



Technische
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Calculating the Frattini subgroup of a polycyclic group

Tagung der Fachgruppe Computeralgebra

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Introduction

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The results presented in this talk are joint work with B. Eick and have been published in the Journal of Algebra [4].

Polycyclic groups: Definition and Examples

Definition: (Hirsch 1938 [5])

A group G is polycyclic if there exists a series of subnormal subgroups

$$G = G_1 > G_2 > \dots > G_n > G_{n+1} = \{1\}$$

so that G_i/G_{i+1} is (finite or infinite) cyclic for $1 \leq i \leq n$.

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Example:

The following groups are polycyclic:

- finite soluble groups
- finitely generated abelian or nilpotent groups
- wallpaper groups, space groups
- extensions of polycyclic groups by polycyclic groups

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Example:

The following groups are **not** polycyclic:

- groups with non-abelian, free subgroups
- groups with non-finitely generated subgroups
- non-soluble groups

The algorithm: basic idea

Let G be a polycyclic group, $N \trianglelefteq G$ a normal subgroup and $\pi : G \rightarrow G/N$ the natural epimorphism.

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so if $\phi(G/N)$ and $I(G, N)$ can be calculated effectively, an effective calculation of $\phi(G)$ is possible.

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- Let $F = \text{Fit}(G)$. If G does not act semisimply on F/F^p for all but finitely many primes p , then there is a normal subgroup $L \trianglelefteq G$ with $L \leq \phi(G)$ that can be calculated from the radical of $F \otimes_{\mathbb{Z}} \mathbb{Q}$. Choose $N = L$.

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- If none of the above approaches yield nontrivial normal subgroups, $\phi(G)$ is trivial.

The algorithm: infinite cyclic central normal subgroups

Definition:

Let G be polycyclic, $N \cong \mathbb{Z}$ and $N \leq Z(G)$. Then $I_p(G, N)$ denotes the intersection of all maximal subgroups M of G having p -power index in G and satisfying $N \not\leq M$.

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Theorem:

Let N be an infinite cyclic, central subgroup of G . Then one of the following cases applies:

1. N/N^p has complements in G/N^p for almost no primes p . There exists a finite set of primes P so that $I(G, N) = \bigcap_{p \in P} I_p(G, N)$.
2. N/N^p has a complement in G/N^p for almost all primes p , and $I(G, N) = G'$ holds.

The algorithm: Semisimplicity

Theorem:

Let G be polycyclic, $N \trianglelefteq G$, $N \cong \mathbb{Z}^d$. Then one of the following two cases holds:

1. G acts semisimply on N/N^p for almost all primes p . This is the case if and only if the induced action of G on $\mathbb{Q}^d = N \otimes_{\mathbb{Z}} \mathbb{Q}$ is semisimple.
2. G acts semisimply on N/N^p for only finitely primes p . This is the case if and only if the induced action of G on $\mathbb{Q}^d = N \otimes_{\mathbb{Z}} \mathbb{Q}$ is not semisimple.

The algorithm: the remaining case

Definition:

Let G be polycyclic and $F = \text{Fit}(G)$. If

- F is free abelian and
- G has no finite non-trivial normal subgroups,
- G has trivial centre,
- G acts semisimply on $F \otimes_{\mathbb{Z}} \mathbb{Q}$

then G has property \mathcal{P} .

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The normal subgroups discussed so far yield an iteration step if and only if G does not have property \mathcal{P} .

The algorithm: the remaining case

Theorem:

Let G have property \mathcal{P} and $F = \text{Fit}(G)$. Then:

- $H^2(G/F, F)$ is finite,
- F/F^p has a complement in G/F^p for almost all primes p and
- $\phi(G)$ is trivial.

The algorithm: the remaining case

Theorem:

Let G have property \mathcal{P} and $F = \text{Fit}(G)$. Then:

- $H^2(G/F, F)$ is finite,
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- $\phi(G)$ is trivial.

The algorithm yields an answer after a finite number of iterations, because

- polycyclic groups are Hopfian and
- every ascending series of (normal) subgroups stabilizes after finitely many steps.

The implementation: runtime

The algorithm described here has been implemented in the GAP 4 [6] package Frattini [7], using the methods of the package Polycyclic [8].

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	dimension 3	dimension 4
minimum	344 ms	308 ms
maximum	660 ms	1572 ms
mean	431 ms	497 ms
median	420 ms	480 ms
standard deviation	53 ms	96 ms

Table 1: Summary statistics for the runtime of the algorithm on all polycyclic space groups of dimensions 3 and 4 (rounded to the nearest millisecond)

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Example	1	2	3	4	5	6	7	8
runtime (ms)	484	484	372	368	556	412	512	536
derived length	2	2	2	2	3	2	3	2
number of generators	4	6	2	3	4	3	4	5
Example	9	10	11	12	13	14	15	16
runtime (ms)	472	604	404	408	504	2456	844	1112
derived length	2	3	2	2	3	3	3	3
number of generators	3	4	5	6	4	24	13	17

Table 2: Runtime of the algorithm on the GAP example library (ExamplesOfSomePcpGroups command)

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