Checking the CB-Property and Related Problems

Le Ngoc Long

Fakultät für Informatik und Mathematik
Universität Passau

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Outline

1. The CB-Property

2. Some Related Problems

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Notation

- $P = K[x_0, \dots, x_n]$ a polynomial ring over a field K
- $\mathbb{X} = \{p_1, ..., p_s\}$ a set of points (0-dim. scheme) in $\mathbb{P}^n \setminus \mathcal{Z}(x_0)$
- $I_{\mathbb{X}} \subseteq P$ the homogeneous vanishing ideal of $\mathbb{X}$
- $R = P/I_{\mathbb{X}}$ the homogeneous coordinate ring of $\mathbb{X}$

Definition 1

Let $d \geq 0$ and $\mathrm{HF}_{\mathbb{X}}(i) = \dim_K(R_i)$ the Hilbert function of $\mathbb{X}$ and $r_{\mathbb{X}} = \min\{i \in \mathbb{N} \mid \mathrm{HF}_{\mathbb{X}}(i) = s\}$ the regular index of $\mathrm{HF}_{\mathbb{X}}$.

(a) We say that $\mathbb{X}$ has the CB-property of degree d (CBP(d)), if every hypersurface of degree d which contains all points of $\mathbb{X}$ but one automatically contains the last point.

(b) $\mathbb{X}$ is called a CB-scheme, if $\mathbb{X}$ has the CBP($r_{\mathbb{X}} - 1$).

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Classical Result

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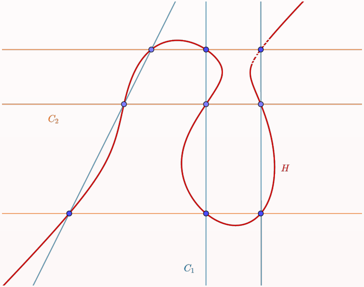
Proposition 2 (Cayley-Bacharach)

Let $d_1, d_2 \geq 0$ and let $\mathbb{X} \subseteq \mathbb{P}^2$ be a set of $d_1 \cdot d_2$ points which are the complete intersection of two algebraic curves C_1 of degree d_1 and C_2 of degree d_2.
Then $\mathbb{X}$ is a CB-scheme, i.e., every algebraic curve of degree $d_1 + d_2 - 3$ which contains $d_1 \cdot d_2 - 1$ points of $\mathbb{X}$ automatically contains the last point of $\mathbb{X}$.



Example for CBP

Example 3

Let $\mathrm{char}(K) \neq 2$, and let $\mathbb{X} = \{p_1, ..., p_8\} \subseteq \mathbb{P}^2$ with $p_1 = (1:0:0)$, $p_2 = (1:1:0)$, $p_3 = (1:0:1)$, $p_4 = (1:1:1)$, $p_5 = (1:2:0)$, $p_6 = (1:0:2)$, $p_7 = (1:1:2)$ and $p_8 = (1:2:2)$. Then:

* $I_{\mathbb{X}} = \langle 2x_0^2x_2 - 3x_0x_2^2 + x_2^3, 2x_0^2x_1 - 3x_0x_1^2 + x_1^3, 2x_1^3x_2 - 3x_1^2x_2^2 + x_1x_2^3 \rangle$.
* $\mathrm{HF}_{\mathbb{X}} : 1\ 3\ 6\ 8\ 8 \cdots$ und $r_{\mathbb{X}} = 3$.

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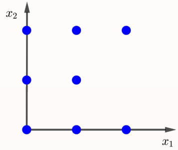

By Bézout's theorem, each curve of degree 2 that contains 7 points of $\mathbb{X}$ has either $\mathcal{Z}(x_1)$ or $\mathcal{Z}(x_1 - x_0)$ as its component. So, there is no curve of degree 2 passing through 7 points of $\mathbb{X}$. Therefore, $\mathbb{X}$ is a CB-scheme.

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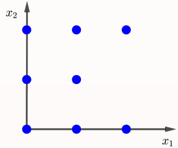
$\mathbb{X} =$ 

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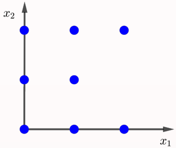
$\mathbb{X} =$ 

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Motivation

- The CB-property comes originally from classical projective geometry and has been used to study curves in projective space, in particular, in $\mathbb{P}^2$ it can be applied to prove other theorems such as Pascal's theorem, Brianchon's theorem, etc.
- This property can be used to describe other nice properties such as complete intersection, arithmetically Gorenstein property.
- This property is one of uniform properties which have some connections to Coding Theory, and they can be used to study Reed-Müller code, MDS code.

