Differential Equations for Feynman Integrals

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IMPRS Young Scientist Workshop
Castle Ringberg, May 9th 2022

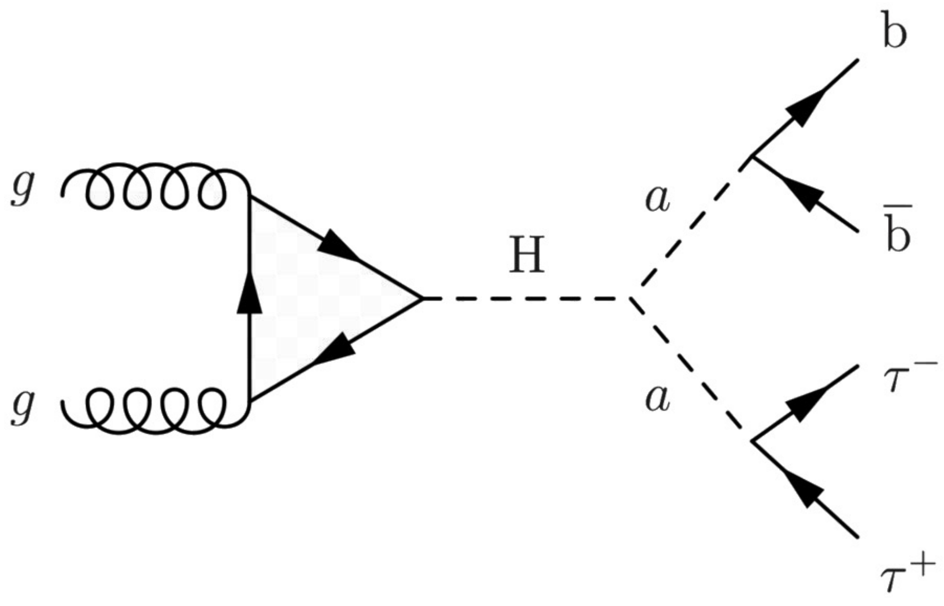
European Research Council
Established by the European Commission

MAX-PLANCK-INSTITUT FÜR PHYSIK

Why Feynman integrals?

$$\mathcal{I}(a_1, ...a_n) = \left(\prod_{j=1}^{L} \int \frac{d^D l_j}{i\pi^{D/2}} \right) \prod_{k=1}^{n} \mathcal{D}_k^{-a_k}$$

- Connection to experiments
- Mathematical properties



Outline

1) Motivation
2) Integrating by differentiating
    1) Simple example
    2) Current project
3) Summary and outlook

Integrating by differentiating

Simple example

Three-mass triangle

- Integral family

$$I_{a_1,a_2,a_3} = \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{(k^2)^{a_1}((k+p_1)^2)^{a_2}((k+p_1+p_2)^2)^{a_3}}$$

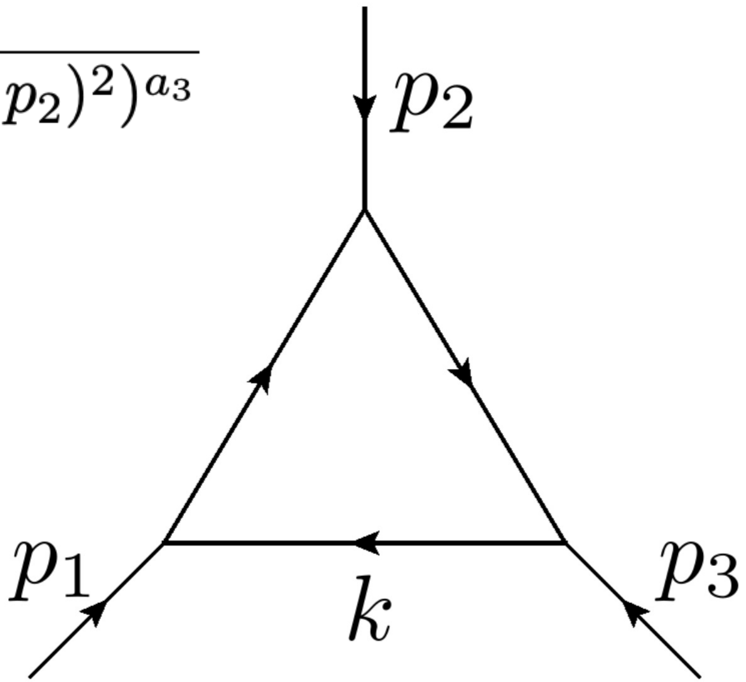
- Kinematics

$$p_i^2 = m_i^2, \quad i = 1, \ldots, 3$$

$$\sum_{i=1}^{3} p_i = 0$$

- Dimensional regularisation

$$D = d_0 - 2\epsilon$$



Integration-by-parts (IBP) relations

[Chetyrkin, Tkachov '81]

