Computing Scattering Amplitudes in N=4 SYM without knowing Feynman diagram

Young Scientists Workshop 2022

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Motivation

Why not Feynman diagram?

- Feynman diagram calculations are inefficient to study multi-loop scattering amplitudes especially in a theory as simple as planar N=4 super-Yang-Mills (SYM) theory.

- Gauge theories have numerous cancellations

Motivation

Why N=4 Super Yang-Mills Theory?

Fundamental Properties

- Maximally supersymmetric Yang-Mills model.
- UV finite to all orders Mandelstam (1982)

→ β -function is zero for all couplings. (Scale Invariant, Theory is conformal)

- AdS/CFT correspondence

+ Sharing Yang-Mills sector with QCD, having the same gluon diagrams.

➡ Good starting Point to study QCD

The Soft-Collinear Bootstrap

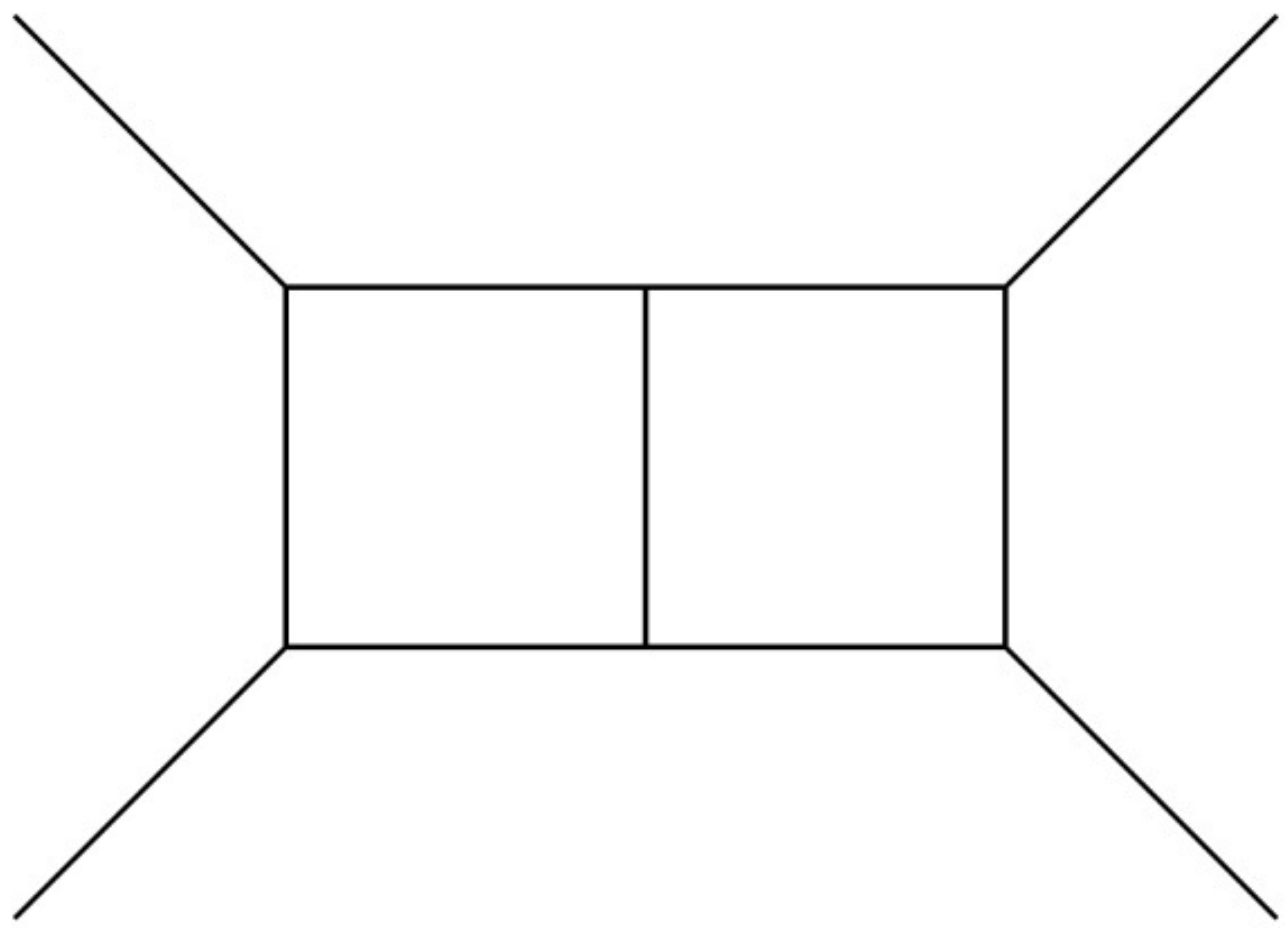
Further properties at ’t Hooft limit

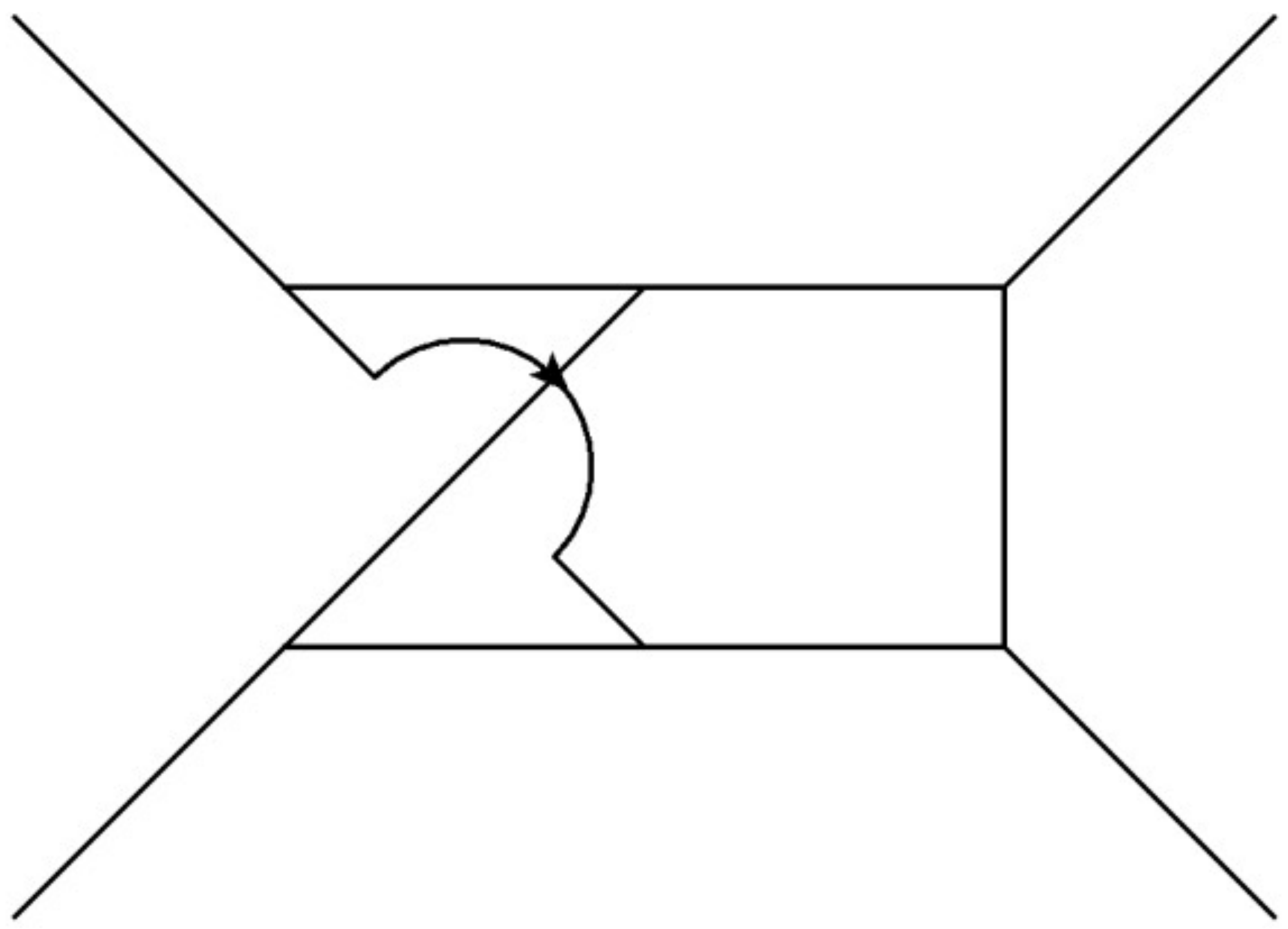
Assumptions
1. Planarity
2. Dual Conformal Invariance
3. IR property