- ▷ Characterize and verify the CB-property.
- ▷ Use the CB-property to describe the complete intersection.
- ▷ Look at the CB-property for sets of points in a multiprojective space.

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Characterization of CBP

Let $p_j \in \mathbb{X}$, $\mathbb{Y} = \mathbb{X} \setminus \{p_j\}$, and $I_{\mathbb{Y}/\mathbb{X}} \subseteq R$ the vanishing ideal of $\mathbb{Y}$.
There is $t \in \mathbb{N}$ such that $0 \le t \le r_{\mathbb{X}}$ and

$$\mathrm{HF}_{\mathbb{Y}}(i) = \begin{cases} \mathrm{HF}_{\mathbb{X}}(i) & \text{für } i < t, \\ \mathrm{HF}_{\mathbb{X}}(i) - 1 & \text{für } i \ge t. \end{cases}$$

- $\deg_{\mathbb{X}}(p_j) := t$ the degree of p_j in $\mathbb{X}$.
- $0 \ne f_j^* \in (I_{\mathbb{Y}/\mathbb{X}})_t$ a minimal separator of p_j in $\mathbb{X}$
- $0 \ne f_j \in (I_{\mathbb{Y}/\mathbb{X}})_{r_{\mathbb{X}}}$ a separator of p_j in $\mathbb{X}$.

Proposition 4 (G-K-R, 1993)

Let $d \ge 0$. The following statements are equivalent:

(a) $\mathbb{X}$ *has the CBP(d).*

(b) $\deg_{\mathbb{X}}(p_j) > d$ *for every point* $p_j \in \mathbb{X}$.

(c) *If* $f_j \in R_{r_{\mathbb{X}}}$ *is a separator of* $p_j \in \mathbb{X}$*, then* $x_0^{r_{\mathbb{X}}-d} \nmid f_j$.

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Canonical Module and CBP

Definition 5

The graded R-module

$$\omega_R = \operatorname{Hom}_{K[x_0]}(R, K[x_0])(-1)$$

is called the canonical module of R.

The canonical module ω_R is finitely generated and

$$\operatorname{HF}_{\omega_R}(i) = \deg(\mathbb{X}) - \operatorname{HF}_{\mathbb{X}}(-i) = s - \operatorname{HF}_{\mathbb{X}}(-i)$$

for all $i \in \mathbb{Z}$.

Theorem 6 (K-L-R, 2019)

For $d \geq 0$, $\mathbb{X}$ has the CBP(d) if and only if $\operatorname{Ann}_R((\omega_R)_{-d}) = \langle 0 \rangle$.

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Checking the CB-scheme

Algorithm 7 (Checking the CB-scheme)

The following algorithm checks whether $\mathbb{X}$ is a CB-scheme and return the corresponding boolean value.

(1) *Compute* $J_{\mathbb{X}} = I_{\mathbb{X}}/(I_{\mathbb{X}} \cap \langle x_0 - 1\rangle)$ *in* $K[x_1, ..., x_n]$ *and* $B = \{b_1, ..., b_s\} = \mathbb{T}^n \setminus \mathrm{LT}_\sigma(J_{\mathbb{X}})$ *w.r.t. a degree-compatible term ordering* σ *on* $\mathbb{T}^n$ *such that* $\deg(b_1) \leq \cdots \leq \deg(b_s)$.

(2) *For* $i = 1, ..., s$, *compute the matrix* $M_i \in \mathrm{Mat}_s(K)$ *of the multiplication by* b_i *in the basis* B.

(3) *Let* $\Delta = \#\{b_i \mid \deg(b_i) = \deg(b_s)\}$. *Form the matrix* $V_j \in \mathrm{Mat}_s(K)$, *whose* i*-th column is the* $(s - \Delta - j)$*-th column of* M_i^{tr} *for* $i = 1, ..., s$ $(j = 1, ..., \Delta)$.