- Vanishing of total derivatives in dim-reg

$$\int \frac{d^D k}{i\pi^{D/2}} \frac{\partial}{\partial k^\mu} \frac{q^\mu}{(k^2)^{a_1}((k+p_1)^2)^{a_2}((k+p_1+p_2)^2)^{a_3}} = 0$$

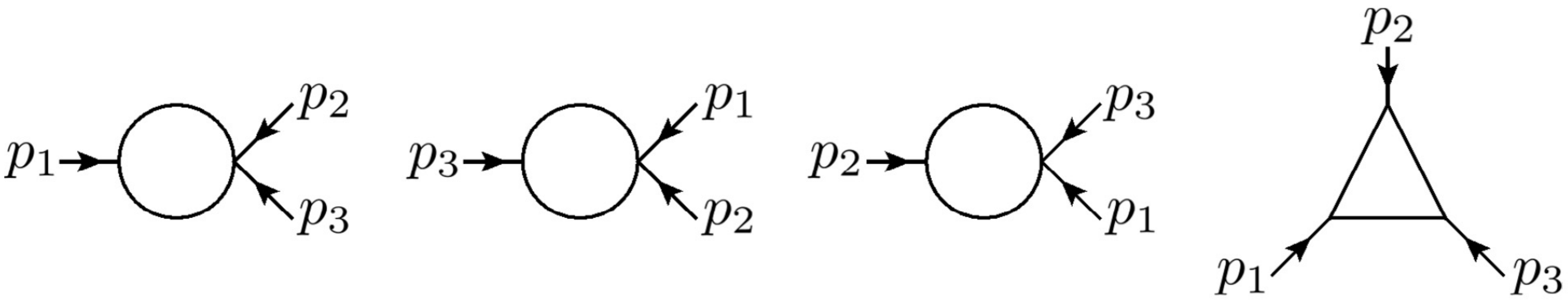
- Relations between integrals with different power of propagators

$$(D-2a_1-a_2-a_3)I_{a_1,a_2,a_3} - a_3 I_{a_1-1,a_2,a_3+1} - a_2 I_{a_1-1,a_2+1,a_3} + m_3^2 a_3 I_{a_1,a_2,a_3+1} + m_1^2 a_2 I_{a_1,a_2+1,a_3} = 0$$

- Finite number of independent integrals → basis of master integrals

  [Smirnov, Petukhov '11]

Canonical basis



$$f_1 = \epsilon m_1^2 I_{2,1,0} \qquad f_2 = \epsilon m_2^2 I_{0,2,1} \qquad f_3 = \epsilon m_3^2 I_{1,0,2} \qquad f_4 = \epsilon^2 \sqrt{\lambda} I_{1,1,1}$$

$$\lambda(m_1^2, m_2^2, m_3^2) = m_1^4 + m_2^4 + m_3^4 - 2m_1^2 m_2^2 - 2m_1^2 m_3^2 - 2m_2^2 m_3^2$$

- Uniform transcendentality

Differential equations

[Kotnikov '91, Remiddi '97, Gehrmann, Remiddi '00]

- System of first-order differential equations

$$\frac{\partial}{\partial m_i^2}\vec{f} = \epsilon A_i \vec{f}, \quad i = 1,2,3$$

- Canonical DE: [Henn '13] $\mathrm{d}\vec{f} = \epsilon\, \mathrm{d}A \cdot \vec{f}$ $\qquad \frac{\partial}{\partial m_i^2} A = A_i$

$$A = \begin{pmatrix} -\log\alpha_1 & 0 & 0 & 0 \\ 0 & -\log\alpha_2 & 0 & 0 \\ 0 & 0 & -\log\alpha_3 & 0 \\ -\log\alpha_4 & -\log\alpha_5 & \log\alpha_4 + \log\alpha_5 & \log\alpha_1 - \log\alpha_2 - \log\alpha_3 + 2\log\alpha_6 \end{pmatrix}$$

$$\vec{\alpha} = \{m_1^2, m_2^2, m_3^2, \frac{m_1^2 - m_2^2 - m_3^2 + \sqrt{\lambda}}{m_1^2 - m_2^2 - m_3^2 - \sqrt{\lambda}}, \frac{-m_1^2 + m_2^2 - m_3^2 + \sqrt{\lambda}}{-m_1^2 + m_2^2 - m_3^2 - \sqrt{\lambda}}, \sqrt{\lambda}\}$$

