Constructive Category Theory and Tilting Equivalences via Strong Exceptional Sequences

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Overview

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Section 1

Constructive Categories and Towers of Categories

Categories

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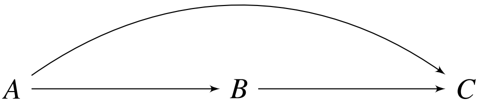

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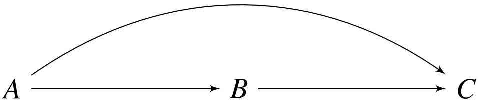

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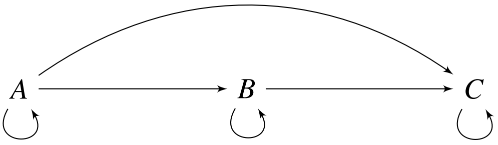

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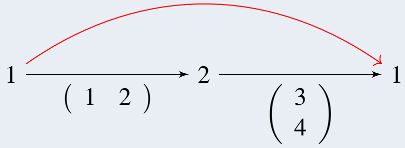

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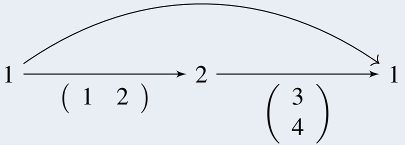

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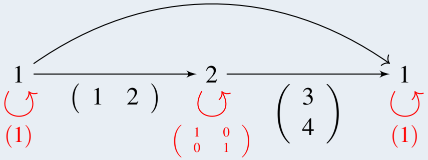

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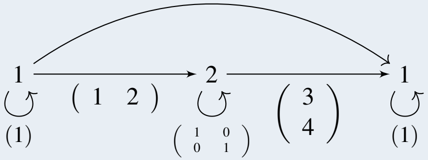

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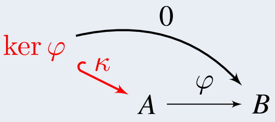

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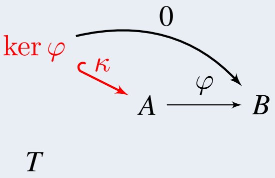

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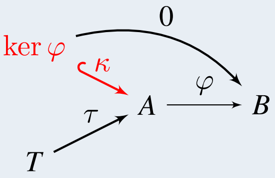

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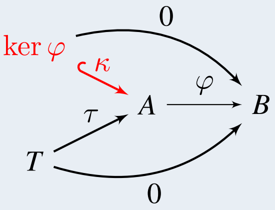

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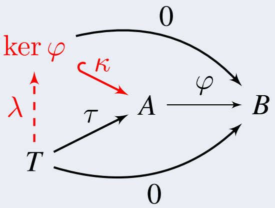

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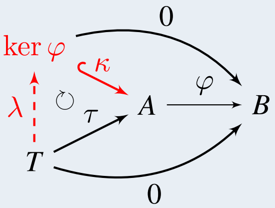

Category Constructors in CAP

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- expresses all defining categorical operations of the doctrine $\mathbf{D}$ (of its output category) as algorithms written in terms of the provided input data.

Examples: An acyclic quiver $\mathfrak{q}$ and a field k

Let k be a field and let $\mathfrak{q}$ be an acyclic quiver.

$\mathfrak{q}$ k

Examples: The free category over $\mathfrak{q}$

Let k be a field and let $\mathfrak{q}$ be an acyclic quiver.

$$\begin{array}{ccc} \mathcal{F}_{\mathfrak{q}} & & \\ \big\downarrow & & \\ \mathfrak{q} & & k \end{array}$$

Examples: The k-linear closure category

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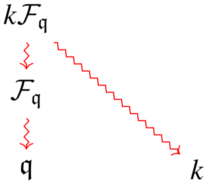

Examples: The quotient category by two-sided ideal

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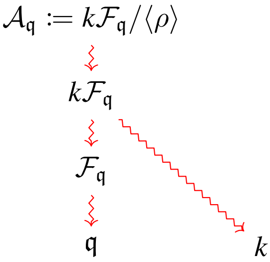

ρ is a finite set in $\mathrm{Mor}(k\mathcal{F}_{\mathfrak{q}})$.

