Online Workshop on Quantum Gravity, Holography and Quantum Information
@MPI,LMU Munich, March 17-19, 2021

# Page Curve from Holographic Moving Mirror and End of the World Brane

Tadashi Takayanagi

Yukawa Institute for Theoretical Physics (YITP), Kyoto Univ.

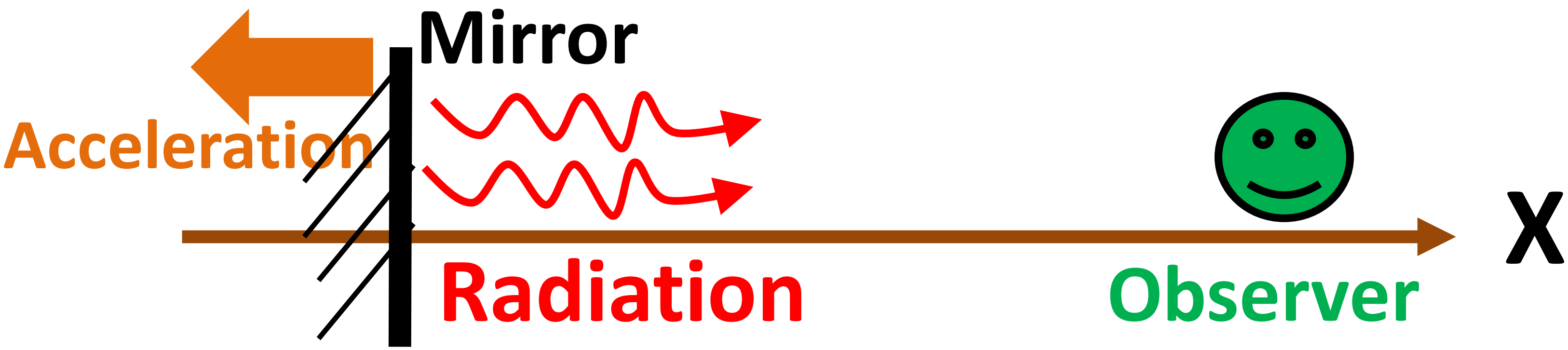
Based on

[1] arXiv: 2011.12005 [Phys. Rev. Lett. 126, 061604] + work in progress
by Ibrahim Akal, Yuya Kusuki, Noburo Shiba, Zixia Wei and TT

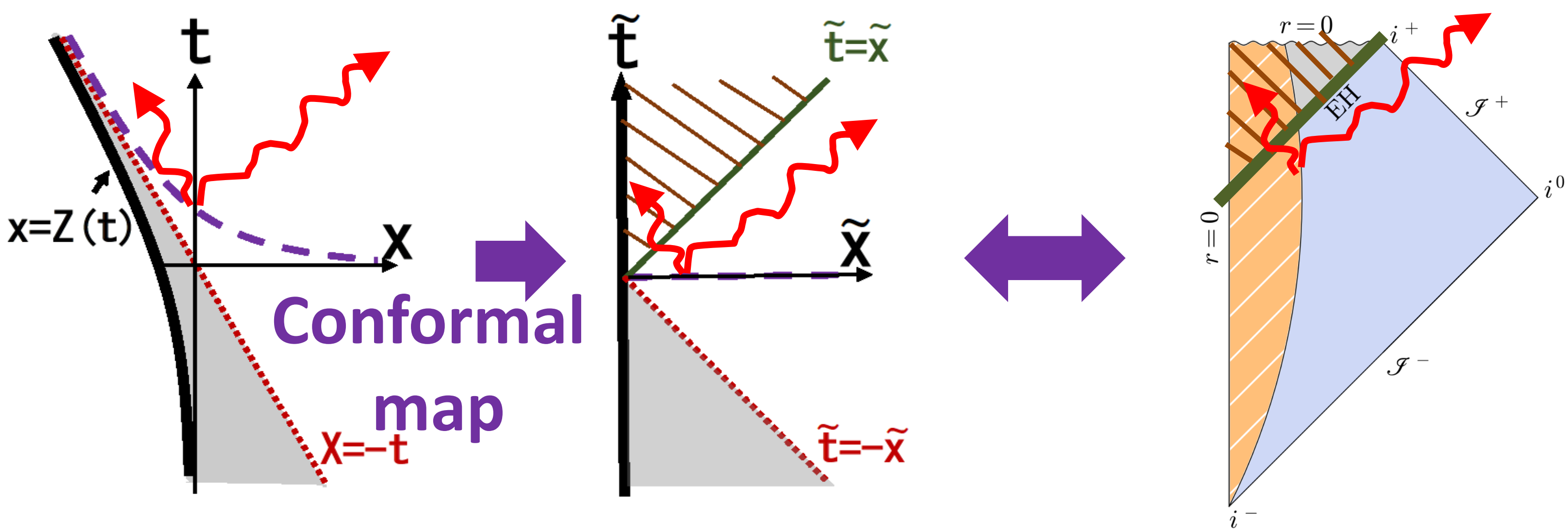
[2]arXiv: 2103.06893 by Masamichi Miyaji, Tomonori Ugajin, and TT

# ① Introduction

## Moving mirror




Moving mirrors have been known for a while as instructive models of particle creations, e.g. setups which mimic the Hawking radiation from black holes. [see e.g. Birrell-Davies text book]



Moreover, moving mirrors also provide us with interesting non-equilibrium quantum processes.

[cf. quantum quenches: Calabrese-Cardy 2005, Abajo-Arrastia-Aparicio-Lopez 2010, Balasubramanian-Bernamonti-de Boer-Copland-Craps-Keski-Vakkuri-Muller-Schafer-Shigemori-Staessens 2010, Hartman-Maldacena 2013,...]

**In this talk, we will analyze the dynamics of quantum entanglement of two dimensional CFT with moving mirrors.**

We will ask

- Do we get Page curves from moving mirrors as in BH evaporation ?

  ⇒ Yes !

- What is the holographic dual of moving mirrors ?

  ⇒ A moving mirror is dual to end of the world (EOW) brane in AdS.

- How EOW branes distribute in holographic CFTs ?

  ⇒ They look `maximally chaotic' !

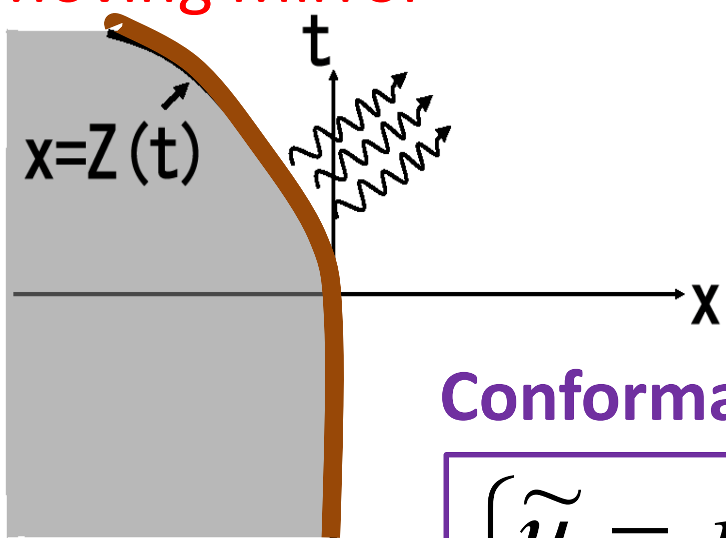
*Often used in recent discussion of BH info.
*Basic object in AdS/BCFT

# Contents

# ② BCFT Description of Moving Mirror

In this talk we focus on two dim. CFTs. Then we can apply conformal mapping to solve the moving mirror problem. We write a mirror trajectory as x=Z(t).

Moving Mirror



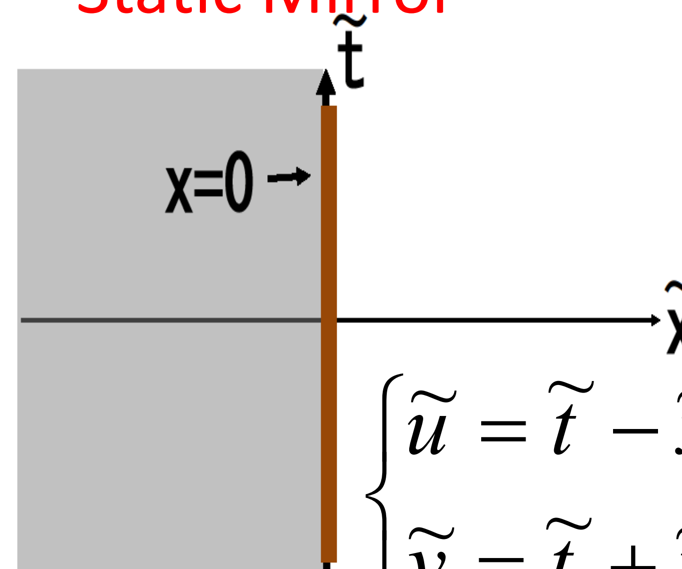

$$\begin{cases} u = t - x \\ v = t + x \end{cases}$$

Conformal Map

$$\begin{cases} \tilde{u} = p(u) \\ \tilde{v} = v \end{cases}$$

In coming vacuum

Static Mirror




$$\begin{cases} \tilde{u} = \tilde{t} - \tilde{x} \\ \tilde{v} = \tilde{t} + \tilde{x} \end{cases}$$

Standard Setup of BCFT

## BCFT (Boundary Conformal Field Theory)

For special choices of boundary conditions, a part of conformal symmetries are preserved.
This is called the **boundary conformal field theory (BCFT)**.

