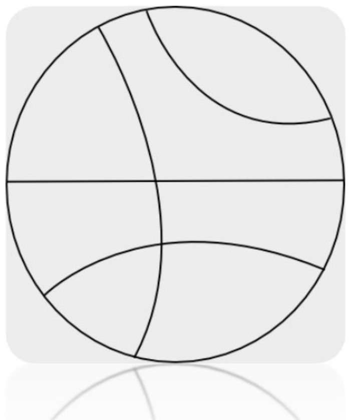
= Two copies of the Poincaré disk

$$\mathbb{H}_2 \times \mathbb{H}_2$$
$$= \frac{SO(2,1)}{SO(2)} \times \frac{SO(2,1)}{SO(2)}$$

Unitarity $\Longrightarrow$ $|\zeta|^2 < 1$

Restricted inside Poincaré disk

$$\boxed{\mathcal{F}_1 = h\left|\frac{\dot{\zeta}\zeta^* - \dot{\zeta}^*\zeta}{1-|\zeta|^2} + i\dot{\gamma}_R + (\zeta \leftrightarrow \bar{\zeta})\right|}$$



Comparison to previous literature

Previously: complexity from the diffeomorphism $f\left(z = e^{i\theta}\right) \in Diff\left(S^1\right)$ *associated to the holomorphic part of the circuit*

$$\mathcal{F}_1 = \left| \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \frac{\partial_\sigma f(\sigma,\theta)}{\partial_\theta f(\sigma,\theta)} \left[-\left(h - \frac{c}{24} \right) + \frac{c}{12} \{f, \theta\} \right] \right|$$

Caputa, Magan (2019); Erdmenger, Gerbershagen, Weigel (2020);

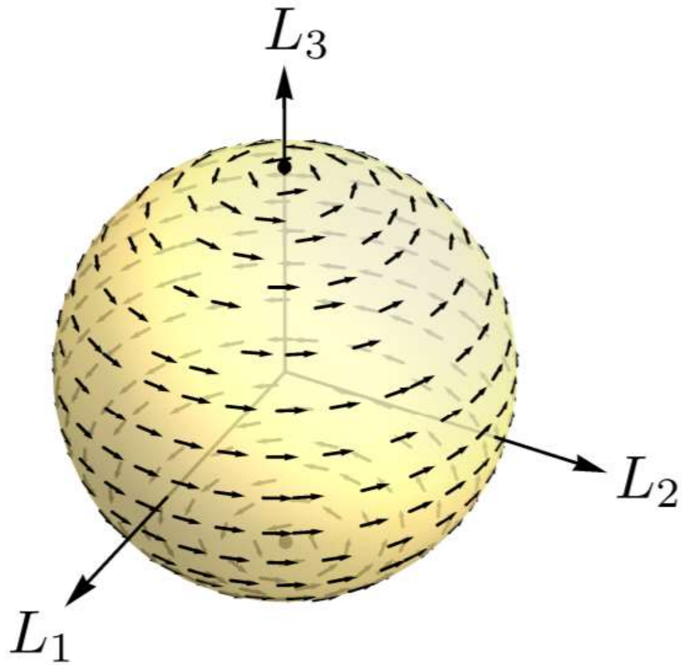
$$\frac{ds_{FS}^2}{d\sigma^2} = \int_0^{2\pi} \frac{\mathrm{d}\theta_1}{2\pi} \frac{\mathrm{d}\theta_2}{2\pi} \frac{\partial_\sigma f(\sigma,\theta_1)}{\partial_{\theta_1} f(\sigma,\theta_1)} \frac{\partial_\sigma f(\sigma,\theta_2)}{\partial_{\theta_2} f(\sigma,\theta_2)} \left[\frac{c}{32 \sin^4[(\theta_1 - \theta_2)/2]} - \frac{h}{2 \sin^2[(\theta_1 - \theta_2)/2]} \right]$$

Heller, Flory (2020)

Diffeomorphism for our circuit

$$f\left(\sigma, e^{i\theta}\right) = \frac{e^{i\gamma^*/2} e^{i\theta} + i\zeta(\sigma) e^{-i\gamma/2}}{-i\zeta^*(\sigma) e^{i\gamma^*/2} e^{i\theta} + e^{-i\gamma/2}}$$