Solving the DE's

$$\vec{f}(m,\epsilon) = \mathbb{P}\exp\left(\epsilon \int_\gamma dA\right) \cdot \vec{b}(\epsilon)$$

- Laurent series around $\epsilon = 0$

$$\vec{f}(m,\epsilon) = \sum_{k\geq 0} \epsilon^k \vec{f}^k(m)$$

- k-fold iterated integral

$$\vec{f}^k(m) = \sum_{j=0}^{k} \int_\gamma \underbrace{dA \cdot \ldots \cdot dA}_{j} \cdot \vec{b}^{(k-j)}$$

Solving the DE’s

- Solving in one region and relating to other regions by analytic continuation or solving in each kinematic region
- Choose a point and fix the boundary constants
  - γ never leaves the region
- Solution in terms of:
  - Chen iterated integrals
  - Goncharov polylogarithms (GPLs)
  - Basis of functions

Basis of functions

- Change of variables

$$z = \frac{m_1^2 + m_2^2 - m_3^2 + \sqrt{\lambda}}{2m_1^2} \qquad \bar{z} = \frac{m_1^2 + m_2^2 - m_3^2 - \sqrt{\lambda}}{2m_1^2}$$

- Solution to $\mathrm{d}\vec{f} = \epsilon\, \mathrm{d}A \cdot \vec{f}$ can be expressed as

$$\vec{f} = (\epsilon^2 w_1^{(2)}, \epsilon w_2^{(1)}, \epsilon w_3^{(1)}, 1)$$

$$w_1^{(2)} = \mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) + \frac{1}{2}\log(z\bar{z})\log(\frac{1-z}{1-\bar{z}}) \qquad w_2^{(1)} = \log(z\bar{z}) \qquad w_3^{(1)} = \log((1-z)(1-\bar{z}))$$

- Classical polylogarithms: $\mathrm{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n} \quad \forall n \in \mathbb{N} \qquad \mathrm{Li}_1(z) = -\log(1-z)$

Integrating by differentiating

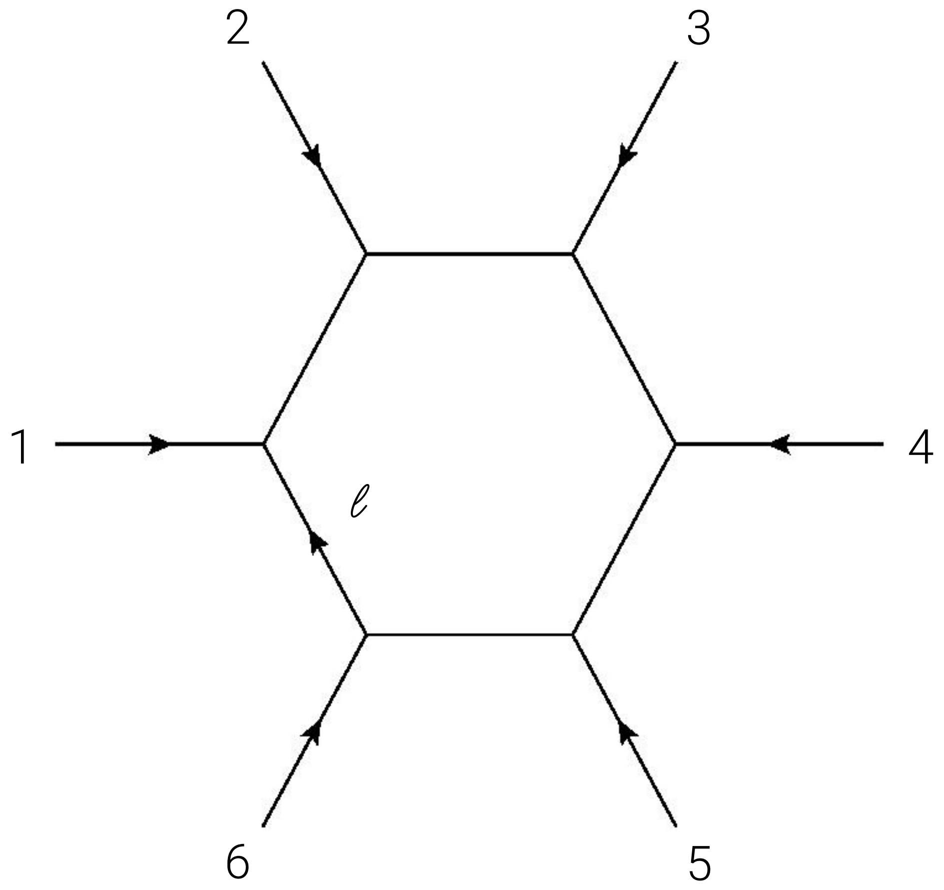
Current project

- With Johannes Henn and Julian Miczajka
- One-loop massless six-particle integral