Examples: The category of matrices

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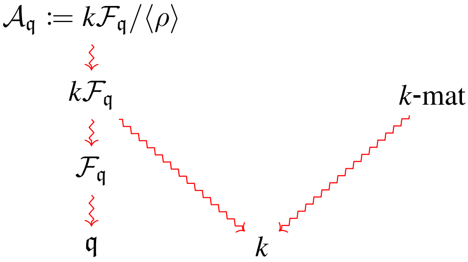

Examples: The f.d. ρ-bounded quiver representations

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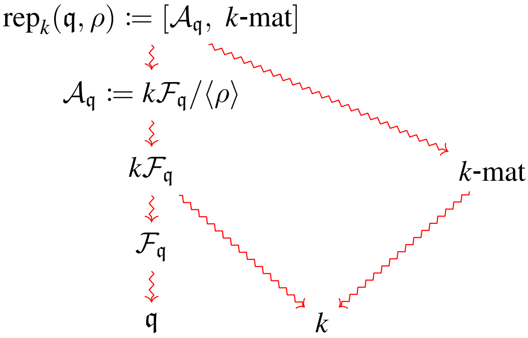

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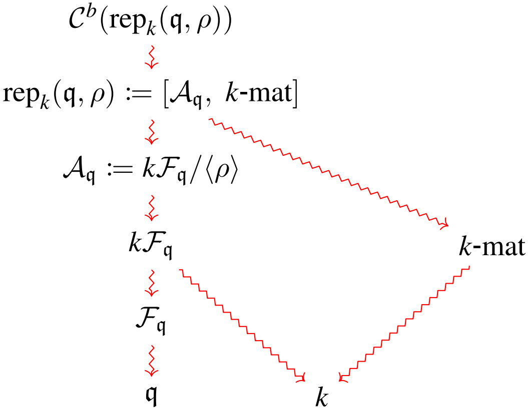

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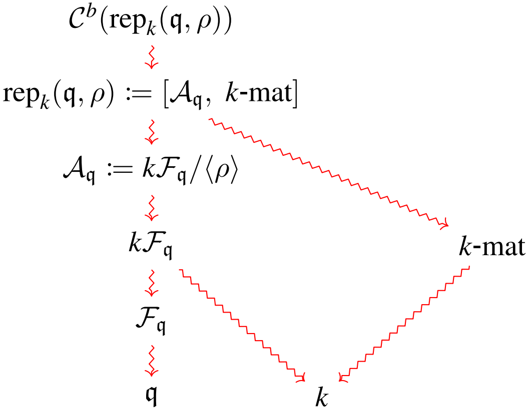

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Examples: The bounded homotopy category?

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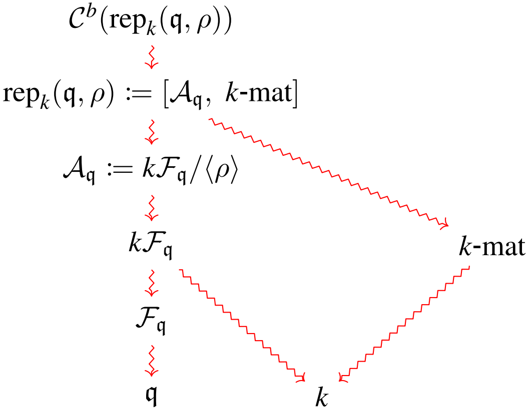

What about the bounded homotopy category $\mathcal{K}^b(\mathrm{rep}_k(\mathfrak{q}, \rho))$?

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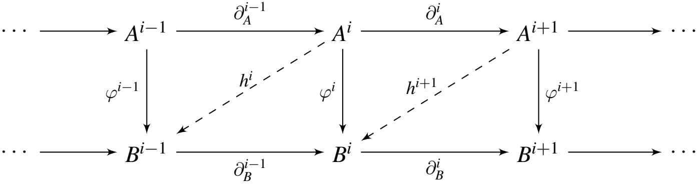

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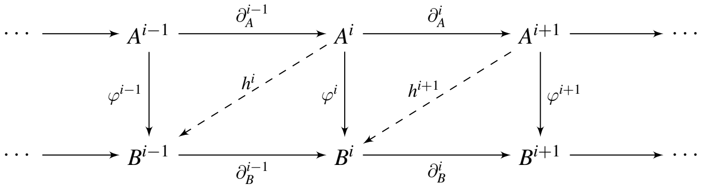

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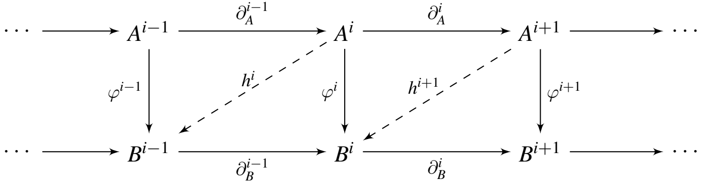

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- That is, two morphisms $\varphi, \psi : A \to B$ in $\mathcal{K}^b(\mathcal{P})$ are equal if and only if there exists a solution to the linear system

$$\partial_A^i \bullet \chi^{i+1} + \chi^i \bullet \partial_B^{i-1} = \varphi^i - \psi^i$$

where i varies over a finite set of values.