- Using knowledge of IR behavior of the theory in addition to the planarity and dual conformal invariance, one can get the scattering amplitudes without drawing Feynman diagram.

Planarity

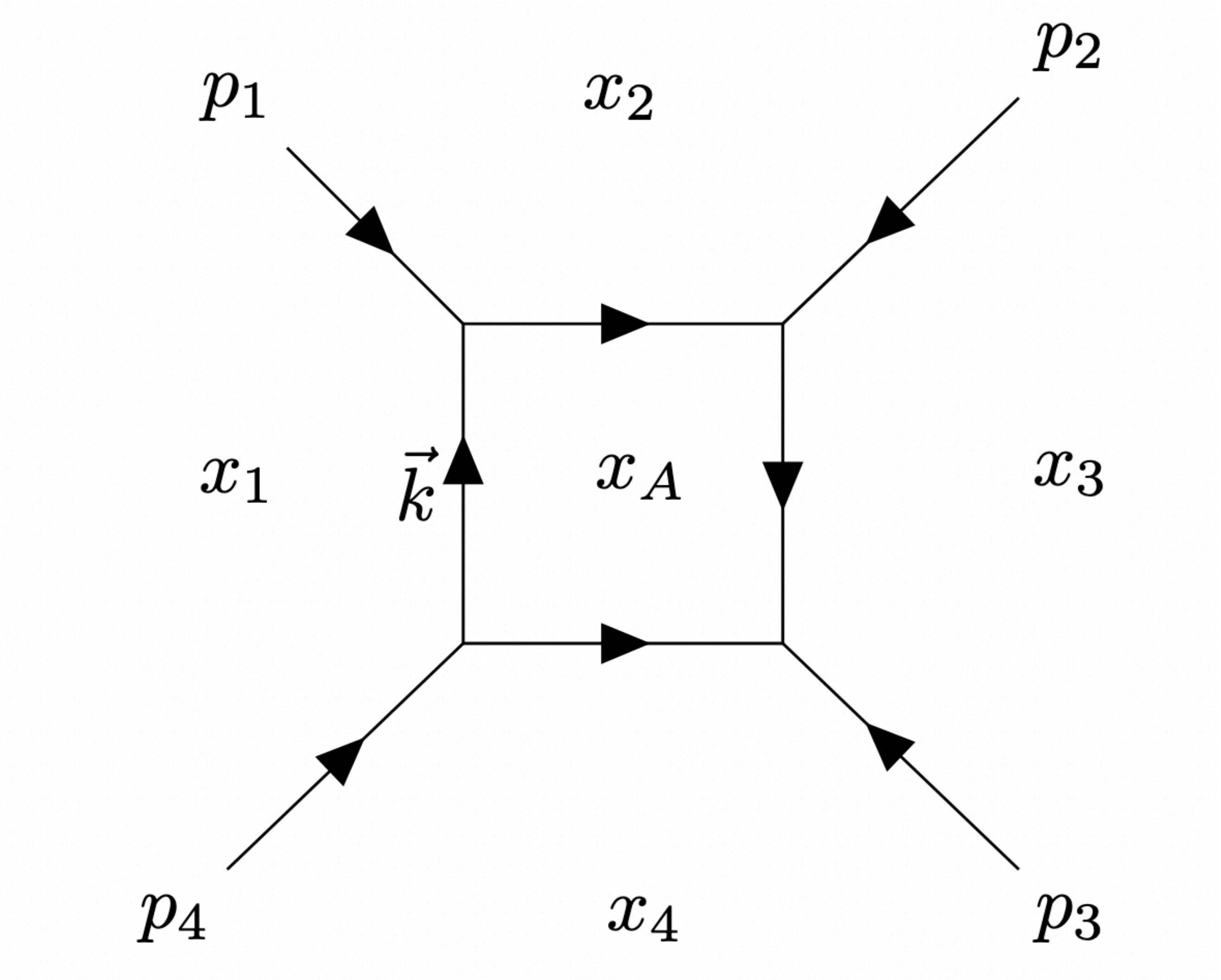
- Planar graph has no crossing edges.



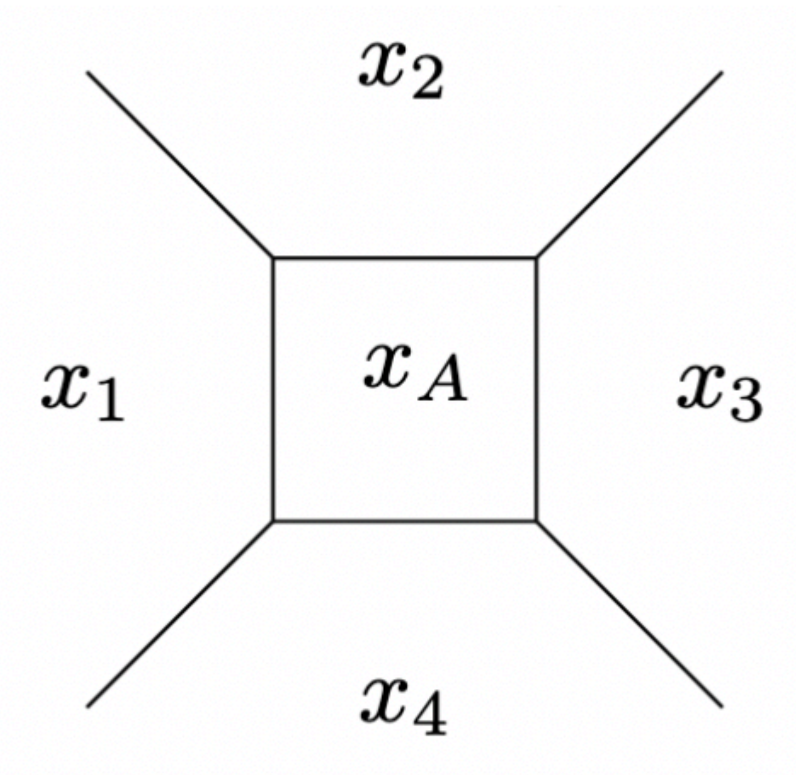


Dual Conformal Invariance

- Conformal invariance in dual space defined by $p_i^\mu = x_i^\mu - x_{i+1}^\mu$
- Conformal transformation : {Inversion, Translation, dilation, rotation}