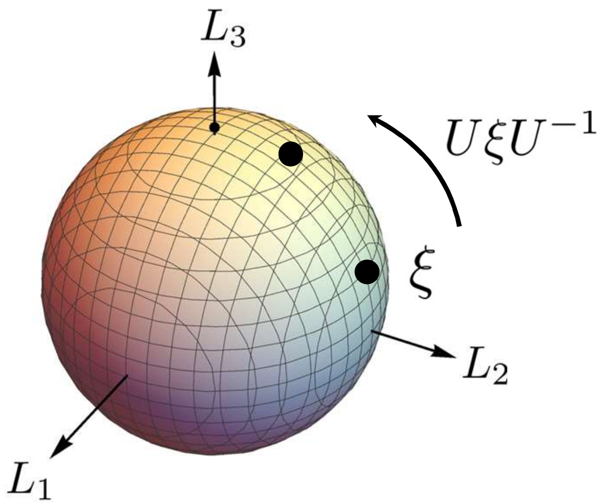
✓



Part IV - Relation to Coadjoint Orbits

Witten (1988)
Kirillov (2004)

Coadjoint orbits



e.g. $SO(3)$

$$[L_i, L_j] = i\epsilon_{ijk}L_k$$

$$\langle L_i, L_j\rangle = \tfrac{1}{2}\mathrm{Tr}(L_iL_j) = \delta_{ij}$$

Symmetry group G, Lie algebra $\mathfrak{g}$.
Dual space $\mathfrak{g}^*$, Pairing $\langle\cdot,\cdot\rangle$

Functions from the algebra to the real numbers

$$\xi \equiv \tfrac{1}{2}\mathrm{Tr}[\xi\ \cdot\ ] \in \mathfrak{g}^*$$

Adjoint action

$$\mathrm{Ad}_U(\xi) = U\xi U^{-1} \qquad U \in G$$

$$\mathrm{ad}_u(\xi) = [u, \xi] \qquad u \in \mathfrak{g}$$

Generates a symplectic manifold that is a coset space G/H

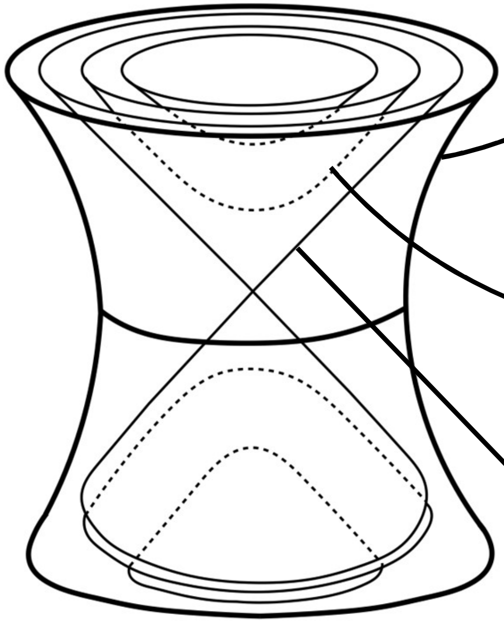
$$\left.\begin{array}{l}\xi = J_1L_1 + J_2L_2 + J_3L_3 \\ \mathrm{Tr}(\xi^2) = \mathrm{Tr}((U\xi U^{-1})^2) = J^2\end{array}\right\}$$

Rotations around "ξ" do nothing

$$SO(3)/SO(2)$$

Coadjoint Orbits

(Co)-adjoint orbits of: $\mathrm{SO}(2,1)$ *Holomorphic copy in 2d CFT*



de Sitter hyperboloid dS_2

$$\frac{SO(2,1)}{SO(1,1)}$$

Hyperboloid $\mathbb{H}_2$

$$\frac{SO(2,1)}{SO(2)}$$

Cone

Geometry of coadjoint orbits

Instantaneous Hamiltonian

- Maurer-Cartan form

$$\Theta \equiv u^{-1}du, \qquad u \in G$$

- Orbit Representative: $\lambda \in \mathfrak{g}^*$
- Stabilizer subgroup: $H_\lambda = \mathrm{stab}(\lambda)$ → Elements that don't modify λ
- Coset space: G/H_λ
- Geometric action: $S_\lambda = \int \mathcal{A}_\lambda \qquad \mathcal{A}_\lambda \equiv \langle \lambda, \Theta \rangle$ → Gauge field
- Symplectic form $\omega_\lambda = \langle \lambda, d\Theta \rangle$ → Field strength/ Curvature
- Coadjoint orbit metric $ds^2_{G/H_\lambda}(x,y) = \omega_\lambda(x, J_\lambda y)$
  where $J_\lambda^2 = -1$ is a complex structure.