Section 2

Homomorphism Structures

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  $$\nu(\alpha \bullet \chi \bullet \beta) = \nu(\chi) \bullet H(\alpha, \beta)$$

  for all compatible triples α, χ, β.

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- The induced equality:

$$\mathbf{vec}(M \bullet U \bullet N) = \mathbf{vec}(U) \bullet (M^T \otimes N)$$

is known as the Kronecker product trick for solving two-sided matrix equations.

Two-sided linear systems

Two-sided linear systems

- Let $\mathcal{P}$ be an additive category. A **linear system** in $\mathcal{P}$ is a collection of morphisms α_{ij}, β_{ij} and γ_i of the following form:

$$
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\alpha_{11} \bullet X_1 \bullet \beta_{11} & + \cdots + & \alpha_{1n} \bullet X_n \bullet \beta_{1n} & = & \gamma_1 \\
\vdots & \vdots & \vdots & & \vdots \\
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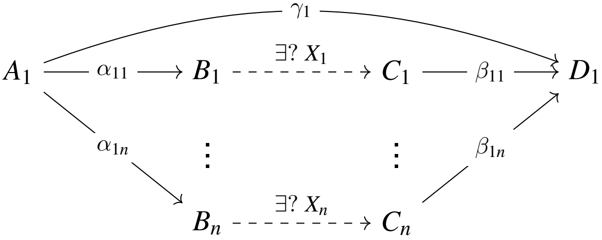

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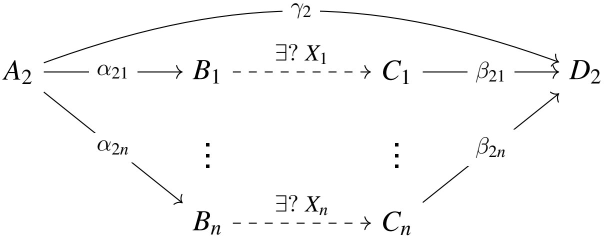

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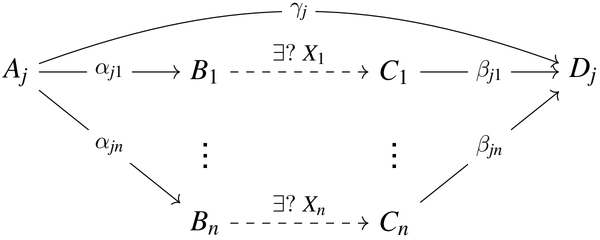

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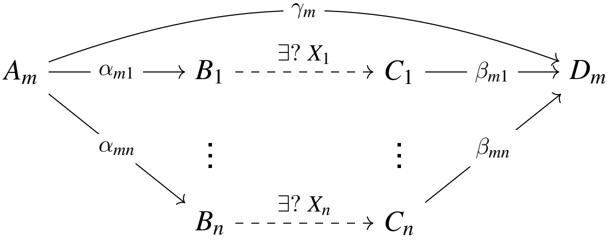

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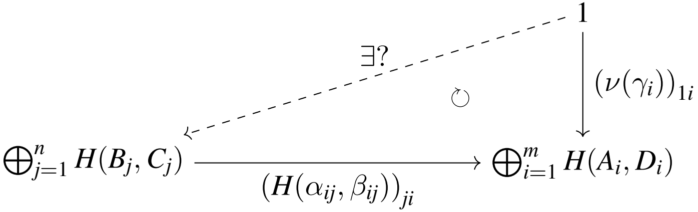

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$$\begin{array}{ccc} & & 1 \\ & \overset{(\nu(X_j))_{1j}}{\swarrow} \quad \circlearrowleft & \downarrow (\nu(\gamma_i))_{1i} \\ \bigoplus_{j=1}^{n} H(B_j, C_j) & \xrightarrow[(H(\alpha_{ij}, \beta_{ij}))_{ji}]{} & \bigoplus_{i=1}^{m} H(A_i, D_i) \end{array}$$