Dual Conformal Invariance



Example - Box diagram

$x_{ij} = x_i - x_j$

- Integral in dual coordinates of box diagram $M_{\text{box}} = \int d^4x_A \frac{N}{x_{1A}^2 x_{2A}^2 x_{3A}^2 x_{4A}^2}$
- Under inversion $M_{\text{box}} \to \int \frac{d^4x_A}{(x_A^2)^4} \frac{N' x_1^2 x_2^2 x_3^2 x_4^2 (x_A^2)^4}{x_{1A}^2 x_{2A}^2 x_{3A}^2 x_{4A}^2}$ as $x_{ij}^2 \to \frac{x_{ij}^2}{x_i^2 x_j^2}$
- Setting normalization factor $N = x_{13}^2 x_{24}^2$, under inversion $N' = \frac{x_{13}^2 x_{24}^2}{x_1^2 x_2^2 x_3^2 x_4^2}$

$$M_{\text{box}} \to \int \frac{d^4x_A}{(x_A^2)^4} \frac{x_{13}^2 x_{24}^2}{x_1^2 x_2^2 x_3^2 x_4^2} \frac{x_1^2 x_2^2 x_3^2 x_4^2 (x_A^2)^4}{x_{1A}^2 x_{2A}^2 x_{3A}^2 x_{4A}^2} = M_{\text{box}}$$

$\Longrightarrow$ The Box diagram integral is invariant under dual conformal transformation

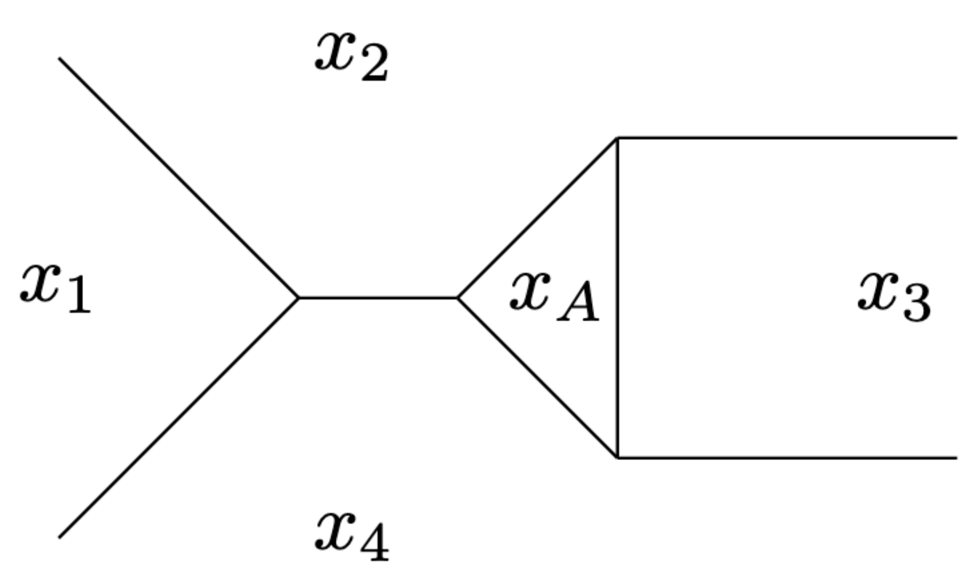
Dual Conformal Invariance

Example - Triangle diagram

- Triangle diagram $M_{\text{tri}} = \int d^4x_A \frac{N'}{x_{24}^2 x_{2A}^2 x_{3A}^2 x_{4A}^2}$

- Under inversion $M_{\text{tri}} \to \int \frac{d^4x_A}{(x_A^2)^{\cancel{4}}} \frac{N'(x_2^2)^2 x_3^2 (x_4^2)^2 \cancel{(x_A^2)}^3}{x_{24}^2 x_{2A}^2 x_{3A}^2 x_{4A}^2}$

Remaining x_A^2 in denominator



$\Rightarrow$ Triangle diagram is not dual conformal invariant

IR exponentiation

$$M_n^{(L)} = A_n^{(L)}/A_n^{(0)} \qquad D = 4 - 2\epsilon$$

$$M_n^{(L)} = \mathcal{O}(\epsilon^{-2L})$$

$$\log[1 + \lambda M_n^{(1)} + \lambda^2 M_n^{(2)} + \ldots] = \mathcal{O}(\epsilon^{-2})$$

$$(\log M_n)^{(1)} = M_n^{(1)}$$

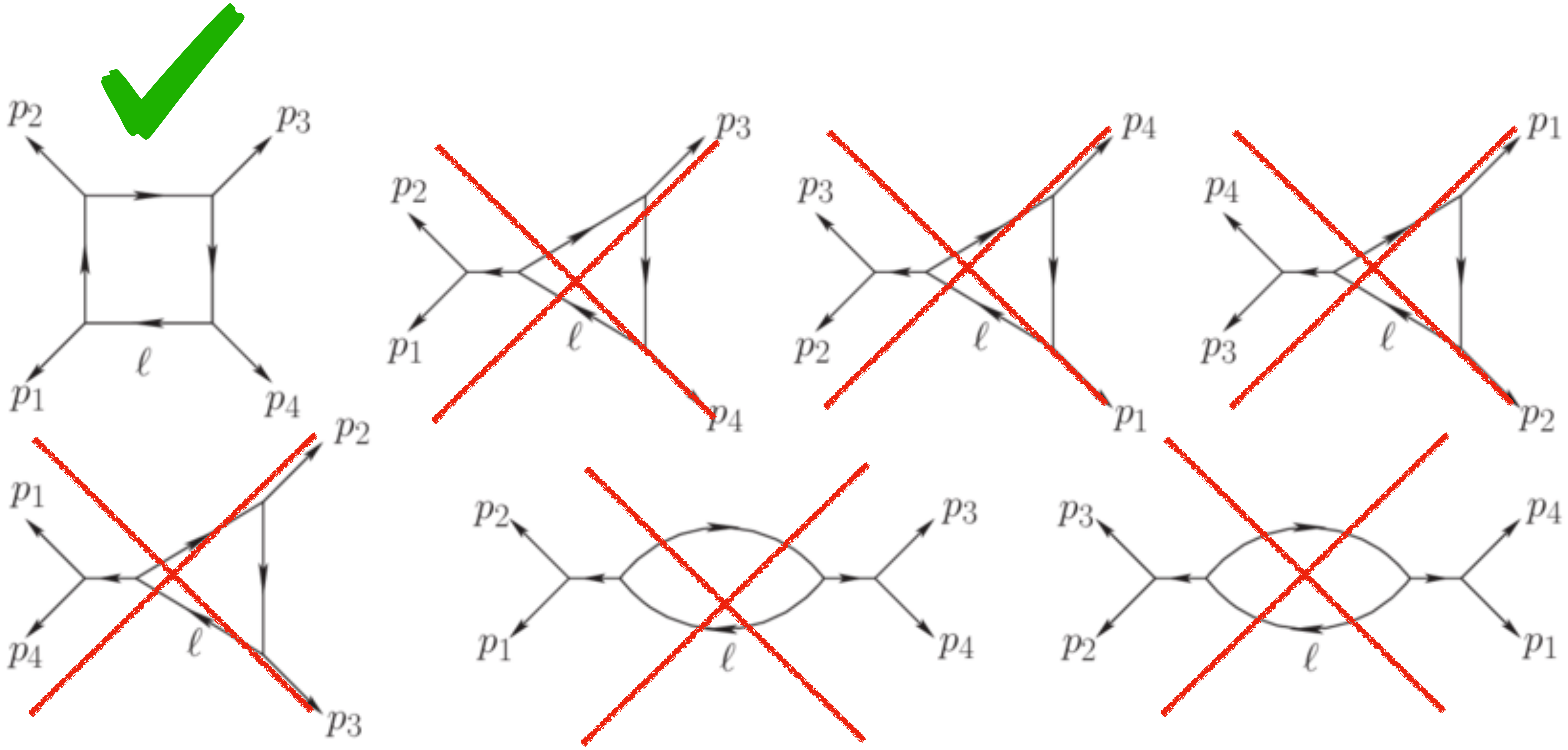
$$(\log M_n)^{(2)} = M_n^{(2)} - \frac{1}{2}\left(M_n^{(1)}\right)^2$$