[Cardy 1984, .., McAvity-Osborn 1995, …., cf. Bachas' talk]

CFTd: SO(2,d)

$\cup$

BCFTd : SO(2,d-1)


**α labels bdy conditions !**

**Boundary α** $\left| B_\alpha \right\rangle$

When d=2, it is called boundary states (Cardy States)

$$(L_n - \tilde{L}_{-n}) \left| B \right\rangle = 0$$

# Example 1 : Constant Radiation from Moving Mirror

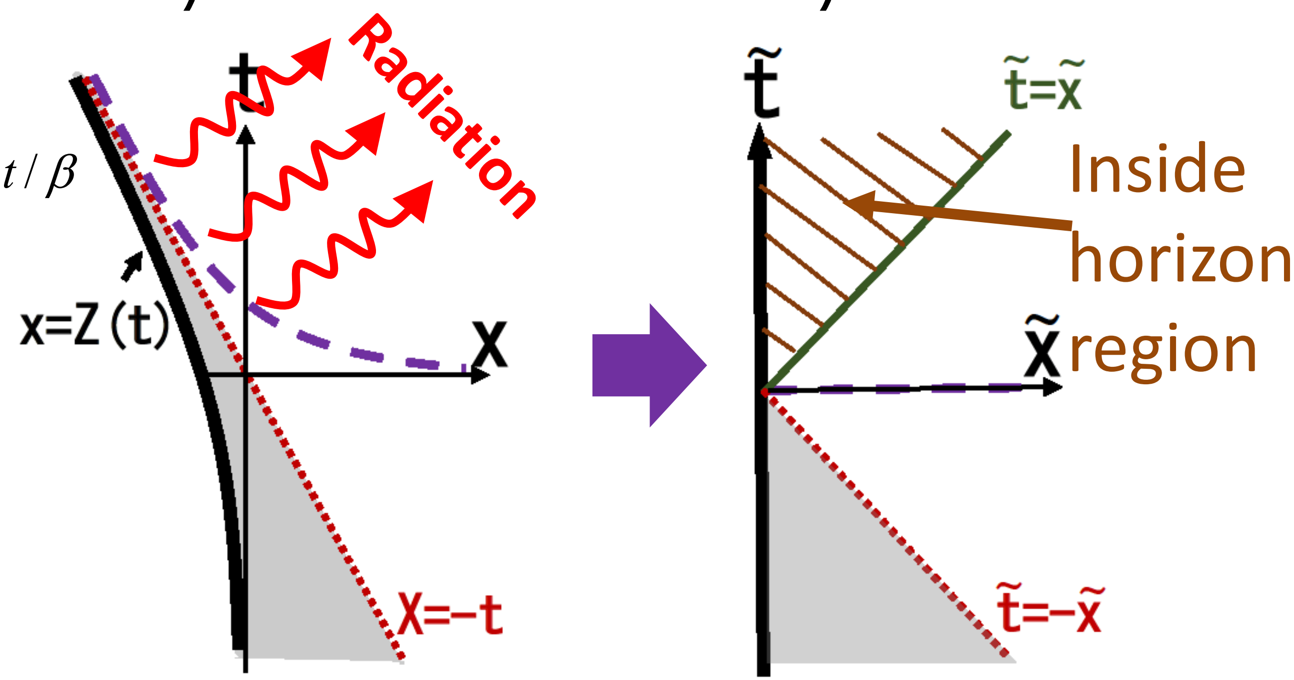
$$p(u) = -\beta \log(1 + e^{-u/\beta})$$

**Thermal flux at temperature T=1/β**

$$\text{Energy flux}: \; T_{uu} = \frac{c}{24\pi}\left(\frac{3}{2}\frac{(P'')^2}{(P')^2} - \frac{p'''}{p'}\right)$$

$$= \frac{c}{48\pi\beta^2}\left(1 - \frac{1}{\left(1+e^{u/\beta}\right)^2}\right) \approx \frac{c}{48\pi\beta^2}$$

$$z(t) \underset{t\to\infty}{\approx} -t - \beta\, e^{-2t/\beta}$$



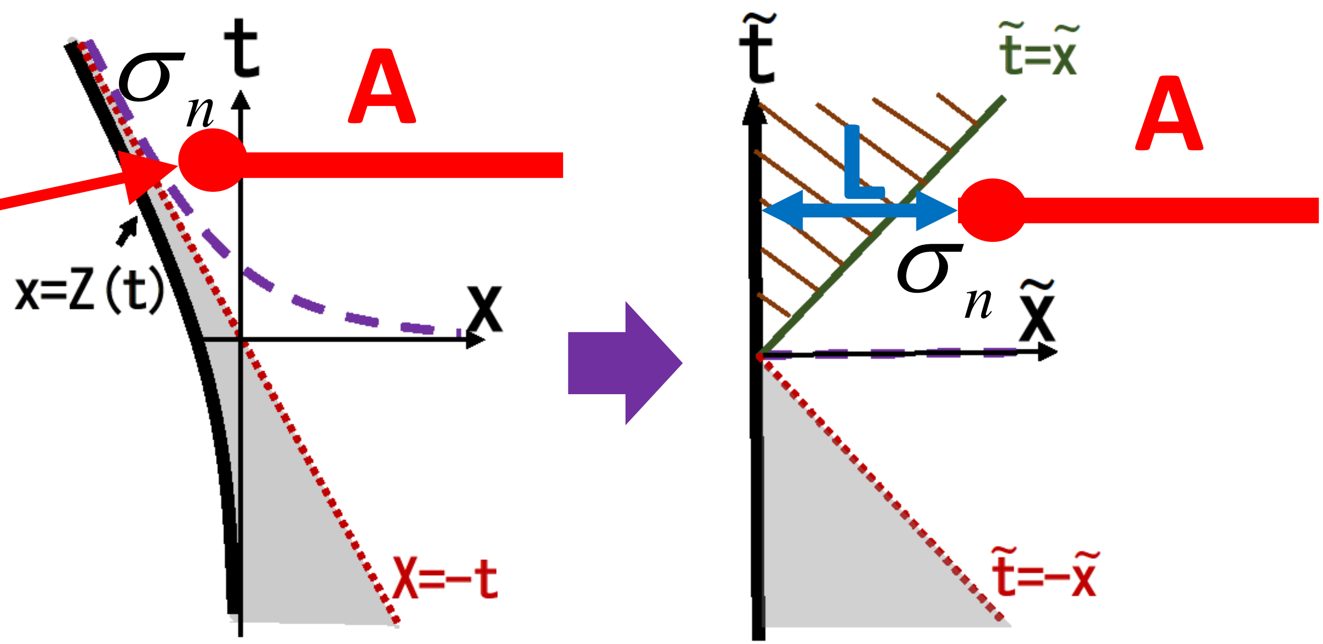
# ③ Moving Mirror and Entanglement Entropy

## Calculation of Entanglement Entropy (EE)

To get a universal result, we choose the subsystem A to be a semi-infinite line A=[x0,∞] at time t.

We consider the EE between A and its compliment.

**A twist operator Insertion at (x0,t)**



We can calculate the EE using the replica method.

$$\langle \sigma_n \rangle = \frac{g}{L^{\Delta_n}}, \qquad \Delta_n = \frac{c}{12}(n - 1/n).$$

where $g = e^{S_{bdy}}$ is the g-function or boundary entropy.
[Affleck-Ludwig 1991]

By applying the conformal transformation, we obtain
(we write the UV cut off or lattice spacing as ε)

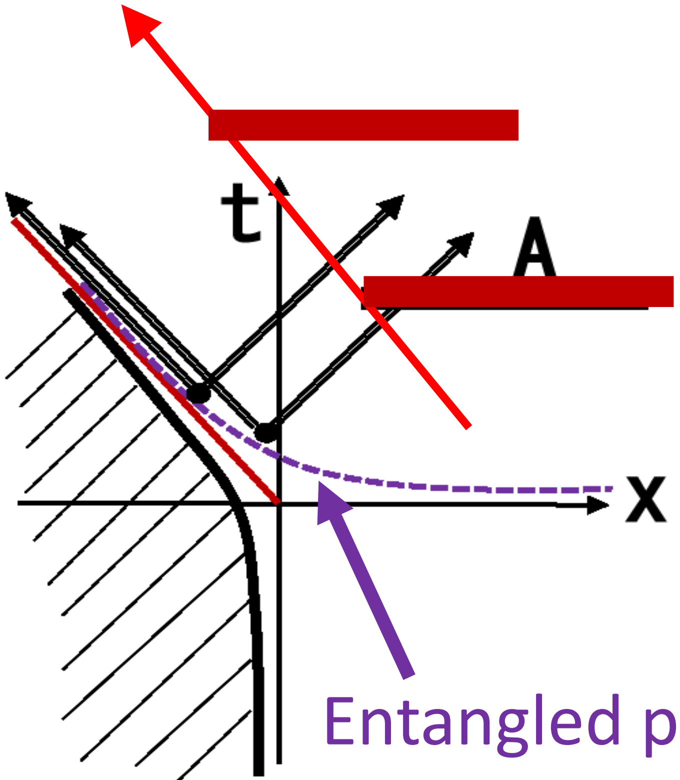
$$S_A = \frac{c}{6}\log\left[\frac{t + x_0 - p(t - x_0)}{\varepsilon\sqrt{p'(t - x_0)}}\right] + S_{bdy}$$

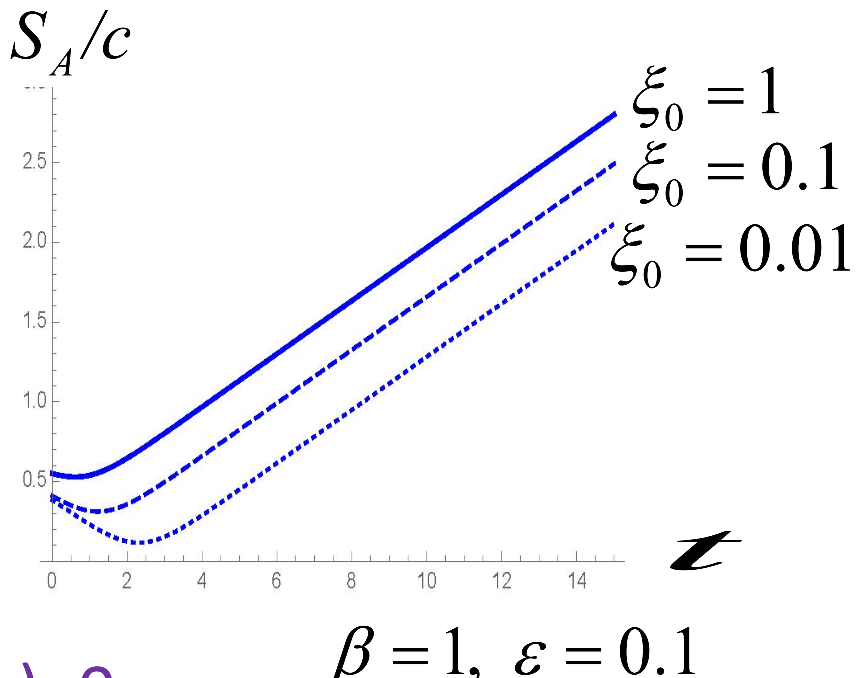
$$\underset{t\to\infty}{\approx} \frac{c}{12\beta}(t - x_0) + \frac{c}{6}\log\frac{t}{\varepsilon} + S_{bdy} \quad .$$

Note that this result is universal for any two dim. CFTs.