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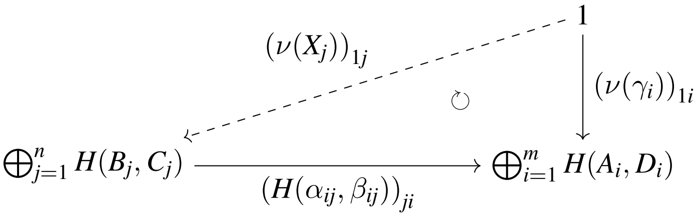

- That is, we can transfer a two-sided linear system in $\mathcal{P}$ to a one-sided linear equation in $\mathcal{D}$.

Equality of morphisms in homotopy categories

Theorem

Let $\mathcal{P}$ be an additive category equipped with a $\mathcal{D}$-homomorphism structure $(1, H(-,-), \nu)$ such that $\mathcal{D}$ has decidable lifts, i.e., we can decide in $\mathcal{D}$ the solvability of the one-sided equation $\mathcal{Y} \bullet \beta = \gamma$. Then

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1. $\mathcal{K}^b(\mathcal{P})$ has decidable equality of morphisms,
2. If $\mathcal{D}$ is Abelian and $1 \in \mathcal{D}$ is a projective object, then $\mathcal{C}^b(\mathcal{P})$ and $\mathcal{K}^b(\mathcal{P})$ can be equipped with a $\mathcal{D}$-homomorphism structure.

Example: An acyclic quiver $\mathfrak{q}$ and a field k

Let k be a field and let $\mathfrak{q}$ be an acyclic quiver.

$\mathfrak{q}$ k

Example: The free category over $\mathfrak{q}$

Let k be a field and let $\mathfrak{q}$ be an acyclic quiver.

$$\begin{array}{ccc} \mathcal{F}_{\mathfrak{q}} & & \\ \rightsquigarrow & & \\ \mathfrak{q} & & k \end{array}$$

Example: The k-linear closure category

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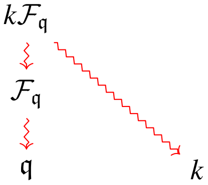

Example: The quotient category by two-sided ideal

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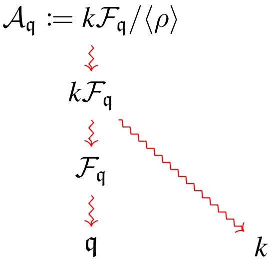

Example: The category of matrices

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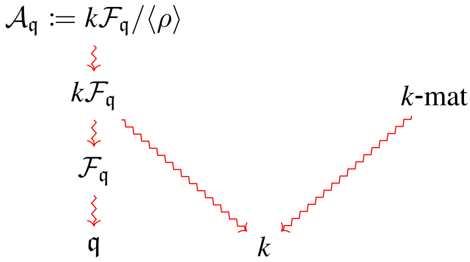

Example: The f.d. ρ-bounded quiver representations

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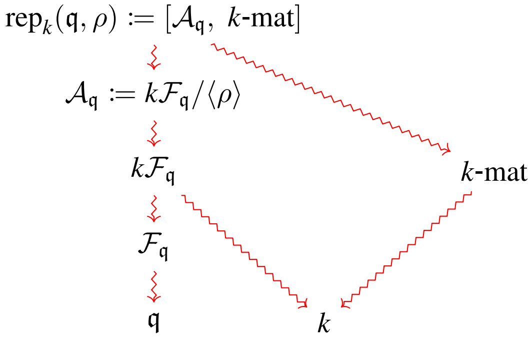

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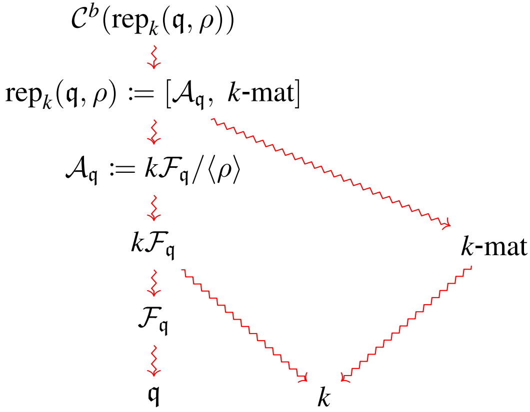