$$(\log M_n)^{(3)} = M_n^{(3)} - M_n^{(2)}M_n^{(1)} + \frac{1}{3}\left(M_n^{(1)}\right)^3$$

J. L. Bourjaily et al. (2014)

1-loop Integrands

1. Planarity
2. Dual Conformal Theory
3. IR property



$$(log M)^{(1)} = M^{(1)} = \int d^4 x_A \frac{N}{x_{1A}^2 x_{2A}^2 x_{3A}^2 x_{4A}^2} \sim \mathcal{O}(\epsilon^{-2})$$

J. L. Bourjaily et al. (2014)

1-loop Integrands

3. IR property

$$M^{(1)} = \int d^4 x_A \frac{N}{x_{1A}^2 x_{2A}^2 x_{3A}^2 x_{4A}^2} \sim \mathcal{O}(\epsilon^{-2})$$

If we define the loop momentum $k^\mu \to 0 + \delta q^\mu$ with arbitrary vector q^μ, In dual coordinate,
$x_{1A}^2 \to \delta^2,\ x_{2A}^2 \to 2x_{21}\delta, x_{4A}^2 \to 2x_{41}\delta, x_{3A}^2 \to x_{31}^2$

Then the integrand $I^{(1)} \sim \mathcal{O}(\delta^{-4})$

One can check that the leading divergence δ^{-4} term can be translated to ϵ^{-2} contribution in the integral

This can be interpreted as there is no sub-divergence appearing in the integrand of the logarithm

J. L. Bourjaily et al. (2014)

2-loop Integrands

- Two loop momentum → two loop dual momentum "x_A, x_B" and they are adjacent.
- Dual conformal invariance →The denominator in the integrand should have 4 more loop momentums.

$$\frac{x_{13}^4 x_{24}^2}{x_{1A}^2 x_{1B}^2 x_{2B}^2 x_{3A}^2 x_{3B}^2 x_{4A}^2 x_{AB}^2} + \text{permutations}$$

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2-loop Integrands

- No sub-divergence check in soft-/collinear-limit
- In the single soft limit (only one loop momentum becomes soft), the divergences in the integrand of the logarithm should be canceled at the leading order

As $x_{1A}^2 \to \delta^2,\ x_{2A}^2 \to 2x_{21}\delta, x_{4A}^2 \to 2x_{41}\delta, x_{3A}^2 \to x_{31}^2, x_{AB}^2 \to x_{1B}^2$

$$I_4^{(2)} = \frac{x_{13}^4 x_{24}^2}{x_{1A}^2 x_{2A}^2 x_{2B}^2 x_{3B}^2 x_{4A}^2 x_{4B}^2 x_{AB}^2} + \text{perm.} \to \frac{x_{13}^4 x_{24}^2}{4\delta^4 x_{12} x_{14} x_{2B}^2 x_{3B}^2 x_{4B}^2 x_{1B}^2} + \mathcal{O}(\delta^{-3})$$

$$(I_4^{(1)})^2 = \frac{x_{13}^2 x_{24}^2}{x_{1A}^2 x_{2A}^2 x_{3A}^2 x_{4A}^2} \times \frac{x_{13}^2 x_{24}^2}{x_{1B}^2 x_{2B}^2 x_{3B}^2 x_{4B}^2} + \text{perm.} \to \frac{x_{24}^2}{4\delta^4 x_{21} x_{41}} \frac{x_{13}^2 x_{24}^2}{x_{1B}^2 x_{2B}^2 x_{3B}^4 x_{4B}^2} + \mathcal{O}(\delta^{-3})$$

$$(\log I_4)^{(2)} = I_4^{(2)} - \frac{1}{2}(I_4^{(1)})^2 \sim \mathcal{O}(\delta^{-3})$$

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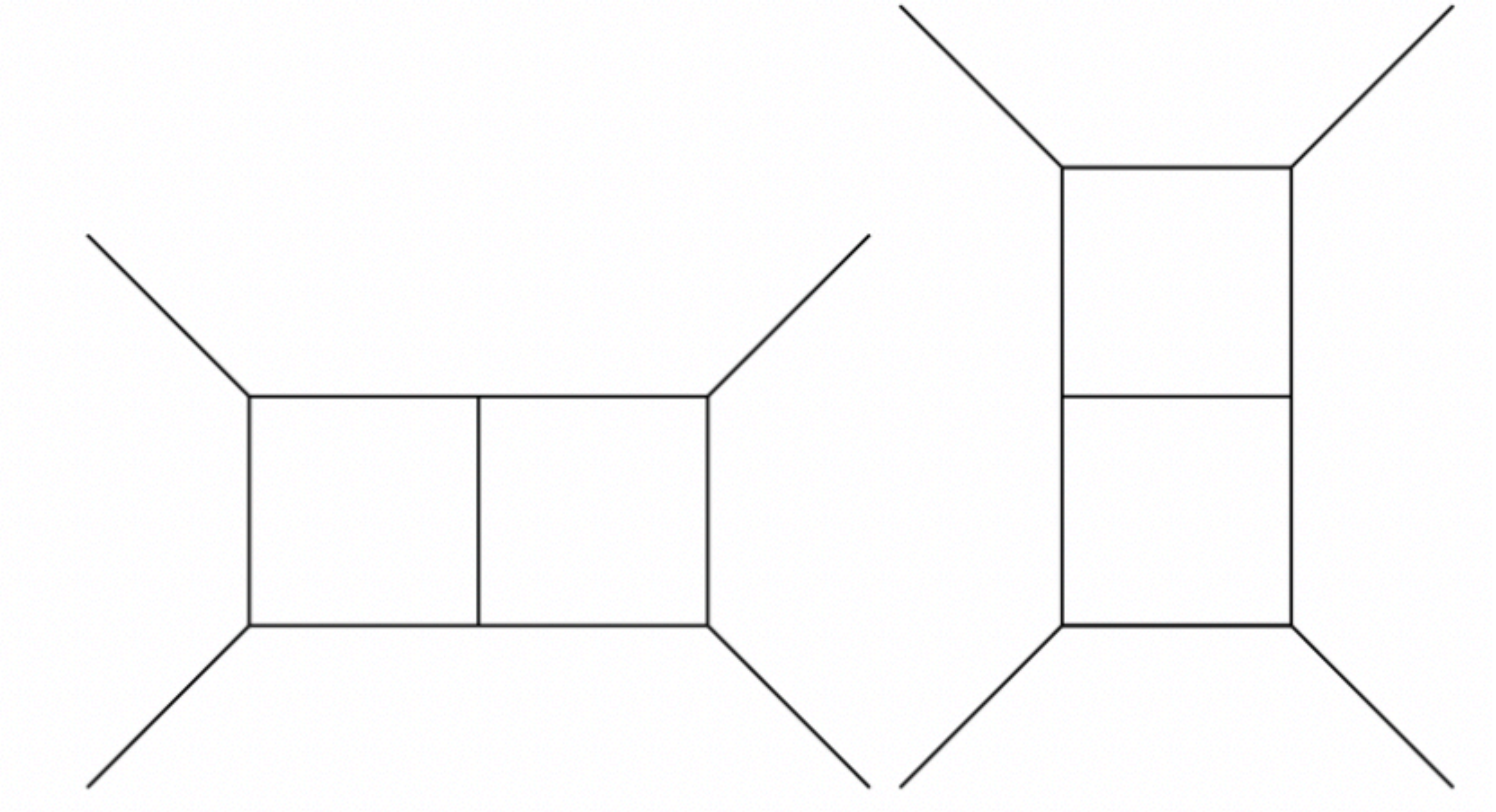
2-loop Integrands