It is instructive to choose the time-dependent subsystem:
A=[x0(t),∞], where x0(t)=-t+ξ0 . In this case we find

$$S_A \underset{t\to\infty}{\approx} \frac{c}{6\beta} t + \frac{c}{6}\log\frac{\xi_0}{\varepsilon} + S_{bdy}.$$



# Entangled pair productions

For simplicity, consider a two dim. Massless free scalar.

$$|0\rangle_{in} \approx \exp\left(\# \int d\omega\, \underbrace{a_{\omega}^{\dagger in} a_{\omega}^{\dagger in}}_{\text{Entangled pair productions}}\right)|0\rangle_{out}.$$

**Entangled pair productions**

We can detect the location of entangled pair creations as

$$\langle 0_{in} | \phi(u_1, v_1)\phi(u_2, v_2) \int d\omega\, a_{\omega}^{\dagger \mathrm{in}} a_{\omega}^{\dagger \mathrm{in}} | 0_{in} \rangle$$

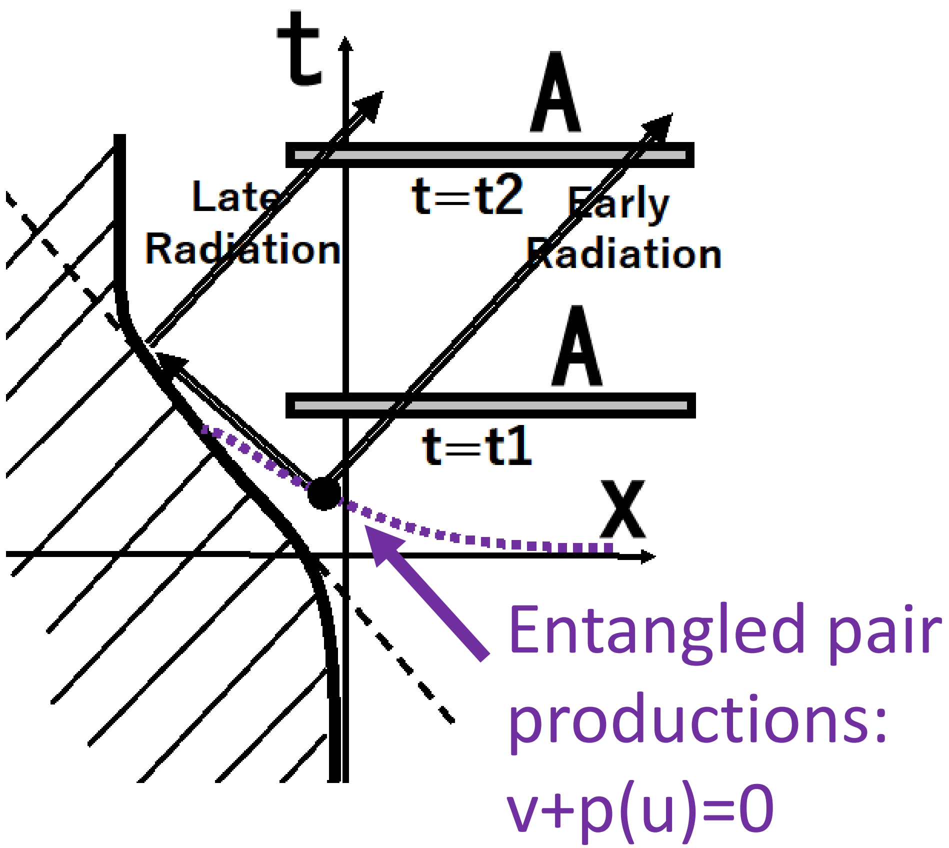
$$= \int \frac{d\omega}{\omega}\left[e^{-i\omega(v_1 + p(u_2))} + e^{-i\omega(v_2 + p(u_1))} - e^{-i\omega(v_1 + v_2)} - e^{-i\omega(p(u_1) + p(u_2))}\right].$$

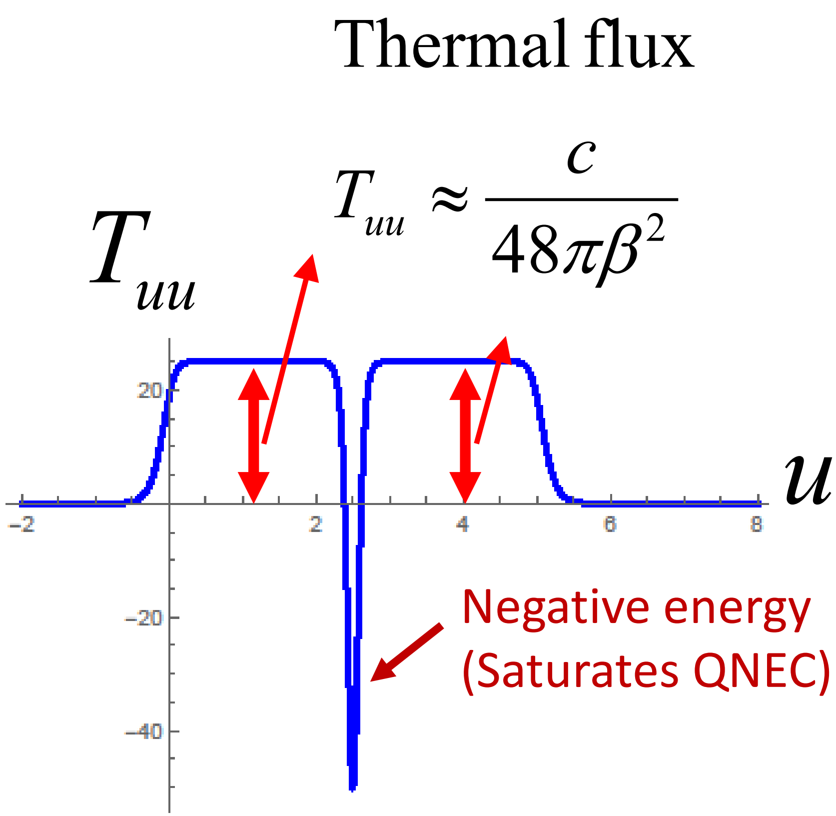
**Entangled pairs are created at v+p(u)=0.**

# Example 2: Model mimicking a BH evaporation

$$p(u) = -\beta \log(1+e^{-u/\beta}) + \beta \log(1+e^{(u-u_0)/\beta}).$$

$$\Rightarrow Z(t) \underset{t\to-\infty}{\approx} 0, \qquad Z(t) \underset{t\to\infty}{\approx} -u_0/2.$$

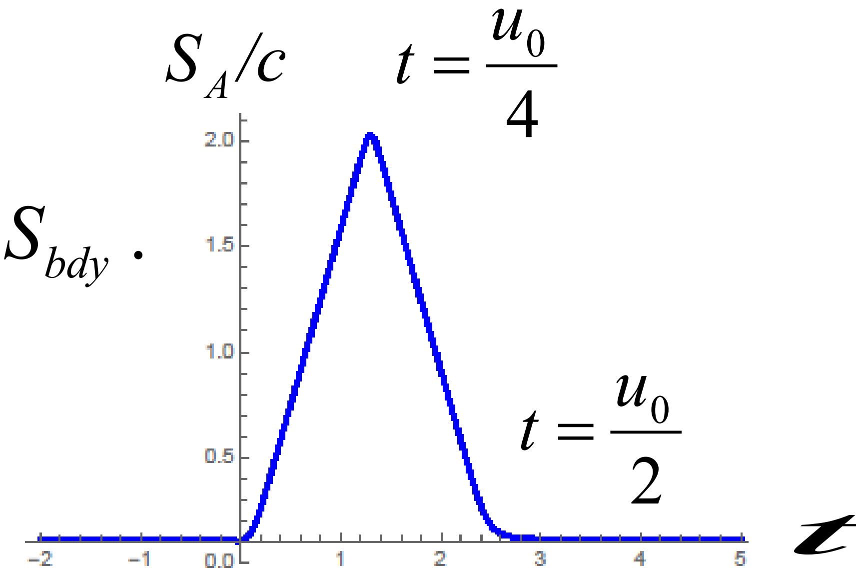


The time evolution of EE

$$S_A = \frac{c}{6}\log\left[\frac{t + x_0 - p(t - x_0)}{\varepsilon\sqrt{p'(t - x_0)}}\right] + S_{bdy}\,.$$

reproduces the perfect page curve:
(We chose A=[Z(t)+0.1,∞].)




$\beta = 1,\ \varepsilon = 0.1,\ u_0 = 5$

Though above results for the semi-infinite subsystem are universal, the EE for a finite interval A depends on CFTs.

Thus, in the next part, we will focus on holographic CFTs.
The above result can be reproduced from *the disconnected geodesic* length which is smaller than the connected one.
[cf. Earlier work : Bianchi-Smerlak 2014, Hotta-Sugita 2015
for the connected counterpart]

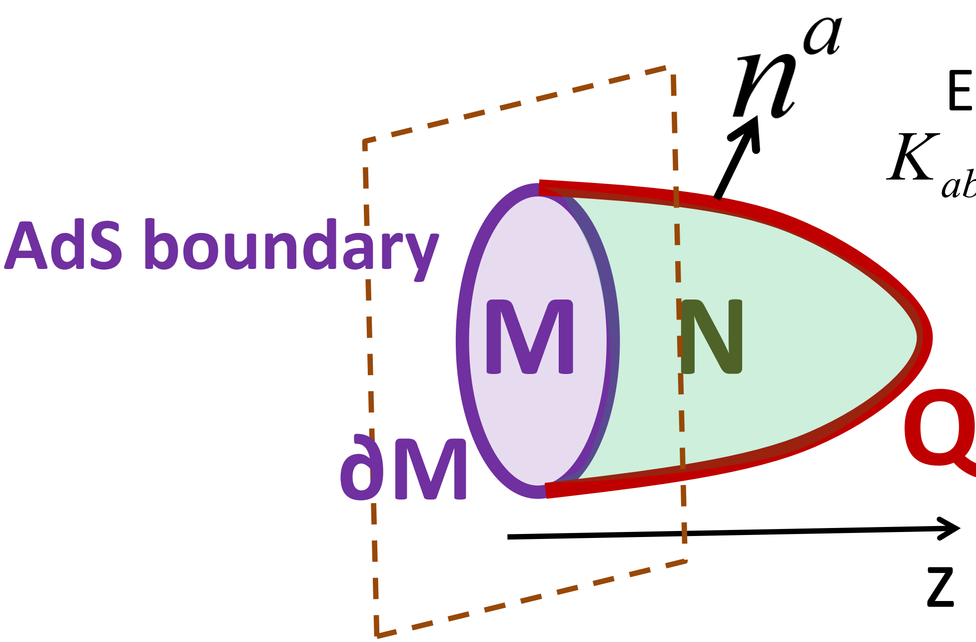
# ④ A Brief Review of AdS/BCFT

For a gravity dual of moving mirror we apply the AdS/BCFT.

## (4-1) AdS/BCFT construction

[TT 2011, Fujita-Tonni-TT 2011, see also Karch-Randall 2001,..]

| CFT on a manifold M with a boundary ∂M | = | Gravity on an asymptotically AdS space N, s.t. ∂N=M∪Q |
|---|---|---|




Extrinsic curvature

$$K_{ab} = \nabla_a n_b, \quad K = h^{ab} K_{ab}$$

←**New surface in the bulk**
(End of the world brane)
We impose *Neumann b.c.*:

$$K_{ab} - K h_{ab} - T^Q_{ab} = 0$$

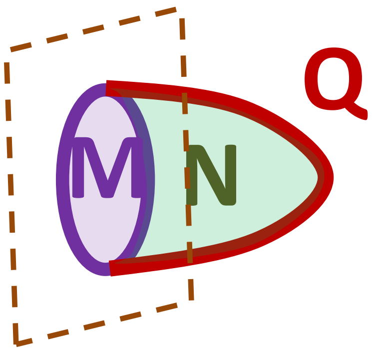
# (4-2) A Basic Example of AdSd+1/BCFTd

To preserve the boundary conformal symmetry, we choose

$$T^{Q}_{ab} \propto h_{ab} \quad \Rightarrow \quad T^{Q}_{ab} = -T\, h_{ab}$$ (T is the tension of Q).