$$p_i^2 = 0 \quad \sum p_i = 0, \qquad i = 1, \ldots, 6$$

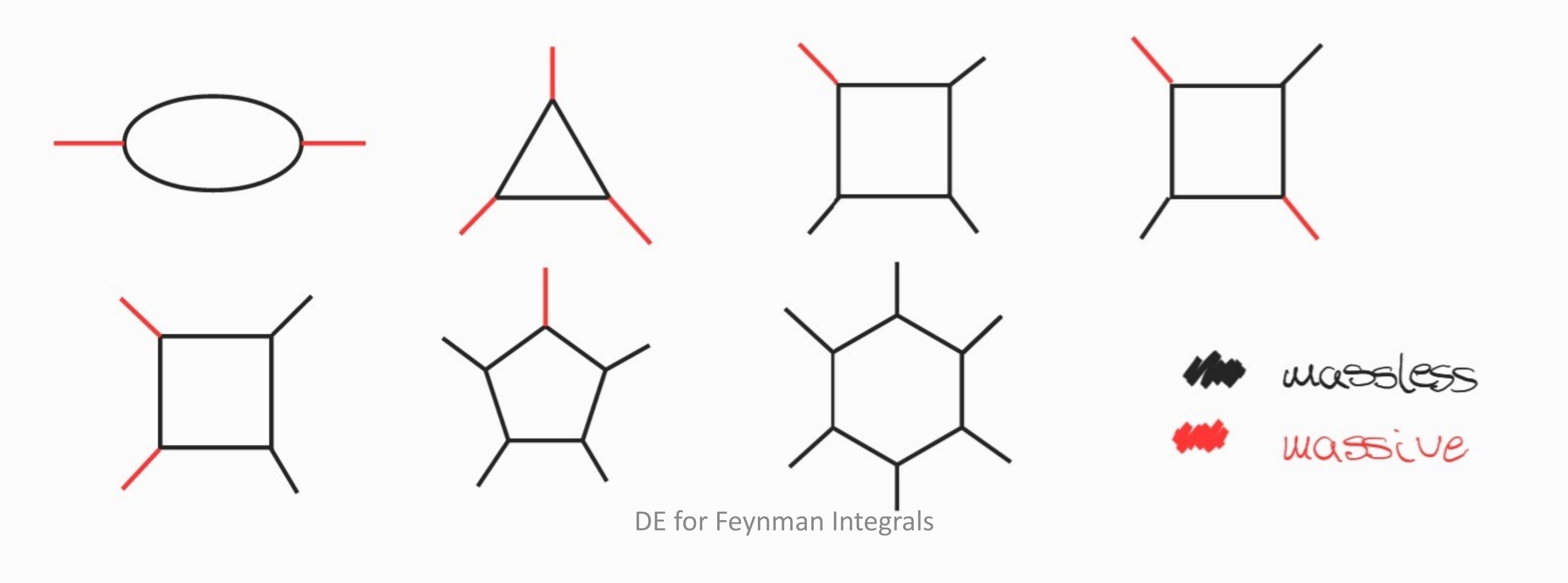
- 9 Lorentz invariants in $D_{ext} = d$

$$v_i = (p_i + p_{i+1})^2, \quad i = 1, \ldots, 6$$
$$v_{i+6} = (p_i + p_{i+1} + p_{i+2})^2, \quad i = 1, \ldots, 3$$



- Performed IBP reduction
- 33 master integrals in UT basis

  = 9 bubbles + 2 three-mass triangles + 6 one-mass boxes + 3 two-mass-easy boxes + 6 two-mass-hard boxes + 6 one-mass pentagons + 1 hexagon



- 33x33 A-matrix in *dlog* form
  - 103 letters (48 even letters, 43 odd letters, and 12 mixed)
- Boundary constants up to weight four
- Analytic solution in Euclidean region up to weight two

Goals

- $D_{ext} \to 4$ limit
- Physical $2 \to 4$ scattering region
  - Analytic continuation
- Numerical checks
- Bonus questions

Summary

- IBP relations
- Basis of master integrals
- Canonical differential equations
- Solution in terms of polylogarithms

Outlook

- Finish our to-do list
- Two-loop six-particles
  - Continue the analysis started in [Henn, Peraro, Xu, Zhang '21]
  - Canonical basis

Thank you for the attention!