$$I_4^{(2)} = \frac{x_{13}^4 x_{24}^2}{x_{1A}^2 x_{2A}^2 x_{2B}^2 x_{3B}^2 x_{4A}^2 x_{4B}^2 x_{AB}^2} + \text{perm.}$$



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3-loop integrands

- Three loop momentum → three loop dual momentum "x_A, x_B, x_C" and they are adjacent to at least one of them.
- Dual conformal invariance →The denominator in the integrand should have 4 more loop momentums.
- Only two available dual conformal invariant integrands
- $I_1^{(3)} = \dfrac{x_{13}^6 x_{24}^2}{x_{1A}^2 x_{1B}^2 x_{1C}^2 x_{2C}^2 x_{3A}^2 x_{3B}^2 x_{3C}^2 x_{4B}^2 x_{AB}^2 x_{AC}^2}$ + sym.
- $I_2^{(3)} = \dfrac{x_{13}^4 x_{24}^2 x_{2A}^2}{x_{1A}^2 x_{1C}^2 x_{2B}^2 x_{2C}^2 x_{3A}^2 x_{3B}^2 x_{4A}^2 x_{AB}^2 x_{BC}^2 x_{AC}^2}$ + sym.
- $I^{(3)} = aI_1^{(3)} + bI_2^{(3)}$

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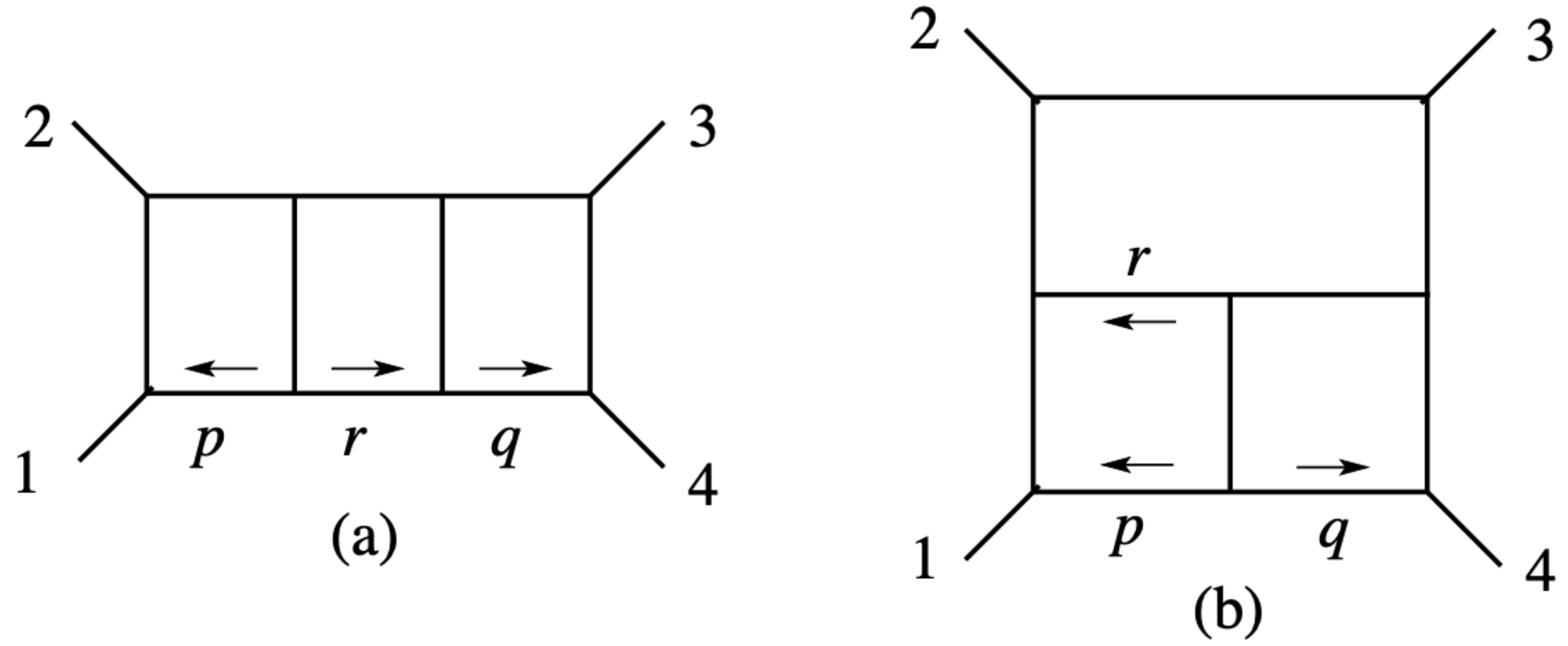
3-loop integrands

- $(\log I_4)^{(3)} = I_4^{(3)} - I_4^{(2)} I_4^{(1)} + \frac{1}{3}(I_4^{(1)})^3$
- $I_1^{(3)} = \frac{x_{13}^6 x_{24}^2}{x_{1A}^2 x_{1B}^2 x_{1C}^2 x_{2C}^2 x_{3A}^2 x_{3B}^2 x_{3C}^2 x_{4B}^2 x_{AB}^2 x_{AC}^2}$ + sym.
- $I_2^{(3)} = \frac{x_{13}^4 x_{24}^2 x_{2A}^2}{x_{1A}^2 x_{1C}^2 x_{2B}^2 x_{2C}^2 x_{3A}^2 x_{3B}^2 x_{4A}^2 x_{AB}^2 x_{BC}^2 x_{AC}^2}$ + sym.
- $I^{(3)} = aI_1^{(3)} + bI_2^{(3)}$
- In soft-/collinear-limit, the cancellation of the leading divergent term only happens at $a = 1, b = 1$



J. L. Bourjaily et al. (2014)
Z. Bern, L. J. Dixon, V. A. Smirnov. (2005)

N-loop integrands?