The Neumann b.c. looks like $K_{ab} = (K - T)\, h_{ab}$

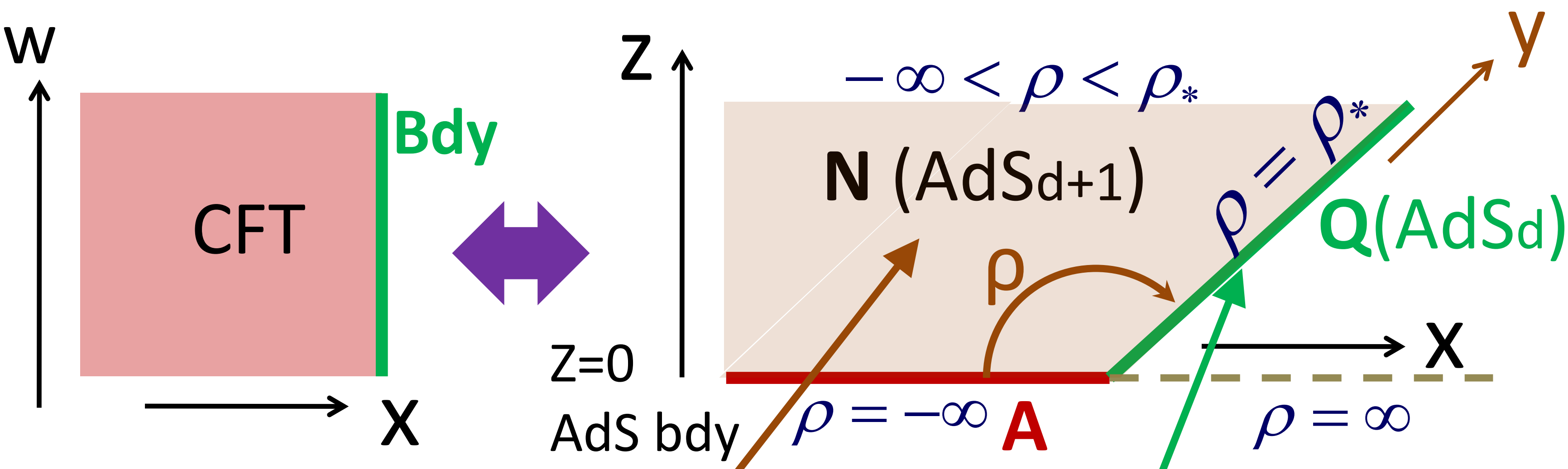
$$\Rightarrow \quad \boxed{K_{ab} = \frac{d}{d-1} T\, h_{ab}}$$




The gravity action takes the simple form

$$\boxed{I_G = \frac{1}{16\pi G_N}\int_N \sqrt{-g}\,(R - 2\Lambda) + \frac{1}{8\pi G_N}\int_Q \sqrt{-h}\,(K - T).}$$

# Construction of a gravity dual of the BCFT




$$ds^2_{M_{d+1}} = d\rho^2 + \cosh^2\left(\frac{\rho}{R}\right) ds^2_{AdS(d)}$$

SO(2,d-1) sym.

$$= R^2\left(\frac{-dt^2 + dz^2 + dx^2 + d\vec{w}^2}{z^2}\right).$$

$$ds^2_{AdS(d)} = R^2\left(\frac{-dt^2 + dy^2 + d\vec{w}^2}{y^2}\right)$$

$$K_{ab} = \frac{1}{R}\tanh\left(\frac{\rho_*}{R}\right) h_{ab},$$

$$z = y/\cosh(\rho/R), \quad x = y\tanh(\rho/R),$$

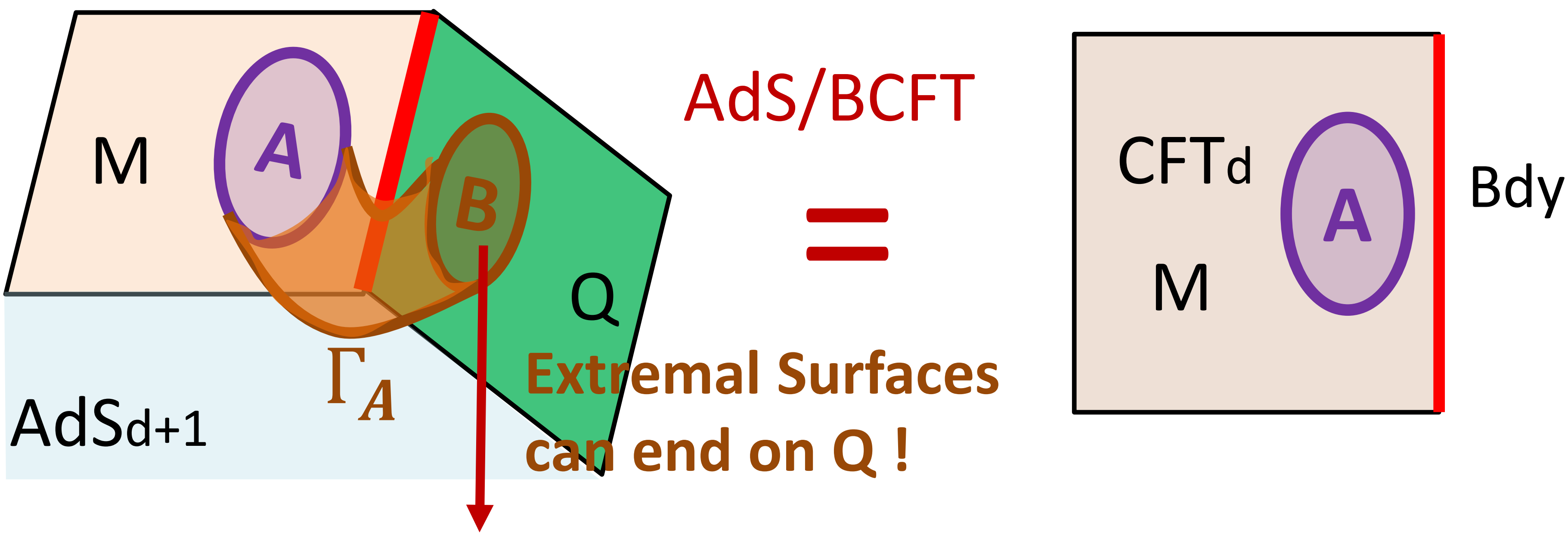
$$\Rightarrow \quad T = \frac{d-1}{R}\tanh\frac{\rho_*}{R} \quad .$$

# (4-3) Holographic Entanglement Entropy in AdS/BCFT

[TT 2011, Fujita-Tonni-TT 2011]

$$S_A = \underset{\Gamma_A, B}{\text{Min Ext}} \left[ \frac{\text{Area}(\Gamma_A)}{4G_N} \right]$$

$$\partial\Gamma_A = \partial A \cup \partial B$$



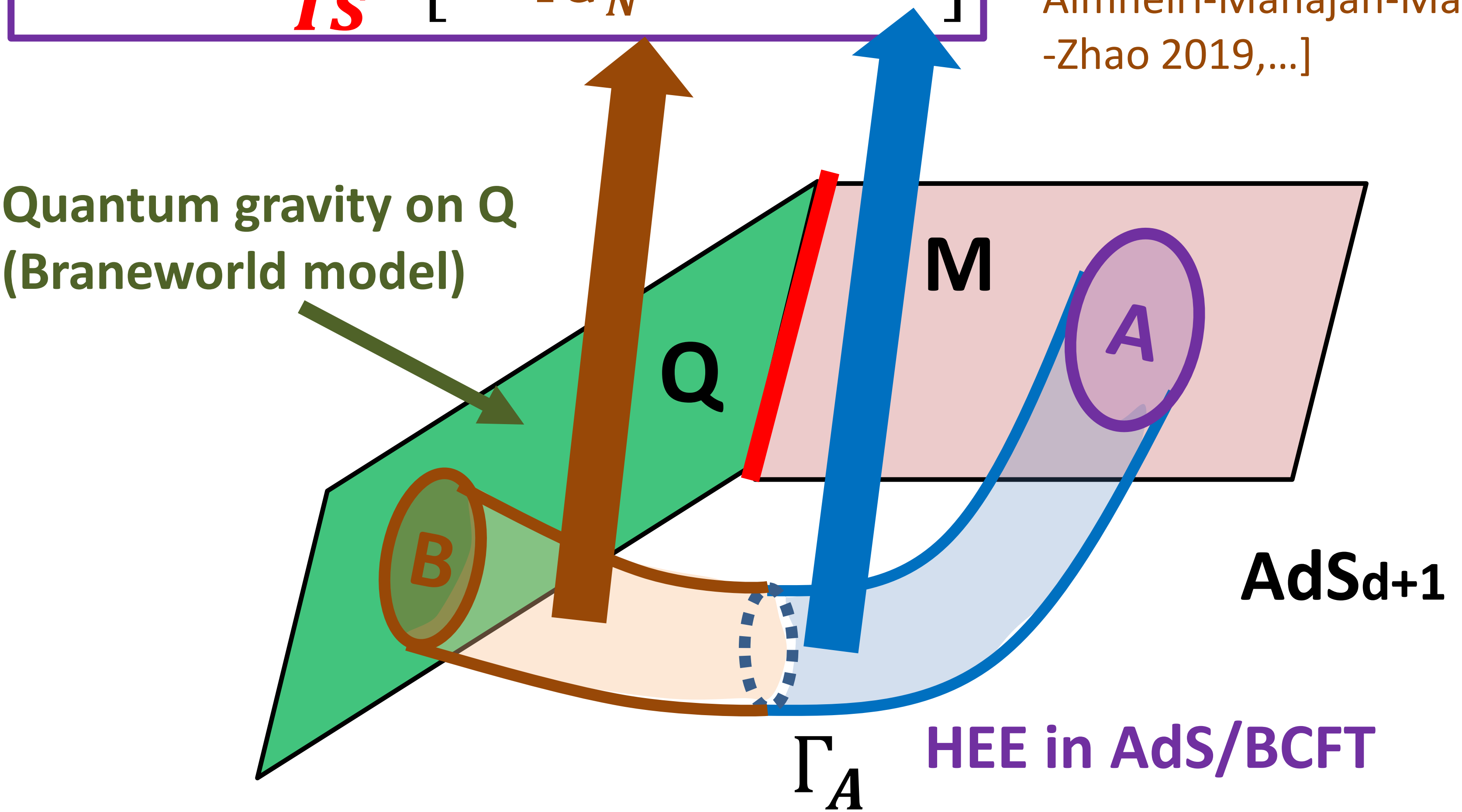

This region B is now known as an **Island** !