Coadjoint orbit comparison

Basis for the conformal algebra:

(-1 , 0 1 , ……… , d)

$$(M_{AB})^C{}_D \equiv \delta_A{}^C g_{BD} - \delta_B{}^C g_{AD} \qquad g = (-,-,+,+,+,\ldots)$$

Representative:

$$\lambda = \Delta M_{-1,0} \qquad \mathfrak{h}_\lambda = \mathrm{stab}(M_{-1,0}) = \mathfrak{so}(d) \times \mathfrak{so}(2)$$

Dilatations generator

Orbit: $$\frac{SO(d,2)}{SO(d) \times SO(2)}$$

coadjoint orbit geometric action = $\mathcal{F}_1$ cost function

$$S_\lambda = \int d\sigma \langle M_{-1,0}, \Theta(\sigma) \rangle = \int d\sigma \mathcal{F}_1$$

✓

coadjoint orbit metric = Fubini-Study metric

Back to the Infinite Dimensional Hilbert Space

- Recast our previous CFT calculations in the language of coadjoint orbit
- For this - need the formalism of coherent states.

Heisenberg algebra: $\{a, a^\dagger, \mathbb{I}\}$ $\quad [a, a^\dagger] = \mathbb{I}$

$$\mathbb{I}\,|0\rangle = |0\rangle$$
$$a\,|0\rangle = 0$$

$$|\alpha\rangle = D(\alpha)\,|0\rangle \qquad D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}$$

Exponentiating $\mathbb{I}$ gives an overall phase

Coherent states: geometric generalization

Symmetry group G , Lie algebra $\mathfrak{g}$

$|\psi_R\rangle$ "base state"

$$H|\psi_R\rangle = e^{i\chi}|\psi_R\rangle \qquad H \subset G$$

$$h|\psi_R\rangle = \chi|\psi_R\rangle \qquad h \in \mathfrak{h}$$

$$|u\rangle = U|\psi_R\rangle \qquad U \in G/H$$

Coherent states are in one-to-one correspondence to points on a coadjoint orbit, where the choice of orbit is set by the base state

Perelmov (1972); Gilmore (1972); Yaffe (1982)

Coherent states: conformal group

$G = SO(d,2)$, primary state $|\Delta\rangle$ base state

	$\underline{d=2}$	$\underline{d>2}$
Subalgebra $\mathfrak{h}$:	$L_0 \, |h\rangle = h \, |h\rangle$	$D \, |\Delta\rangle = \Delta \, |\Delta\rangle$ $L_{\mu\nu} \, |\Delta\rangle = 0$
Raising/lowering operators:	L_1, L_{-1}	P_μ, K_μ

$$U \text{ ``}{=}\text{''} e^{\eta P + \eta^* K}$$

$$\boxed{G/H = \frac{SO(d,2)}{SO(d) \times SO(2)}}$$

Coherent states: conformal group

Instantaneous Hamiltonian

Maurer Cartan Form $$\Theta = U^{-1}dU = U^{\dagger}dU$$

Representative $$\lambda(\mathcal{O}) = i\mathrm{Tr}\left[|\psi_R\rangle\langle\psi_R|\mathcal{O}\right]$$

Coadjoint action $$\lambda(\mathcal{O}) = i\mathrm{Tr}\left[U|\psi_R\rangle\langle\psi_R|U^{-1}\mathcal{O}\right]$$

Stabilizer acts trivially $$U|\psi_R\rangle = e^{i\chi}|\psi_R\rangle$$

Coadjoint action $$S_\lambda = \int \mathcal{A}_\lambda = \int \langle\lambda, \Theta\rangle = \int \mathrm{Tr}\left[|\psi_R\rangle\langle\psi_R|U^{-1}dU|\right]$$
$$= \int \langle\psi_R|U^{-1}dU|\psi_R\rangle = \int \mathcal{F}_1$$