Loop	# of DCI candidates	# of denominator topologies	# of integrands with coefficient:			
			1	**-1**	**2**	**0**
1	1	1	1	0	0	0
2	1	1	1	0	0	0
3	2	2	2	0	0	0
4	8	6	6	2	0	0
5	34	30	23	11	0	0
6	256	197	129	99	1	27
7	2329	1489	962	904	7	456

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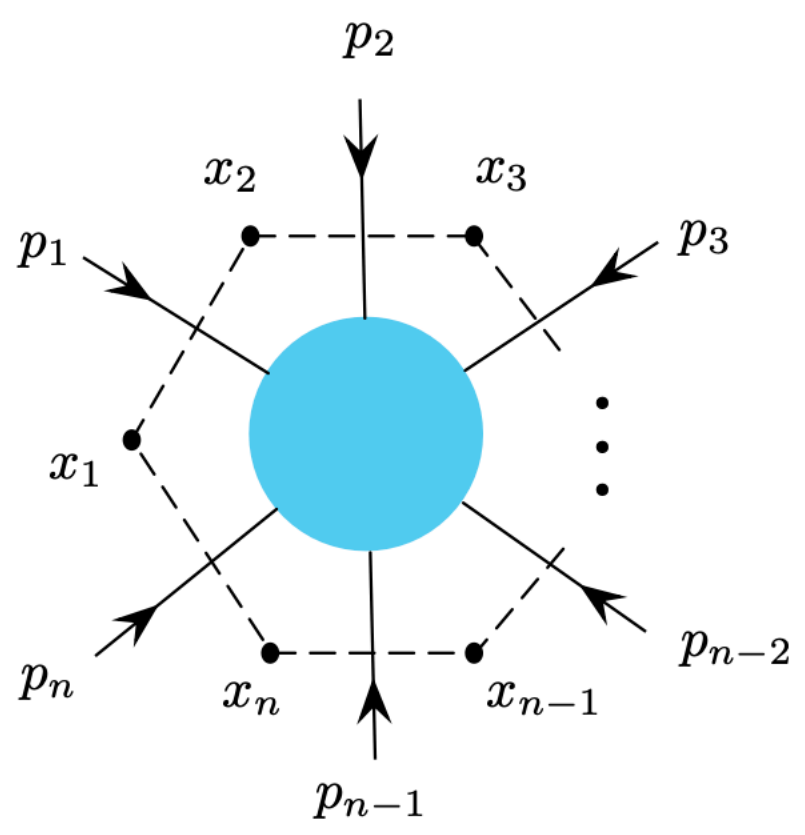
Amplituhedron

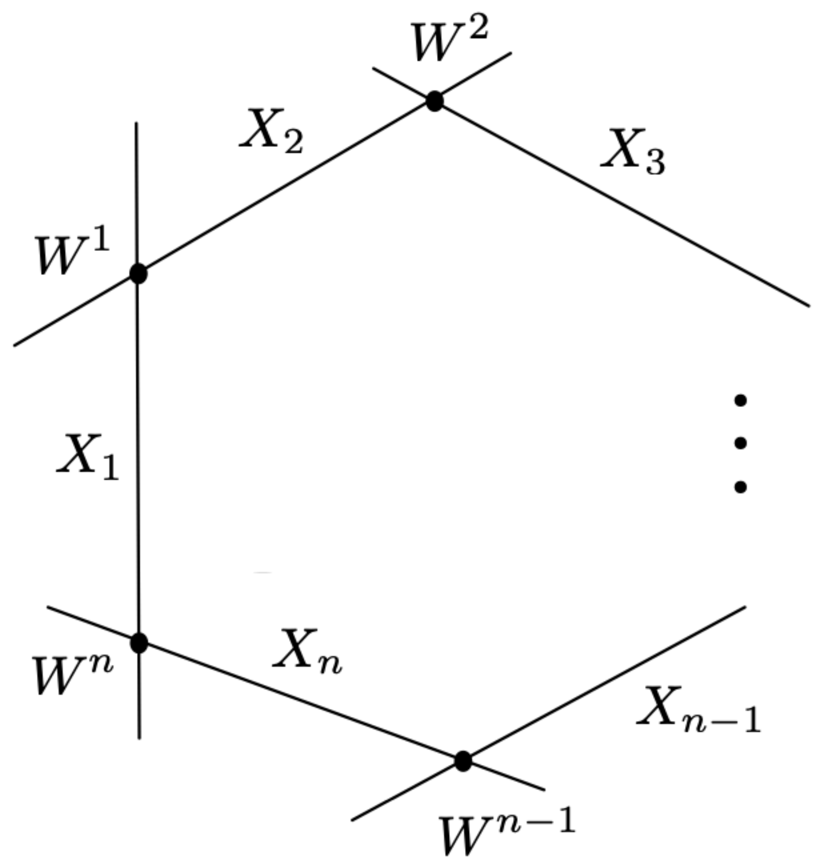
- Grassmannian generalization of polygons and polytopes. Same way that convex plane polygons generalize triangles.
- Neither space-time nor Hilbert space make any appearance.
- Only the associated physics of locality and unitarity arise as consequence of the geometry (Positive Grassmannian)
- Meromorphic differential form is crucial to convert these geometries into physical scattering amplitudes.
- The calculation of scattering amplitudes is then reduced the mathematical question of determining this canonical form.

Momentum Twistor

- Twistor in the dual coordinate (Conformally compactified space)
- Twistor space W_i, with $i = 1,2,3,\ldots,n$
- The dual space coordinate x_1 and x_2 are represented by the twistor lines $[W_n \wedge W_1]$ and $[W_1 \wedge W_2]$
- Their separation $(x_1 - x_2)^2 = \dfrac{\epsilon(W_n, W_1, W_2, W_3)}{\langle W_n W_1 \rangle \langle W_2 W_3 \rangle}$

, where $\epsilon(W_1, W_2, W_3, W_4) \equiv \epsilon^{\alpha\beta\gamma\delta} W_{1,\alpha} W_{2,\beta} W_{3,\gamma} W_{4,\delta}$ and $\langle W_1 W_2 \rangle \equiv I^{\alpha\beta} W_{1,\alpha} W_{2,\beta}$





One-Loop Geometry

- At one loop we have a single line corresponding to the loop called “(AB)”
- The geometry is given by the positive Grassmannian $G_+(2,4)$
- The external data form a polygon in $\mathbb{P}^3$ with the vertices Z_1, Z_2, Z_3, Z_4 and edges Z_1Z_2, Z_2Z_3, Z_3Z_4, Z_1Z_4.
- The line $AB = \mathscr{L}_1\mathscr{L}_2$ is parametrized as $\mathscr{L}_\gamma^I = D_{\gamma a}Z_a^I$
- $D = \begin{pmatrix} 1 & x & 0 & -w \\ 0 & y & 1 & z \end{pmatrix}$ where $x, y, z, w > 0$
- $\langle AB12\rangle = w, \langle AB23\rangle = z, \langle AB34\rangle = y, \langle AB14\rangle = x$

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Canonical form of positive geometry

- The form is fixed by the requirement of having simple poles on all the boundaries of the geometry.
- When we define the closed interval $[a, b] \subset \mathbb{P}^1(\mathbb{R})$ to be the set of points $\{(1,x) | x \in [a, b]\} \subset \mathbb{P}^1(\mathbb{R})$, where $a < b$.

$$\Omega([a, b]) = \frac{dx}{x-a} - \frac{dx}{x-b} = \frac{(b-a)}{(b-x)(x-a)}dx$$