# Relation to Island formula

## Island Formula

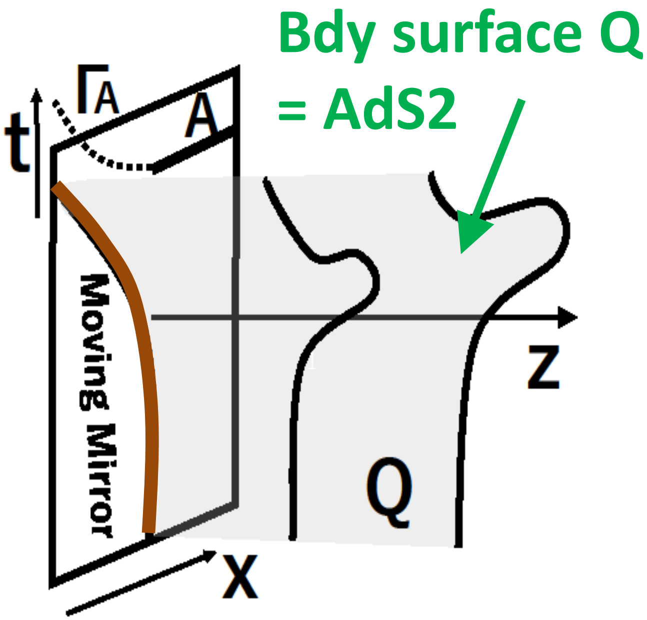
$$S_A = \underset{Is}{\text{Min Ext}}\left[\frac{Area(Is)}{4G_N} + S_{A \cup Is}\right]$$

[Penington 2019,
Almheiri-Engelhardt-Marolf
-Maxfield 2019,
Almheiri-Mahajan-Maldacena
-Zhao 2019,...]



# ⑤ Holographic Moving Mirror

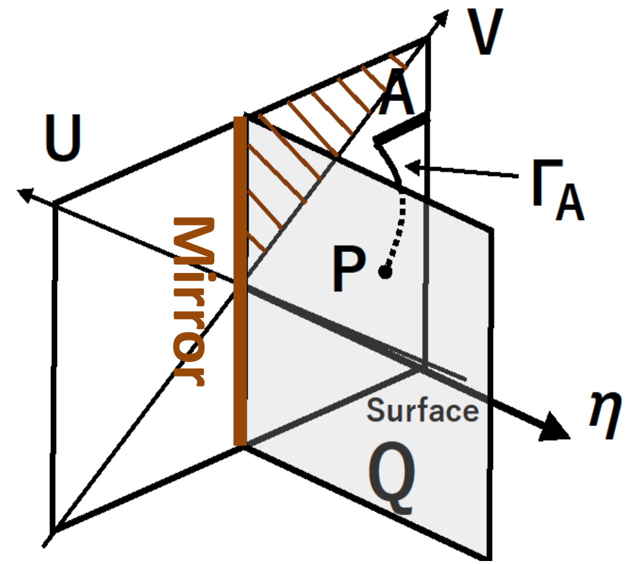
We apply AdS/BCFT to get a gravity dual of moving mirror.




$$\begin{cases} U = p(u) \\ V = v + \frac{p''(u)}{2p'(u)} z^2 \\ \eta = z\sqrt{p'(u)} \end{cases}$$

**Coordinate transformation**

[Banados 1999, Roberts 2012]




$$ds^2 = \frac{dz^2}{z^2} - \frac{dudv}{z^2} + \frac{12\pi}{c} T_{uu}(u) du^2$$

$$ds^2 = \frac{d\eta^2 - dUdV}{\eta^2}$$

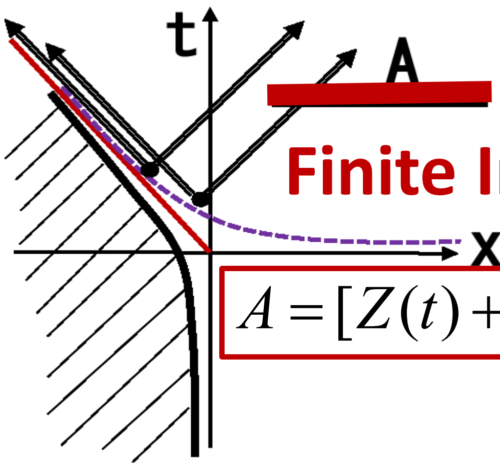
**Standard AdS/BCFT setup for BCFT on a half plane**

# Example 1: Constant Radiation from Moving Mirror

$$p(u) = -\beta \log(1 + e^{-u/\beta})$$

The HEE can be computed as

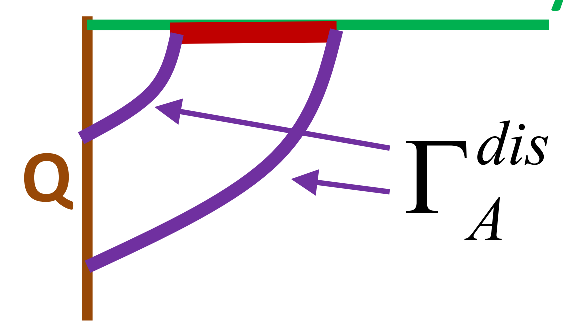
$$S_A = Min\left\{ \frac{L(\Gamma_A^{con})}{4G_N}, \frac{L(\Gamma_A^{dis})}{4G_N} \right\}.$$



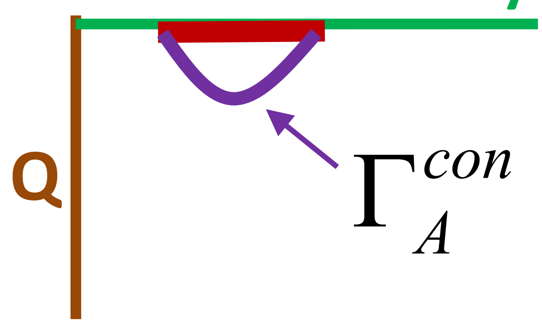

**Finite Interval**

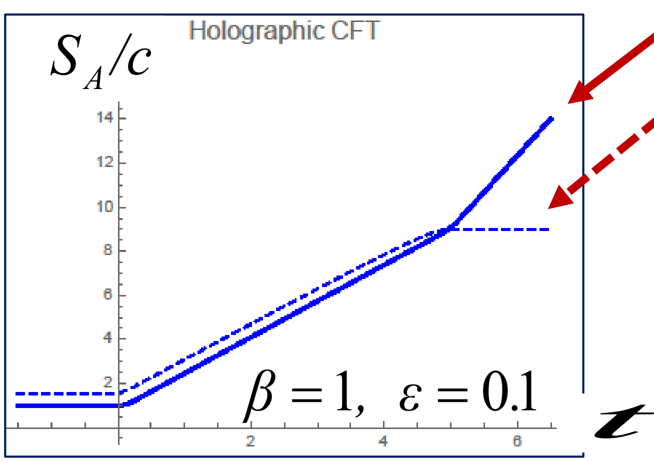
$$A = [Z(t) + 0.1, Z(t) + 10]$$

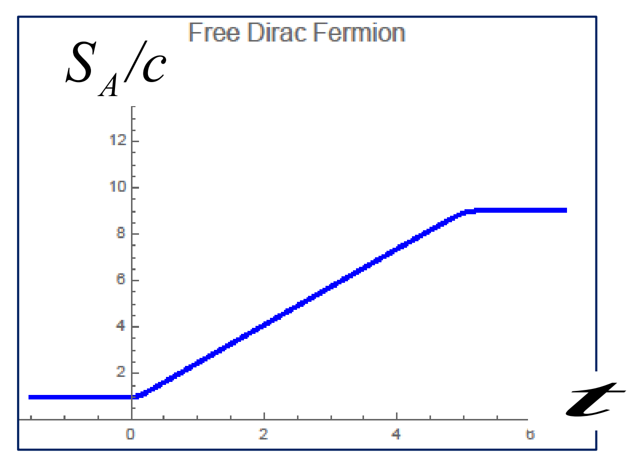
**Disconnected**



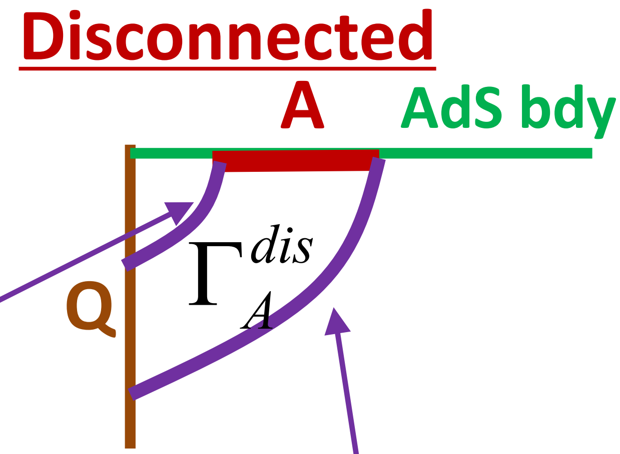

**Connected**



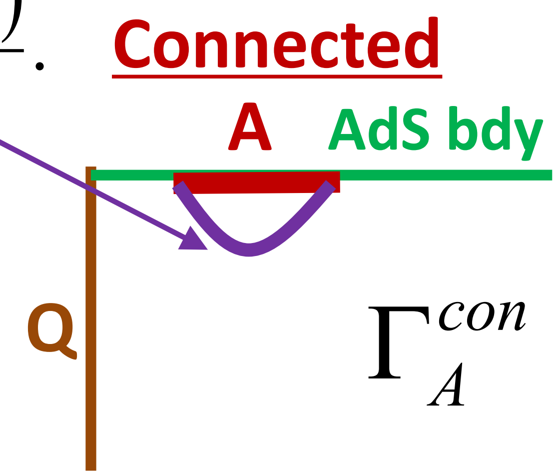
# Explicit Formula for the HEE




**A=[x0,x1] at time t**

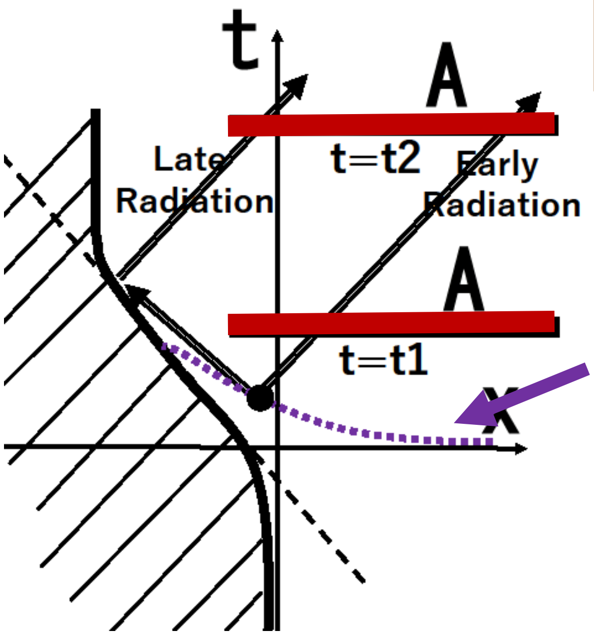
$$S_A^{dis} = \frac{c}{6}\log\left[\frac{t+x_0-p(t-x_0)}{\varepsilon\sqrt{p'(t-x_0)}}\right] + \frac{c}{6}\log\left[\frac{t+x_1-p(t-x_1)}{\varepsilon\sqrt{p'(t-x_1)}}\right] + 2S_{bdy}$$

$$S_A^{con} = \frac{c}{6}\log\frac{(x_1-x_0)\big(p(t-x_0)-p(t-x_1)\big)}{\varepsilon^2\sqrt{p'(t-x_0)p'(t-x_1)}}.$$