(4) *Form the block-matrix* $W = \mathrm{Col}(V_1, ..., V_\Delta)$, *calculate* $\mathrm{Ker}(W)$. *If* $\mathrm{Ker}(W) = \langle 0\rangle$, *return True; otherwise return False.*

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Example 3 (continued)

Let $\sigma = \texttt{DegLex}$, and let $\mathbb{X} = \{p_1, \ldots, p_8\} \subseteq \mathbb{P}^2$ (as in the figure)

(1) $J_{\mathbb{X}} = \langle x_2^3 - 3x_2^2 + 2x_2, x_1^3 - 3x_1^2 + 2x_1, x_1^2x_2^2 - 2x_1^2x_2 - x_1x_2^2 + 2x_1x_2 \rangle$.

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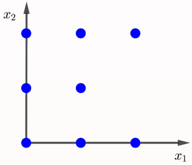


(2) A basis of $K[x_1, x_2]/J_{\mathbb{X}}$: $B = \{1, x_2, x_1, x_2^2, x_1x_2, x_1^2, x_1x_2^2, x_1^2x_2\}$, $s = 8$, and $\Delta = 2$.

(3)

$$V_1 = \begin{pmatrix} 0&0&0&0&0&0&1&0\\ 0&0&0&0&1&0&3&1\\ 0&0&0&1&0&0&1&0\\ 0&0&1&0&3&1&7&3\\ 0&1&0&3&1&0&3&1\\ 0&0&0&1&0&0&1&0\\ 1&3&1&7&3&1&7&3\\ 0&1&0&3&1&0&3&1 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&1&0&2\\ 0&0&0&0&1&0&2&3\\ 0&0&0&0&0&2&0&4\\ 0&0&1&0&2&3&4&6\\ 0&1&0&2&3&0&6&7\\ 0&0&2&0&4&6&8&12\\ 1&2&3&4&6&7&12&14 \end{pmatrix}$$

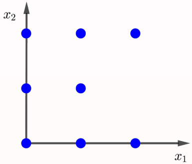
(4) $W = \mathrm{Col}(V_1, V_2)$, $\mathrm{Ker}(W) = \langle 0 \rangle$. Hence $\mathbb{X}$ is a CB-scheme.

Example 3 (continued)

Let $\sigma = \texttt{DegLex}$, and let $\mathbb{X} = \{p_1, ..., p_8\} \subseteq \mathbb{P}^2$ (as in the figure)

(1) $J_{\mathbb{X}} = \langle x_2^3 - 3x_2^2 + 2x_2, x_1^3 - 3x_1^2 + 2x_1, x_1^2x_2^2 - 2x_1^2x_2 - x_1x_2^2 + 2x_1x_2 \rangle$.

(2) A basis of $K[x_1, x_2]/J_{\mathbb{X}}$: $B = \{1, x_2, x_1, x_2^2, x_1x_2, x_1^2, x_1x_2^2, x_1^2x_2\}$, $s = 8$, and $\Delta = 2$.

$\mathbb{X} =$ 

(3)

$$V_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 3 & 1 & 7 & 3 \\ 0 & 1 & 0 & 3 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 3 & 1 & 7 & 3 & 1 & 7 & 3 \\ 0 & 1 & 0 & 3 & 1 & 0 & 3 & 1 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 4 \\ 0 & 0 & 1 & 0 & 2 & 3 & 4 & 6 \\ 0 & 1 & 0 & 2 & 3 & 0 & 6 & 7 \\ 0 & 0 & 2 & 0 & 4 & 6 & 8 & 12 \\ 1 & 2 & 3 & 4 & 6 & 7 & 12 & 14 \end{pmatrix}$$

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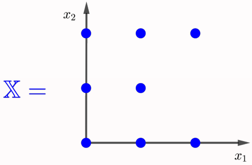

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Remark 8

- The notion of CB-property was generalized for an arbitrary 0-dimenisonal scheme in $\mathbb{P}^n$ (see [K-L-R, 2019][1]).
  Furthermore, a 0-dimenisonal scheme $\mathbb{X}$ has the CBP(d) if and only if $\mathrm{Ann}_R((\omega_R)_{-d}) = \langle 0 \rangle$.
- Algorithm 7 can be also used to check whether an arbitrary 0-dimensional scheme in $\mathbb{P}^n$ is a CB-scheme.

[1] M. Kreuzer, L.N. Long, L. Robbiano, On the Cayley-Bacharach Property, Commun. Algebra 47 (2019), 328-354.