<u>Example</u>

Consider $0 < x_1 < x_2$, the form is $\Omega = \dfrac{dx_1 dx_2}{x_1(x_2 - x_1)}$

Consider $0 < x_1 < x_2 < a$, the form is

$$\Omega = \left(\frac{dx_1}{x_1} - \frac{dx_1}{x_1 - a}\right)\left(\frac{dx_2}{x_2 - x_1} - \frac{dx_2}{x_2 - a}\right) = \frac{dx_1 dx_2}{x_1(x_2 - x_1)(a - x_1)}$$

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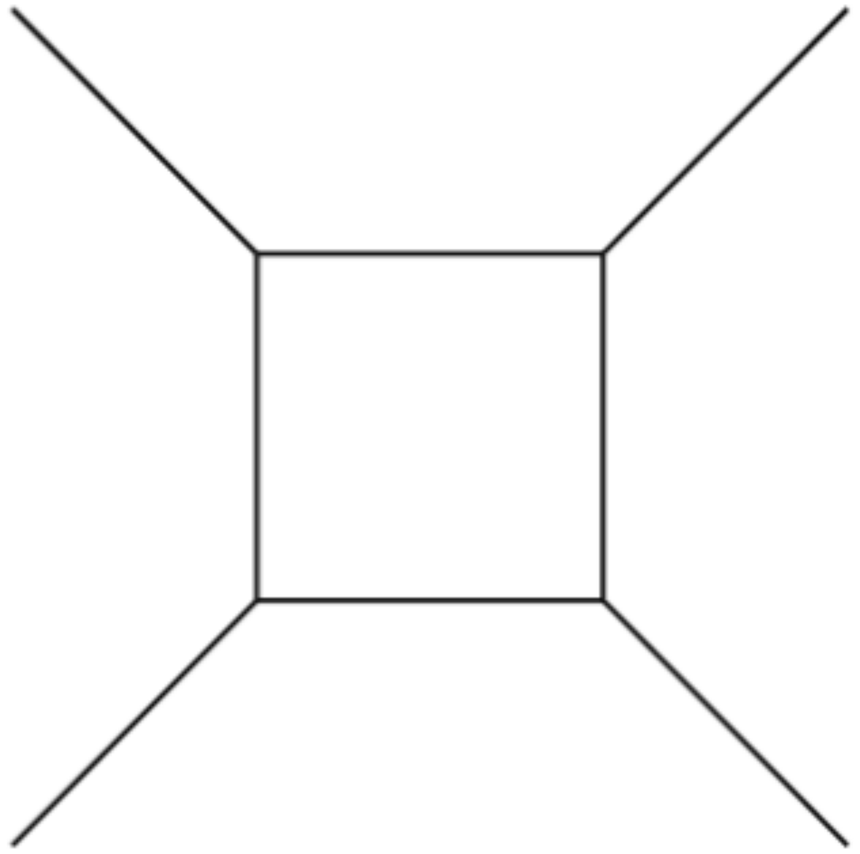
One-Loop Geometry

- $D = \begin{pmatrix} 1 & x & 0 & -w \\ 0 & y & 1 & z \end{pmatrix}$ where $x, y, z, w > 0$
- The form is $\Omega = \frac{dx}{x}\frac{dy}{y}\frac{dw}{w}\frac{dz}{z}$
- In momentum twistors, $\Omega = \frac{\langle 1234 \rangle^2}{\langle AB12 \rangle \langle AB23 \rangle \langle AB34 \rangle \langle AB14 \rangle}$



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Two-Loop Geometry

- At two loop we have two lines corresponding to the loops called “(AB)” , “(CD)”
- The lines AB and CD are parametrized as $\mathscr{L}^I_\gamma = D^{(i)}_{\gamma a} Z^I_a$
- $D^{(i)} = \begin{pmatrix} 1 & x_i & 0 & -w_i \\ 0 & y_i & 1 & z_i \end{pmatrix}$ where $x_i, y_i, z_i, w_i > 0$ with $i = 1{,}2$
- $\langle AB12 \rangle = w_1, \langle AB23 \rangle = z_1, \langle AB34 \rangle = y_1, \langle AB14 \rangle = x_1$
- $\langle CD12 \rangle = w_2, \langle CD23 \rangle = z_2, \langle CD34 \rangle = y_2, \langle CD14 \rangle = x_2$

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Two-Loop Geometry

- A single mutual positivity condition
  $(x_1 - x_2)(z_1 - z_2) + (y_1 - y_2)(w_1 - w_2) < 0$
- Without loss of generality, we can take $x_1 < x_2$.
- Then we have $z_1 - z_2 > \dfrac{(y_1 - y_2)(w_1 - w_2)}{x_2 - x_1}$

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Two-Loop Geometry

- Case 1. $(y_1 - y_2)(w_1 - w_2) > 0$,
- the form is

$$[x_1, x_2]\frac{1}{z_2}\frac{1}{z_1 - z_2 - \frac{(y_1 - y_2)(w_1 - w_2)}{x_2 - x_1}}\left([y_1, y_2][w_1, w_2] + [y_2, y_1][w_2, w_1]\right)$$

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Two-Loop Geometry

- Case 2. $(y_1 - y_2)(w_1 - w_2) < 0$,
- The form is

$$\frac{1}{x_1}\frac{1}{x_2 - x_1}\frac{1}{z_1}\left(\frac{1}{z_2} - \frac{1}{z_2 - z_1 + \frac{(y_1 - y_2)(w_1 - w_2)}{x_2 - x_1}}\right)\big([y_1, y_2][w_1, w_2] + [y_2, y_1][w_2, w_1]\big)$$

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Two-Loop Geometry

- With the term swapping $1 \leftrightarrow 2$, the sum of these terms is then
- $$\left(\frac{1}{x_2y_1y_2z_1w_1w_2\left[(x_1-x_2)(z_1-z_2)+(y_1-y_2)(w_1-w_2)\right]}+1\leftrightarrow 2\right)+$$
$$\left(\frac{1}{x_1x_2y_2z_1z_2w_1\left[(x_1-x_2)(z_1-z_2)+(y_1-y_2)(w_1-w_2)\right]}+1\leftrightarrow 2\right)$$