# Example 2: Model mimicking a BH evaporation

$$p(u) = -\beta \log(1+e^{-u/\beta}) + \beta \log(1+e^{(u-u_0)/\beta})$$




Entangled pair productions: v+p(u)=0

**A=Semi infinite interval**

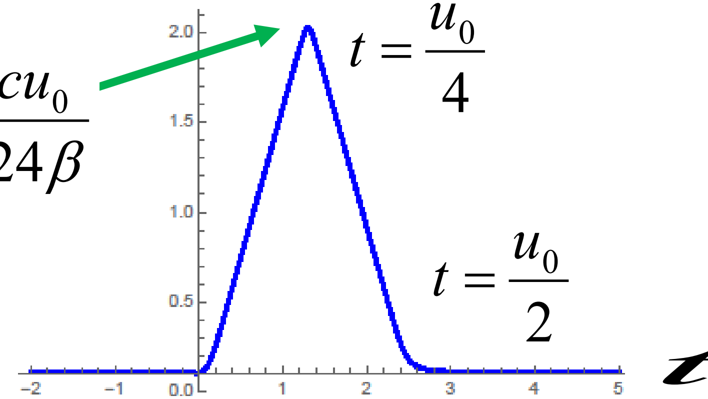
$$A = [Z(t)+0.1, \infty]$$

$S_A/c$

$$S_A = \frac{c u_0}{24\beta}$$

$$t = \frac{u_0}{4}$$

$$t = \frac{u_0}{2}$$

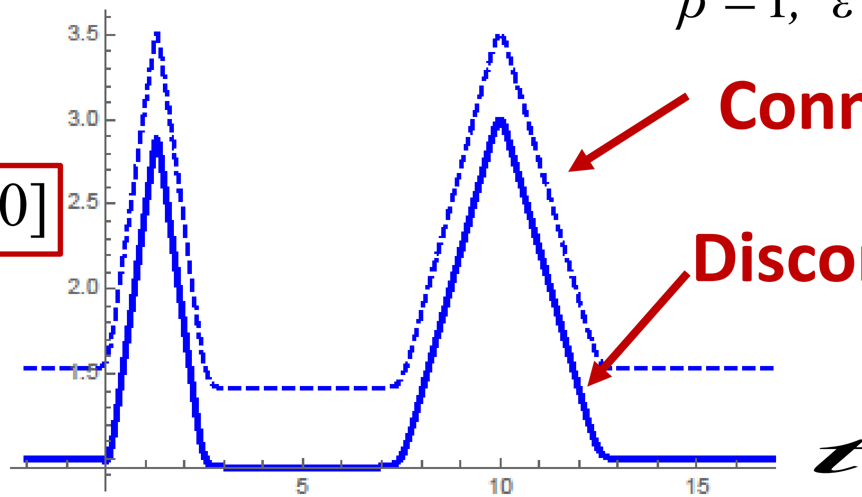



$$\beta = 1, \ \ \varepsilon = 0.1, \ \ u_0 = 5$$

**A=Finite Interval**

$$A = [Z(t)+0.1, Z(t)+10]$$

$S_A/c$

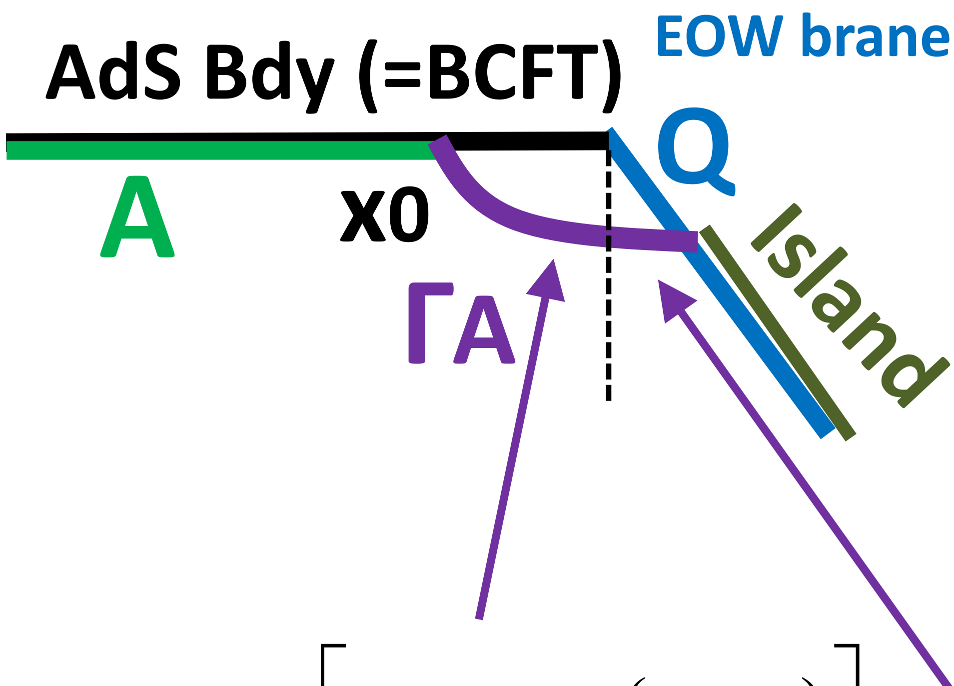



**Connected** $\Gamma_A^{con}$

**Disconnected** $\Gamma_A^{dis}$

# Brane-world and Island for Moving Mirror

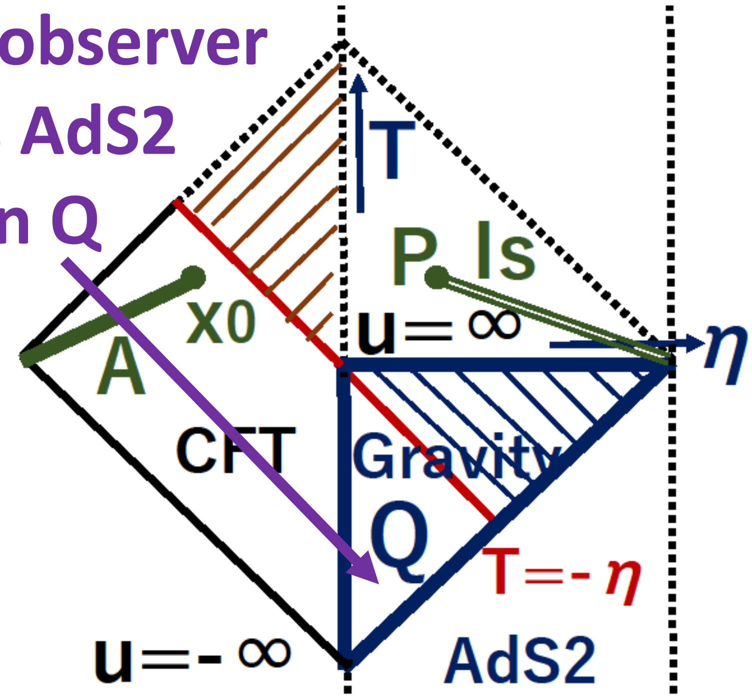
Let us examine the spacetime structure of holographic moving mirror.




$$S_A = \frac{c}{6}\log\left[\frac{t+x_0-p(t-x_0)}{\varepsilon\sqrt{p'(t-x_0)}}\right]+S_{bdy}.$$

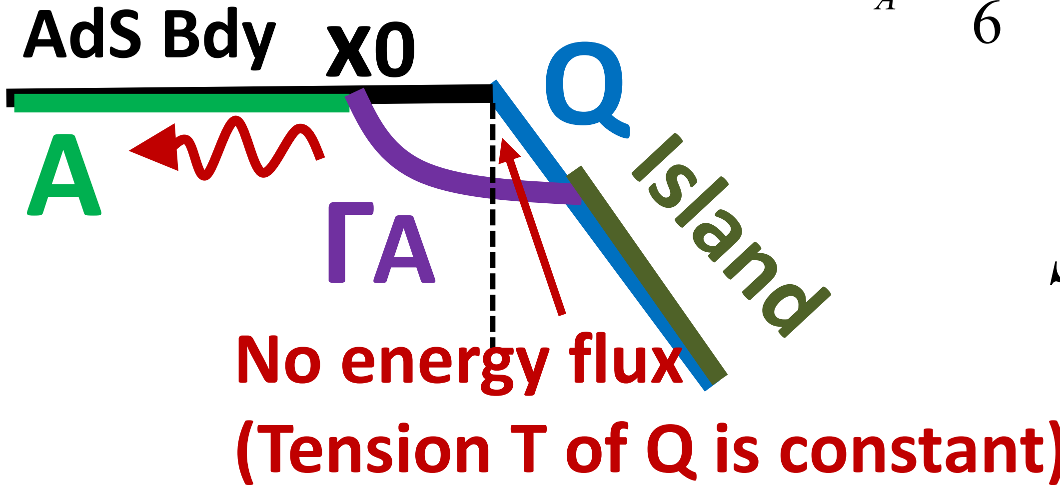
## Causal Structure of Ex.1

CFT observer sees AdS2 BH in Q




When A=semi-infinite,
**an Island always exists !**
The island is in horizon.

# Holographic Moving Mirror




$$S_A = \frac{c}{6}\log\left[\frac{t+x_0-p(t-x_0)}{\epsilon\sqrt{p'(t-x_0)}}\right] + S_{bdy}.$$

$S_{A\cup Is}$ $\qquad$ $\frac{1}{4G_N^{(2)}}$

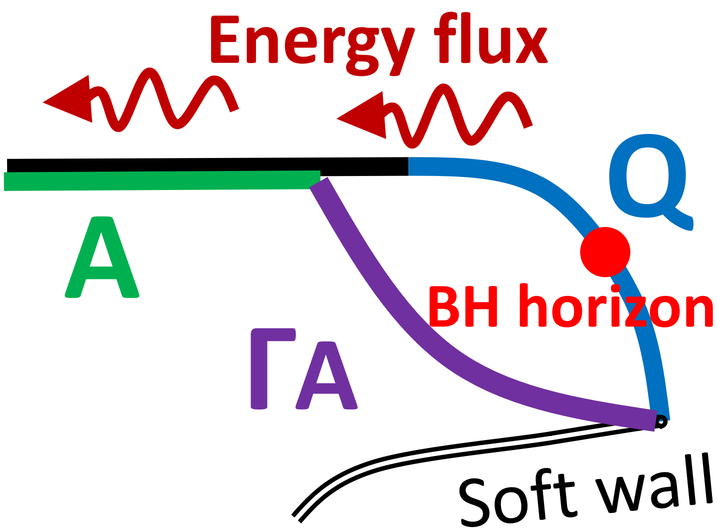
AdS2 Entropy

(time-independent)