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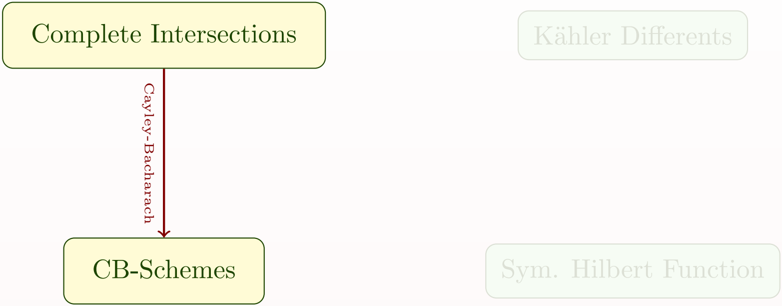
Characterization of CI



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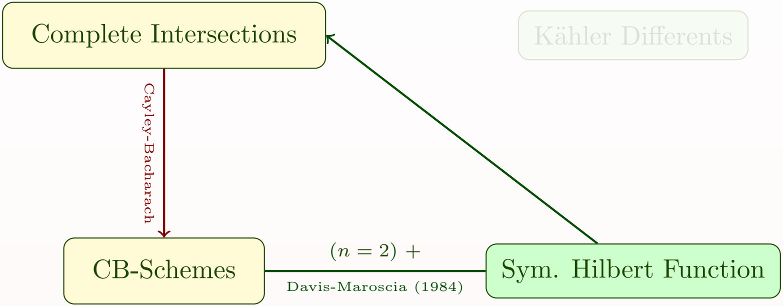

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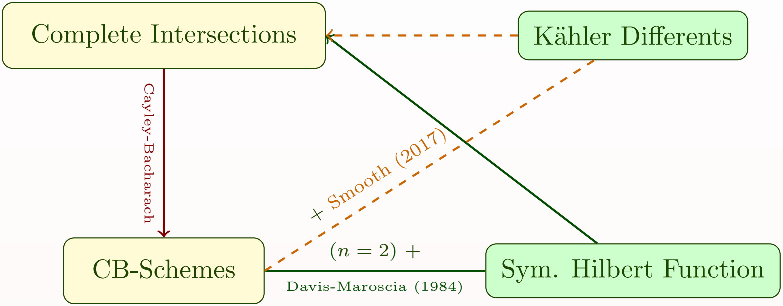

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Characterization of CI

Definition 9

Let $\mathbb{X} = \{p_1, ..., p_s\}$ be a 0-dimensional scheme in $\mathbb{P}^n$, and let $\{F_1, ..., F_r\}$ be a homogeneous system of generators of $I_{\mathbb{X}}$.

(a) $\mathbb{X}$ is called a complete intersection if $I_{\mathbb{X}}$ is generated by a homogeneous regular sequence of length n.

(b) The Kähler different of $\mathbb{X}$, denoted by $\vartheta_{\mathbb{X}}$, is the homogeneous ideal of the ring R generated by n-minors of the Jacobian matrix $\left(\frac{\partial F_j}{\partial x_i}\right)_{i=1,\dots,n, j=1,\dots,r}$.

(c) The Hilbert function of $\vartheta_{\mathbb{X}}$ satisfies

$$\mathrm{HF}_{\vartheta_{\mathbb{X}}}(i) \leq \mathrm{HF}_{\vartheta_{\mathbb{X}}}(i+1) \leq \cdots \leq \mathrm{HP}_{\vartheta_{\mathbb{X}}} \leq \deg(\mathbb{X}), \quad \forall i \in \mathbb{Z}.$$

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Characterization of CI

We give a characterization of a complete intersection as follows.

Theorem 10

For each $j \in \{1, ..., s\}$, *let* $d_j = \dim_K(\mathcal{O}_{\mathbb{X},p_j}/\mathfrak{m}_{\mathbb{X},p_j})$, $d = \sum_{j=1}^{s} d_j$, *and assume* $\mathrm{char}(K) \nmid \deg(\mathbb{X})$. *The following conditions are equivalent.*

(a) $\mathbb{X}$ *is a complete intersection.*

(b) $\mathbb{X}$ *is a CB-scheme,* $\mathrm{HP}_{\vartheta_{\mathbb{X}}} = d$ *and* $\mathrm{HF}_{\vartheta_{\mathbb{X}}}(r_{\mathbb{X}}) \neq 0$.

Sets of Points in $\mathbb{P}^m \times \mathbb{P}^n$

Now let $\mathbb{X} = \{p_1, ..., p_s\}$ be a set of points in $\mathbb{P}^m \times \mathbb{P}^n$ and let $S = K[x_0, ..., x_m, y_0, ..., y_n]$ be the bi-graded polynomial ring with $\deg(x_i) = (1, 0)$ and $\deg(y_j) = (0, 1)$.