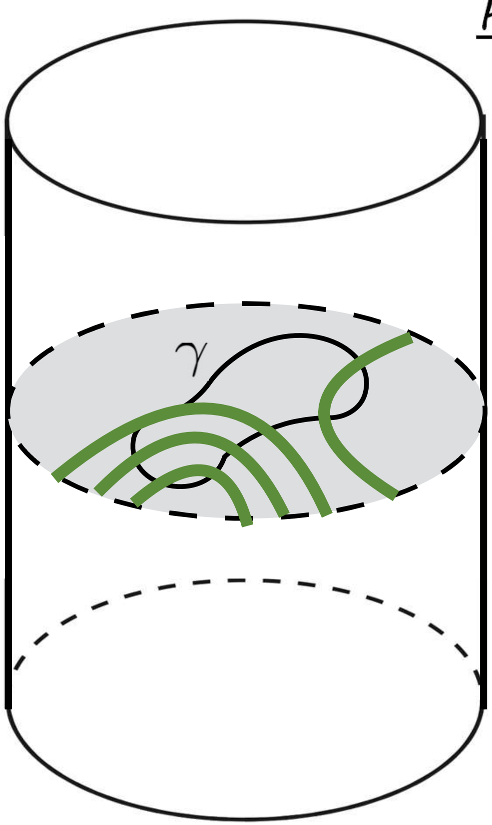
Similar logic applies for the metric $$ds^2_{FS} = ds^2_{G/H_\lambda}$$

Future Directions

Future directions

Czech, Lamprou, McCandlish, Sully (2015);
Czech, Lamprou, McCandlish, Mosk, Sully (2016);
de Boer, Haehl, Heller, Myers (2016)

1) Timelike version of kinematic space



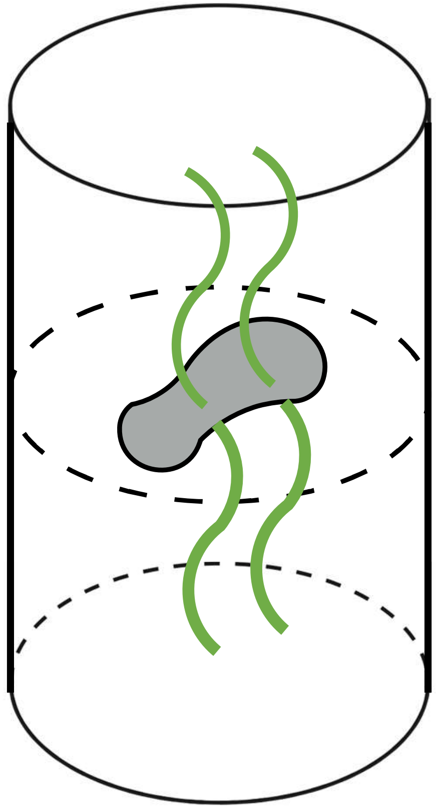
Kinematic space = space of spacelike geodesics in AdS

$$= \boxed{\frac{SO(d,2)}{SO(d-1,1)\times SO(1,1)}}$$

another coadjoint orbit of the conformal group!
Penna, Zukowski (2018)

Crofton formula: computes lengths in AdS in terms of an integral over spacelike geodesics that intersect a curve

Future directions



Space of timelike geodesics in AdS

$$= \boxed{\frac{SO(d,2)}{SO(d)\times SO(2)}}$$

the coadjoint orbit relevant
for complexity of CFTs!

Gibbons (1999)

Crofton-like formula for lengths/areas of
surfaces pierced by timelike geodesics?
Bulk interpretation of circuits

2) Connection to complexity=volume?

Relation to bulk symplectic form? (Belin, Lewkowycz, Sárosi)

Future directions

3) Moving outside a given conformal family.

4) A more realistic complexity metric:
add penalty factors, the true $\mathcal{F}_1$ norm

5) More general states
Mixed states - thermal, subregions?
States created by non-local insertions - the role of the OPE
Spinning representations can be considered with the same technology

6) Complexity as an order parameter signaling phase transitions.

Summary

- We computed the circuit complexity of CFTs in general dimensions along trajectories starting from a primary state. We used two different cost functions, the $\mathcal{F}_1$ and Fubini-Study metric.

- These respectively match the geometric action and metric for a coadjoint orbit of the conformal group which is also associated to the space of timelike geodesics in AdS.

- Thus the geometry of coadjoint orbits provides a unifying geometrical framework for different choices of cost functions.

- The coadjoint orbit connection generalizes for other symmetry groups, using a group theoretic generalization of coherent states.

- Many interesting future directions to explore!

Thank you!