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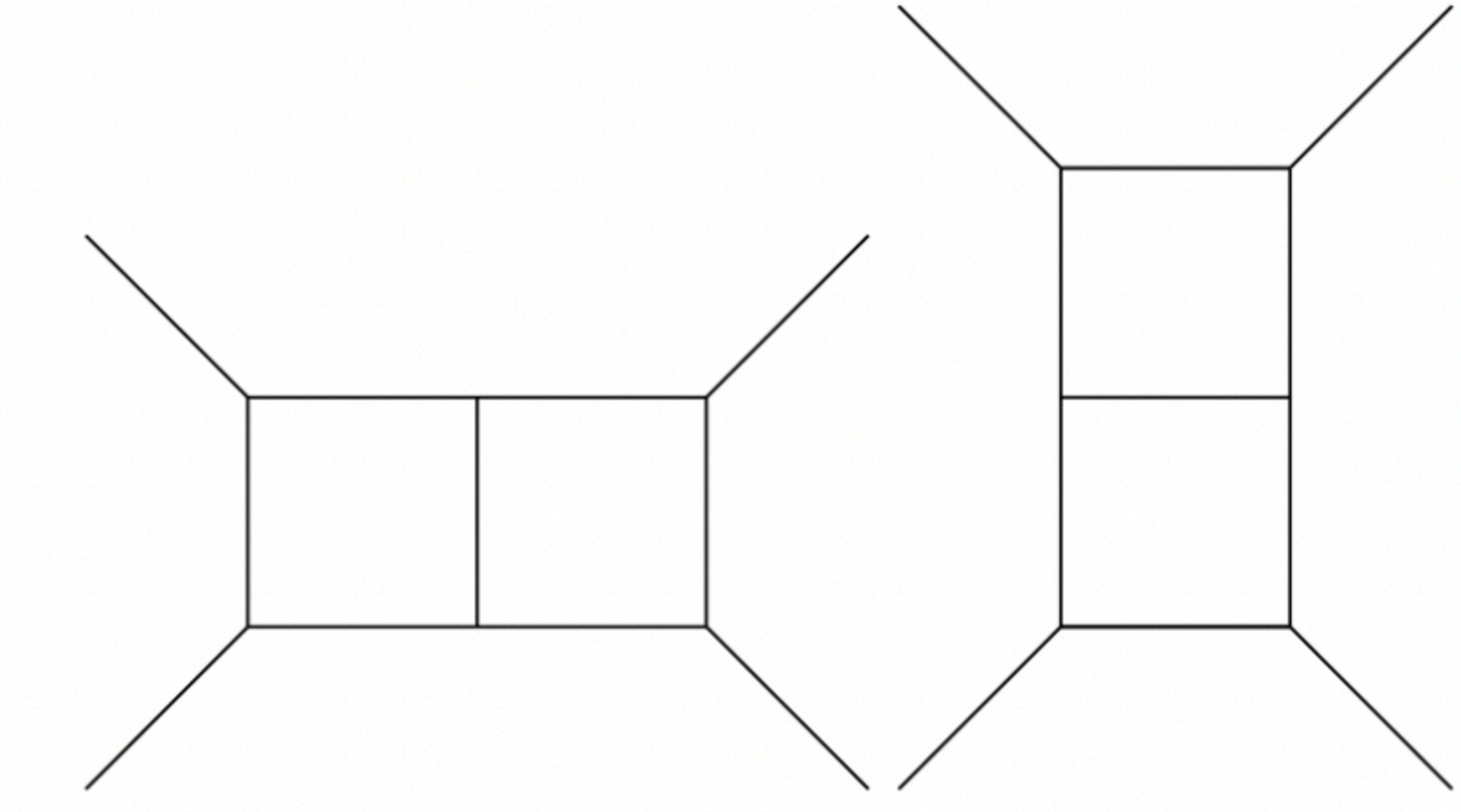
Two-Loop Geometry

- In terms of momentum twistors
- $$\left[\frac{\langle 1234\rangle^3}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle ABCD\rangle\langle CD34\rangle\langle CD14\rangle\langle CD12\rangle}+\frac{\langle 1234\rangle^3}{\langle AB23\rangle\langle AB34\rangle\langle AB14\rangle\langle ABCD\rangle\langle CD14\rangle\langle CD12\rangle\langle CD23\rangle}\right]+(AB)\leftrightarrow(CD)$$



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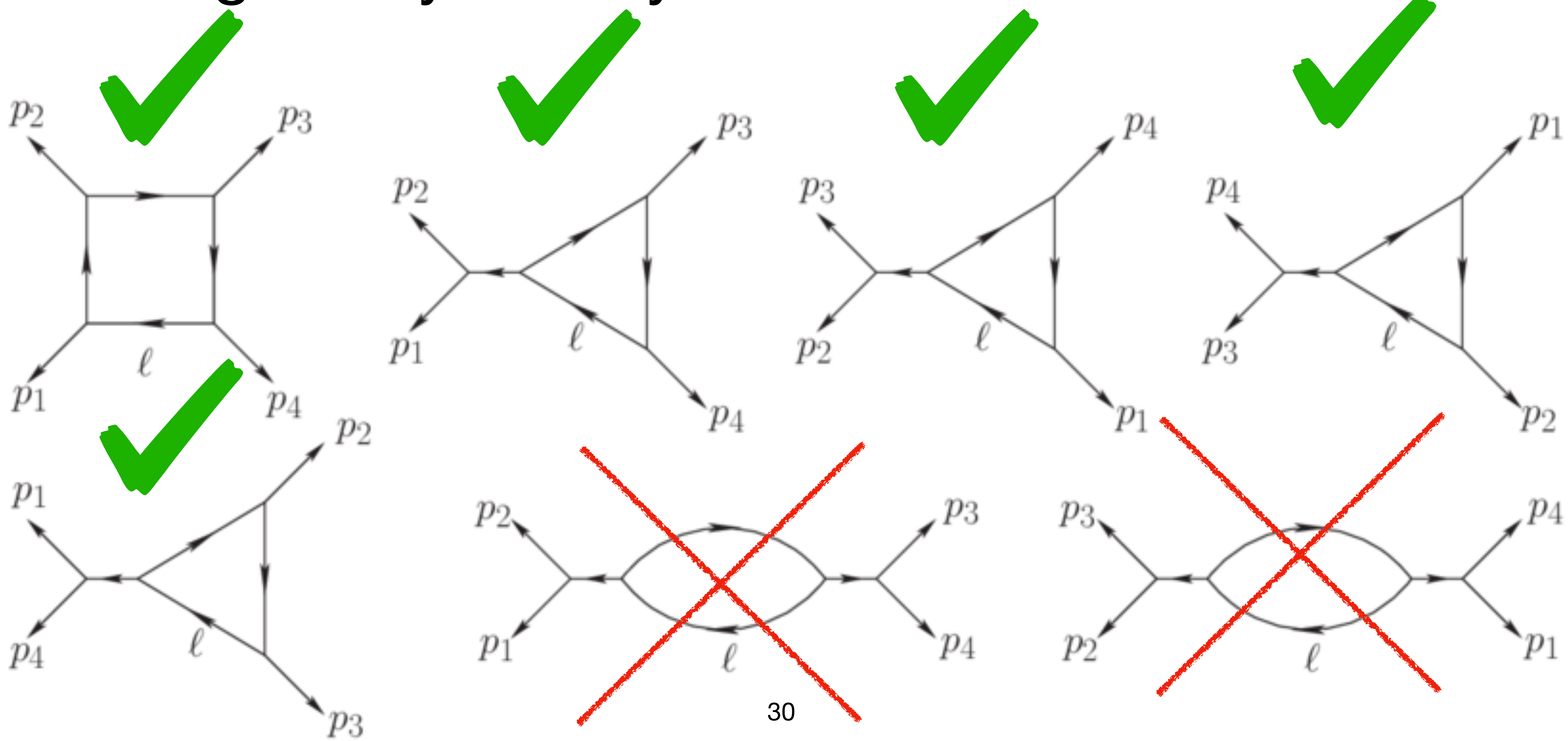
QCD = Peturbation from N=4 SYM?

- QCD can be viewed as containing a "conformal limit terms" and "conformal-breaking terms"
- Example) Loosening the symmetry : remove "Dual Conformal Theory"



Summary

- N=4 SYM is a theory that can understand QCD.
- With the symmetry and the IR property, bootstrapping techniques can get up to 7-loop integrand.
- Amplituhedron is the geometrical approach with its underlying properties.
- By loosening the symmetry of the theory, one can get closer to QCD.