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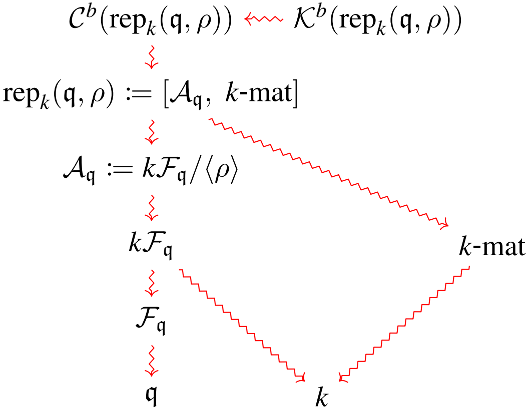

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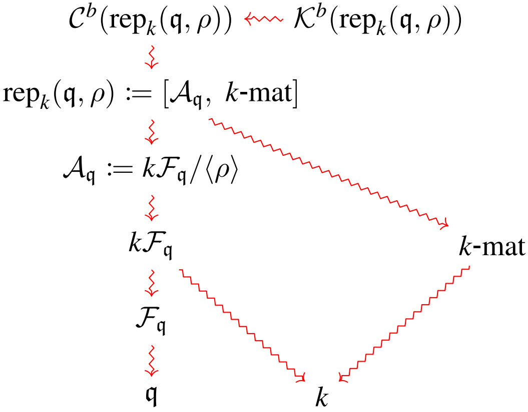

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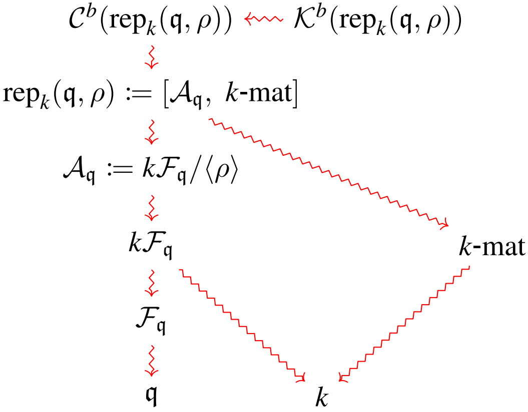

What about the bounded derived category $\mathcal{D}^b(\mathrm{rep}_k(\mathfrak{q},\rho))$?

Example: The subcategory of projective objects

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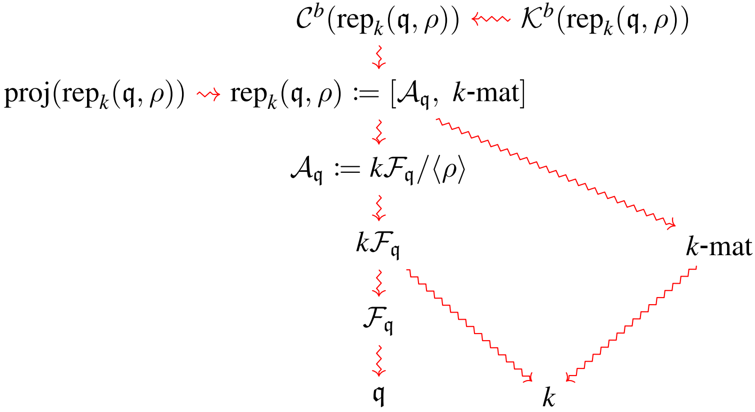

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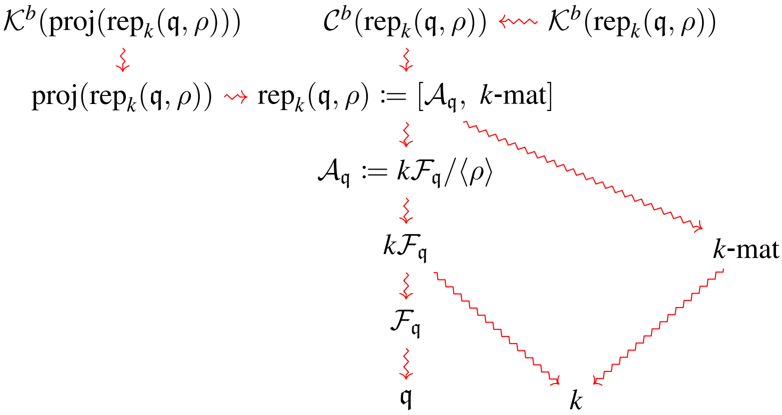