Deformation

# Braneworld Derivation of Page Curve

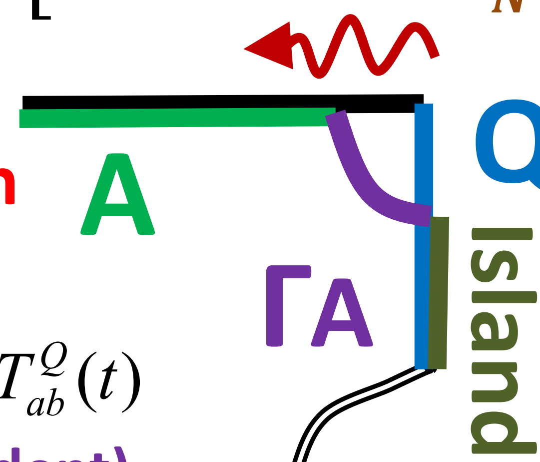
$$S_A = \text{Min Ext}\left[S_{A\cup Is} + \frac{Area(Is)}{4G_N}\right]$$




Time evolution

$$K_{ab} - Kh_{ab} = T^Q_{ab}(t)$$

(time-dependent)



# ⑥ Spectrum of End of the World Branes

As in our moving mirror example, we find the holographic duality:

Boundary states (Cardy states) ⟷ End of the world-branes

**AdS/BCFT**

## (6-1) Boundary states and Open-Closed Duality

**Cardy states:** $|B_a\rangle = \sum_k c_a^k |I_k\rangle$

**Ishibashi states:** $|I_k\rangle = \sum_n |n,k\rangle_L |n,k\rangle_R$

Maximally entangled state of left-right desendants for a primary k

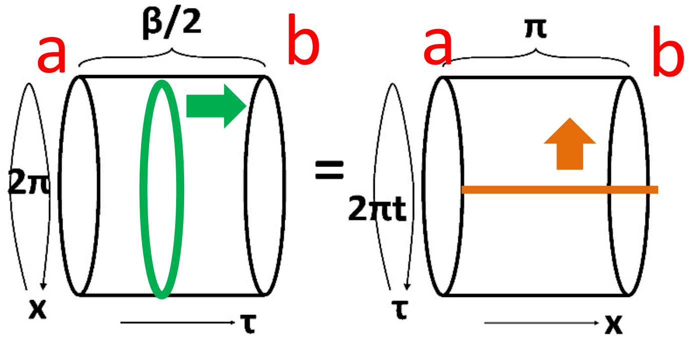
**Thermal pure state Teff=1/β:**
**(Regularized boundary state)**

$$|\Psi_a\rangle \propto e^{-\frac{\beta}{4}H}|B_a\rangle, \quad (\beta \to 0)$$

We normalize $\langle \Psi_a | \Psi_a \rangle = 1.$ $\Rightarrow$ How does $\langle \Psi_a | \Psi_b \rangle$ behave ?

# Open-closed duality

$$\langle B_a | e^{-\frac{\beta}{2}H_C} | B_b \rangle = \sum_k N_{ab}^{(k)} \mathrm{Tr}_k \left[ e^{-2\pi t H_O} \right], \qquad \beta = \frac{2\pi}{t}.$$




**Lowest energy of Open string between |a> and |b>**

**β→0 (t→∞)**

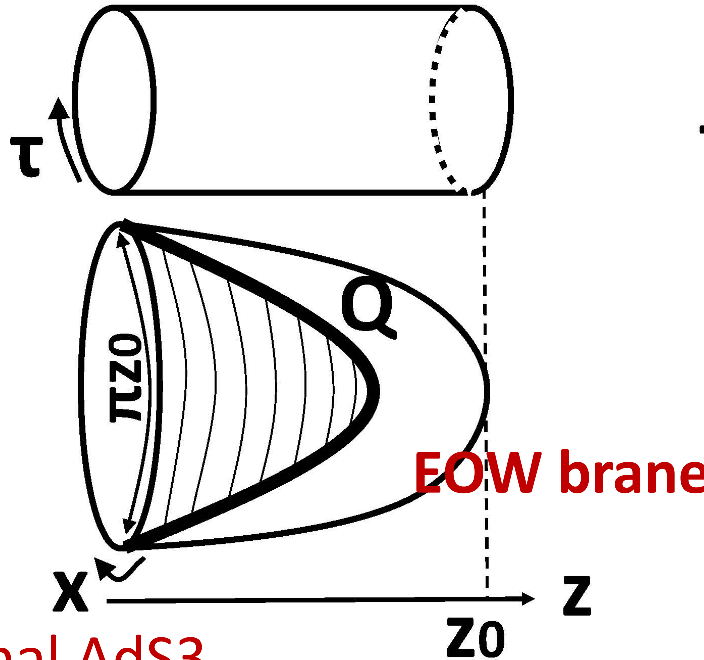
$$\begin{cases} \langle B_a | e^{-\frac{\beta}{2}H_C} | B_a \rangle \approx e^{\frac{\pi c t}{12}} \\ \langle B_a | e^{-\frac{\beta}{2}H_C} | B_b \rangle \approx N_{ab}^{k\,\min} \cdot e^{-2\pi t \left( h_{ab}^{\min} - \frac{c}{24} \right)} \end{cases}$$

$$\langle \Psi_a | \Psi_b \rangle \approx \delta_{ab} + N_{ab}^{k\,\min} \cdot e^{-4\pi^2 \frac{h_{ab}^{\min}}{\beta}}.$$

# Holographic calculation of inner products

$$\langle B_a | e^{-\frac{\beta}{2}H_C} | B_a \rangle \qquad\qquad \langle B_a | e^{-\frac{\beta}{2}H_C} | B_b \rangle$$

**Connected Phase**

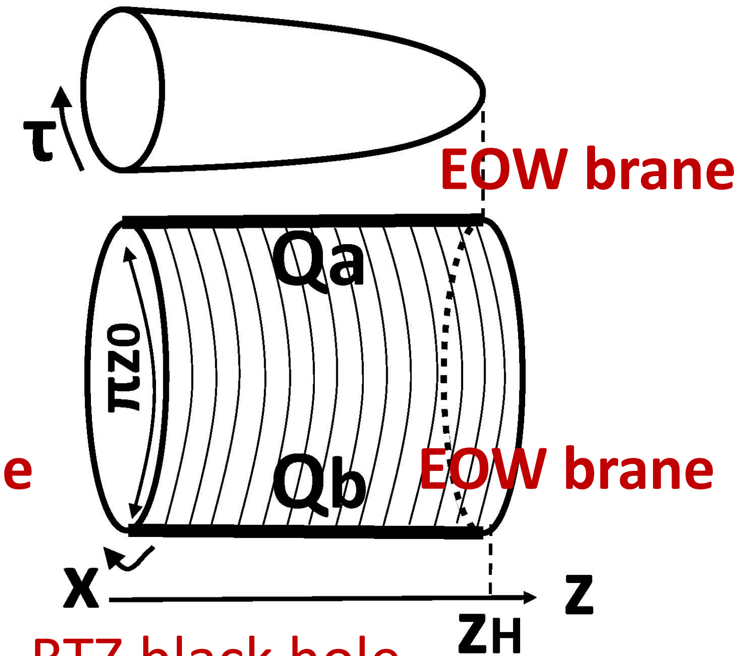



Thermal AdS3

$$ds^2 = \frac{d\tau^2}{z^2} + \frac{dz^2}{h(z)z^2} + \frac{h(z)dx^2}{z^2},$$

$$h(z) = 1 - (z/z_0)^2.$$

**Disconnected Phase**




BTZ black hole

$$ds^2 = \frac{g(z)d\tau^2}{z^2} + \frac{dz^2}{g(z)z^2} + \frac{dx^2}{z^2},$$

$$g(z) = 1 - (z/z_H)^2.$$

This AdS/BCFT gravity dual analysis leads to

$$\langle B_a | e^{-\frac{\beta}{2}H_C} | B_a \rangle \approx e^{\frac{\pi^2 c}{6\beta}}, \qquad \langle B_a | e^{-\frac{\beta}{2}H_C} | B_b \rangle \approx e^{\frac{c\beta}{12} + S^{(a)}_{bdy} + S^{(b)}_{bdy}}.$$

This shows that the minimal energy is given by

$$h^{\min}_{ab} = \frac{c}{24} \quad (\text{when } a \neq b)$$

In this case we have

**Maximally chaotic !**

Log[ # of boundary states]

$$\langle \Psi_a | \Psi_b \rangle \approx \delta_{ab} + e^{-\frac{S_{BS}}{2}}$$

, where $S_{BS} = \frac{S_{th}}{2} = \frac{\pi^2 c}{3\beta}.$

Boundary states distribute randomly in holographic CFTs.

Boundary states are limited to left-right symmetric states !

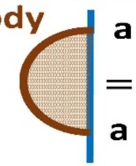
[Similar to BH microstates Pennington-Shenker-Stanford-Yang 2019, Analogous to ETH, Balasubramanian’s and Belin’s talk]

# Comparison with microstates average arguments in JT gravity

JT gravity computes averaged quantities.

[Pennington -Shenker-Stanford -Yang 2019]

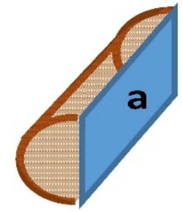
**AdS2** $i, j = 1,2, \ldots, e^{S_{th}}$

$\overline{\langle \phi_i | \phi_i \rangle} =$  $= 1$

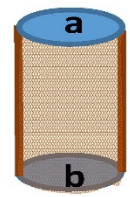


$\overline{\langle \phi_i | \phi_j \rangle} = 0$

**AdS3** $a, b = 1,2, \ldots, e^{S_{BS}}$

$\langle \psi_a | \psi_a \rangle =$  $= 1$

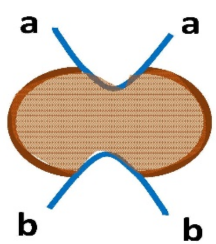


$\langle \psi_a | \psi_b \rangle =$  $= e^{-S_{BS}/2}$



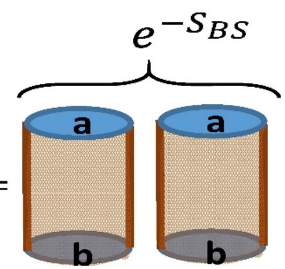
In AdS3/BCFT2, we can calculate off diagonal inner products directly from the gravity.

**AdS2**

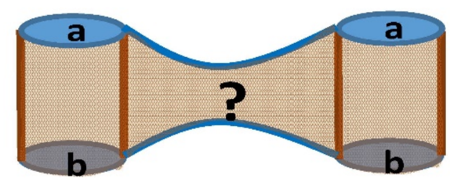
$\overline{|\langle \phi_i | \phi_j \rangle|^2} =$  $\approx e^{-S_{th}}$



**AdS3**

$\overline{|\langle \psi_a | \psi_b \rangle|^2} =$ 



$+$ 



Off diagonal inner product can be indirectly computed from replica wormholes

# ⑦ Conclusions

- Moving mirrors provide a class of non-equilibrium setups, analogous to Hawking radiations and BH evaporations.
- We computed the time evolution of EE and interpreted this in terms of entangled pair productions.
- In a moving mirror model which mimics a BH evaporation, we showed that the EE follows an ideal Page curve.
- Moving mirrors are dual to end of the world (EOW) branes.
- In holographic CFTs, we evaluate the inner products of boundary states dual to EOW branes which distributed in random manner. We found the open string gap h=c/24 and argued that this is the maximal value.