- Writing $p_i = q_i \times q_i' \in \mathbb{P}^m \times \mathbb{P}^n$, $I_{p_i} = I_{q_i} S + I_{q_i'} S$, $I_{\mathbb{X}} = \bigcap_{i=1}^{s} I_{p_i}$.
- $I_{\mathbb{X}} \subseteq S$ is bi-homogeneous, and so $R = S/I_{\mathbb{X}}$ is bi-graded.

Definition 11

(a) $F \in S$ is a separator of p_i in $\mathbb{X}$, if F is bi-homogeneous, $F(p_i) \neq 0$ and $F(p_j) = 0$ for $j \neq i$.

(b) A separator $F \in S$ of p_i in $\mathbb{X}$ is minimal, if there is no separator G of p_i in $\mathbb{X}$ such that $\deg(G) \prec \deg(F)$. Here we write $(i_1, i_2) \prec (j_1, j_2)$ for $(i_1, i_2), (j_1, j_2) \in \mathbb{N}^2$ with $i_1 < j_1$ and $i_2 < j_2$.

(c) The degree of $p_i \in \mathbb{X}$ is defined as

$$\deg_{\mathbb{X}}(p_i) = \{\deg(F) \mid F \text{ is a minimal separator of } p_i\}.$$

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CB-Schemes in $\mathbb{P}^m \times \mathbb{P}^n$

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Definition 12

The set $\mathbb{X} \subseteq \mathbb{P}^m \times \mathbb{P}^n$ is called a CB-scheme, if all points of $\mathbb{X}$ have the same degree.

Remark 13

- $\mathbb{X}$ is a CB-scheme, if the Hilbert function of $\mathbb{X} \setminus \{p_i\}$ does not depends on the choice of the point $p_i \in \mathbb{X}$.
- When $m = n = 1$ and $\mathbb{X}$ is ACM (i.e. R is a Cohen-Macaulay ring), $\mathbb{X}$ is a CB-scheme if and only if $I_{\mathbb{X}}$ is generated by two bi-homogeneous polynomials (see [G-K-L-L, 2021][3]).

[3] E. Guardo, M. Kreuzer, T.N.K. Linh, L.N. Long, Kähler differentials for fat point schemes in $\mathbb{P}^1 \times \mathbb{P}^1$, J. Commut. Algebra 13 (2021), 179–207.

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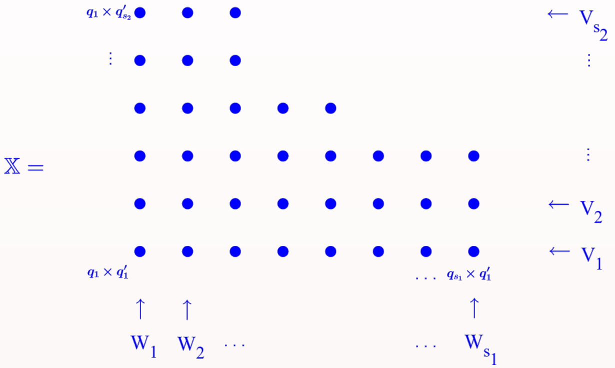
The $(\star)$-Property

Definition 14

We say that $\mathbb{X}$ has the $(\star)$-property, if for two points $q_1 \times q_1'$ and $q_2 \times q_2'$ of $\mathbb{X}$ with $q_1 \neq q_2$ and $q_1' \neq q_2'$, $q_1 \times q_2'$ or $q_2 \times q_1'$ belong to $\mathbb{X}$.



Characterizing CBP in $\mathbb{P}^m \times \mathbb{P}^n$

Proposition 15 (HKLNN, 2022)

Suppose $\mathbb{X} \subseteq \mathbb{P}^m \times \mathbb{P}^n$ has the $(\star)$-property. Then $\mathbb{X}$ is a CB-scheme if and only if the following conditions are satisfied:

(a) $V_1, ..., V_{s_2}$ *are CB-schemes in* $\mathbb{P}^m$ *and* $r_{V_1} = \cdots = r_{V_{s_2}}$;

(b) $W_1, ..., W_{s_1}$ *are CB-schemes in* $\mathbb{P}^n$ *and* $r_{W_1} = \cdots = r_{W_{s_1}}$.

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Thank you for your attention!