# Further directions

- Double Moving Mirrors ?    [→Our forthcoming longer paper]
- Higher Dimensional Generalizations
- Precise connection between an evaporating BH and
  a gravity dual of a moving mirror ?
- BH singularities ?
- Condensed Matter Applications ?
- Tensor Network Interpretation ?

# Thank you！

# Formulation of AdS/BCFT

The gravity action in Euclidean signature looks like

Gibbons
-Hawking term

$$I = \frac{1}{16\pi G_N}\int_N \sqrt{-g}\,(R-2\Lambda+L_{matter}) + \frac{1}{8\pi G_N}\int_Q \sqrt{-h}\,(K+L^Q_{matter}).$$

Bulk matter fields

Matter fields
localized on Q

The coordinate and induced metric of Q are $x^a$ and $h^{ab}$.

We define the extrinsic curvature and its trace

$K_{ab} = \nabla_a n_b,\quad K = h^{ab}K_{ab}$ . ($n^a$ is a unit vector normal to Q.)

e.g. Gaussian normal coordinate: $ds^2 = d\rho^2 + h_{ab}(\rho,x)dx^a dx^b$

$$\Rightarrow K_{ab} = \frac{1}{2}\partial_\rho h_{ab}(\rho,x).$$

Variation: $\delta I = \frac{1}{16\pi G_N}\int_Q \sqrt{-h}\,(K_{ab} - K h_{ab} - T^Q_{ab})\,\delta h^{ab}.$

At the AdS boundary M, we impose Dirichlet boundary condition $\delta h^{ab} = 0$ as in the standard AdS/CFT.

On the other hand, at the new boundary Q, we argue to require the Neumann b.c. :

$$K_{ab} - K h_{ab} - T^Q_{ab} = 0$$

`boundary Einstein eq.'

Why Neumann b.c. ? [closely related to brane-world models]

(1) Keep the boundary dynamical. New data at Q should not be required.

(2) Orientifolds in string theory leads to this condition.

# AdS3/BCFT2 and Boundary Entropy

## Boundary Entropy

We focus on the d=2 case (AdS3/BCFT2).


**α labels bdy conditions !** → **Boundary α**

**Boundary entropy**: A measure of the degrees of freedom at the boundary [Affleck-Ludwig 1991]

$$g = e^{S_{bdy}}$$

→ **This value depends on α !**

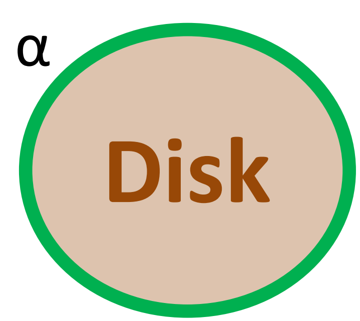
**g-theorem**: Sbdy or g decreases under the bdy RG flow.

[Proof: Friedan-Konechny 2004]

# Three Definitions of Boundary Entropy

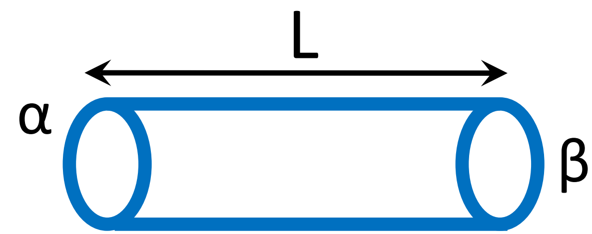
## Def 1 (Disk Amplitude)

$$S_{bdy(\alpha)} = \log g_\alpha \,, \qquad g_\alpha \equiv \langle 0 | B_\alpha \rangle .$$




## Def 2 (Cylinder Amplitude)

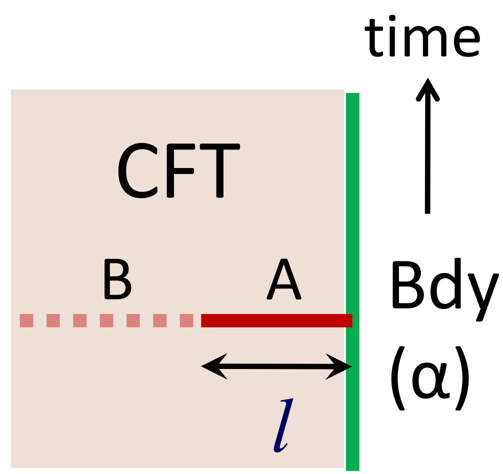
$$Z^{cylinder}_{(\alpha,\beta)} = \langle B_\alpha | e^{-HL} | B_\beta \rangle \underset{L\to\infty}{\approx} \underbrace{g_\alpha g_\beta}_{\text{Boundary Part}} \underbrace{e^{-E_0 L}}_{\text{Bulk Part}} .$$




## Def 3 (Entanglement Entropy)

In 2D BCFT, the EE behaves like

$$S_A = \underbrace{\frac{c}{6}\log\frac{l}{\varepsilon}}_{\text{Bulk Part}} + \underbrace{\log g_\alpha}_{\text{Boundary Part}} \quad .$$




[Calabrese-Cardy 2004]

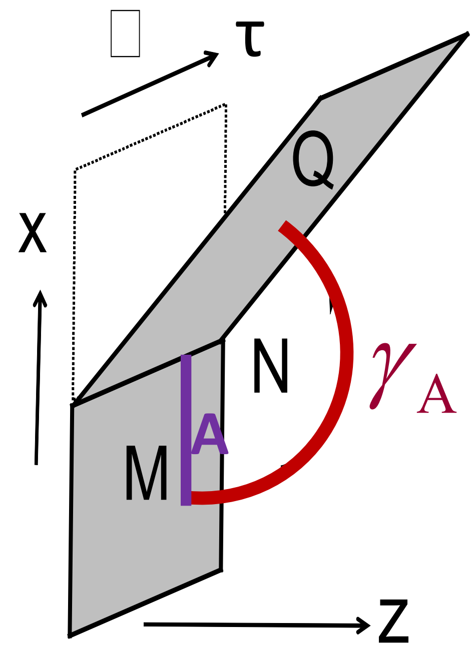
# Bdy entropy from HEE in AdS/BCFT

**New Aspect in AdS/BCFT: Extremal Surfaces end on Q !**

The holographic EE is obtained as

$$S_A = \frac{\text{Length}}{4G_N} = \frac{1}{4G_N}\int_{-\rho_\infty}^{\rho_*} d\rho$$

$$= \frac{\rho_\infty + \rho_*}{4G_N} = \frac{c}{6}\log\frac{l}{\varepsilon} + \frac{\rho_*}{4G_N} \quad .$$

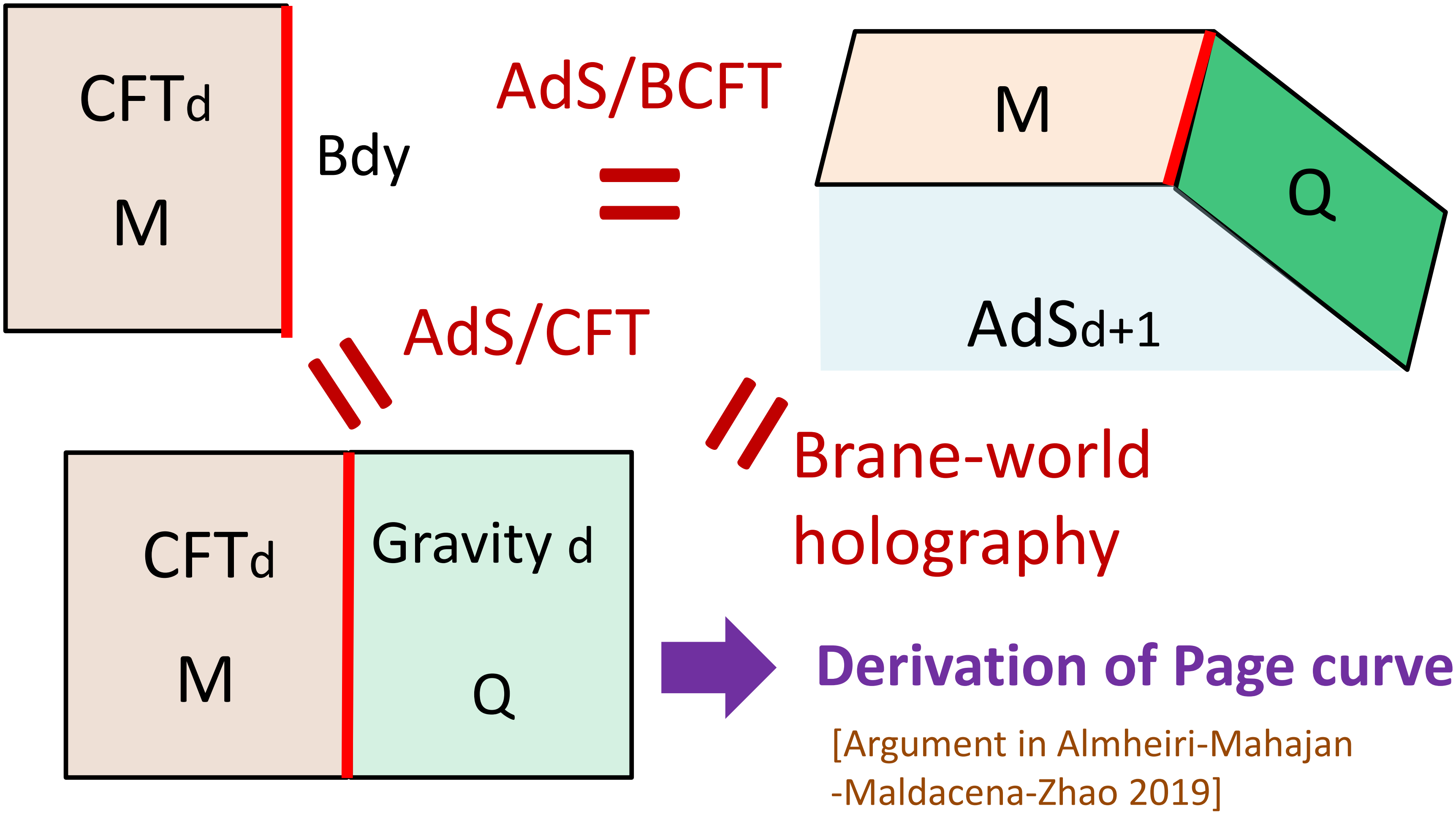



Thus we reproduced the same boundary entropy:

$$S_{bdy} = \frac{\rho_*}{4G_N}.$$

# Holographic Triality

We can apply the brane-world holography to AdS/BCFT as follows.




[More Progresses: Rozali-Sully-Raamsdonk-Waddell-Wakeham 2019, Chen-Fisher-Hernandez -Myers-Ruan 2019, Almheiri-Mahajan-Santos 2019, Chen-Myers-Neuenfeld-Reyes-Sandor 2020, Chen-Gorbenko-Maldacena 2020, Geng-Karch 2020, Balasubramanian-Kar-Ugajin 2020, ……]