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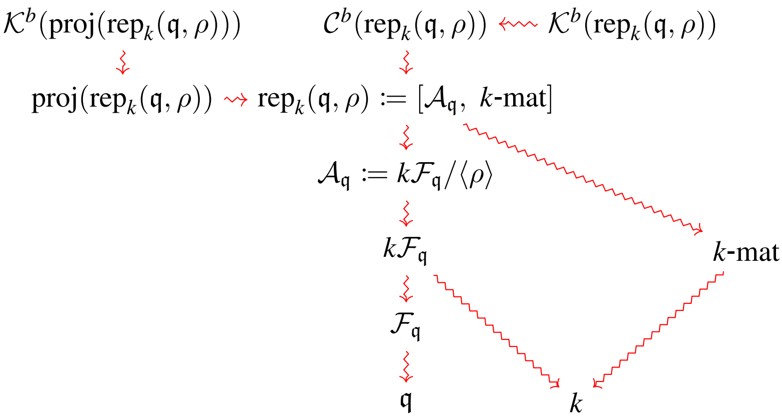

If $\mathrm{rep}_k(\mathfrak{q},\rho)$ has a f.g. dimension then $\mathcal{D}^b(\mathrm{rep}_k(\mathfrak{q},\rho)) \simeq \mathcal{K}^b(\mathrm{proj}(\mathrm{rep}_k(\mathfrak{q},\rho)))$.

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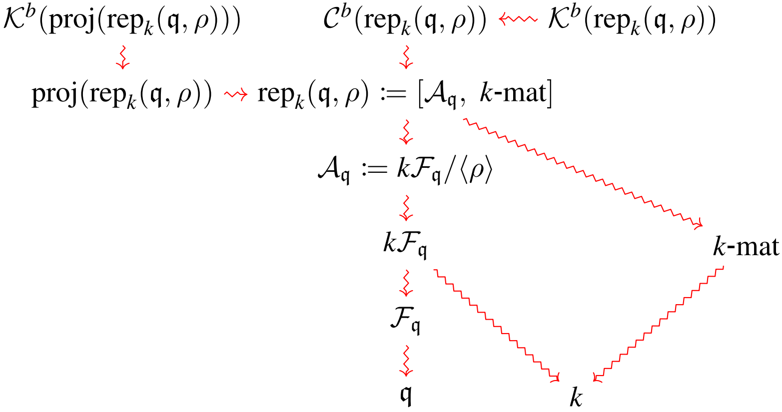

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Section 3

Tilting Equivalences in Bounded Homotopy Categories

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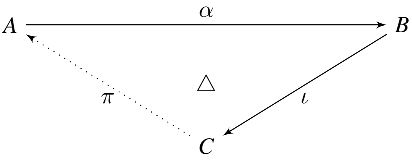

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- The rotation axiom, the octahedral axiom, etc ...

Example

(Bounded) homotopy and derived categories are triangulated.

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- An exceptional sequence is called **full** if the smallest triangulated subcategory generated by $\mathcal{E}$, denoted by $\mathcal{E}^{\triangle}$, equals $\mathfrak{T}$.

Theorem: Equivalences induced by strong exceptional sequences

Let k be a field, $\mathcal{P}$ a k-linear Hom-finite category and $\mathcal{E}$ is strong exceptional sequence in $\mathcal{K}^b(\mathcal{P})$. Then, there exists an acyclic quiver $\mathfrak{q}_{\mathcal{E}}$ and set of relations $\rho \subset \mathrm{Mor}(k\mathcal{F}_{\mathfrak{q}_{\mathcal{E}}})$ such that

$$\mathcal{D}^b(\mathrm{rep}_k(\mathfrak{q}_{\mathcal{E}}, \rho)) \simeq \mathcal{K}^b(\mathcal{E}^{\oplus}) \simeq \mathcal{E}^{\triangle} \subseteq \mathcal{K}^b(\mathcal{P})$$

where $\mathcal{E}^{\oplus}$ is the full subcategory of $\mathcal{K}^b(\mathcal{P})$ generated by all finite direct sums of objects in $\mathcal{E}$.

The convolution functor in one slide

The convolution functor is the functor $\mathcal{K}^b(\mathcal{E}^{\oplus}) \xrightarrow{F} \mathcal{E}^{\triangle} \subseteq \mathcal{K}^b(\mathcal{P})$. Let U be an object in $\mathcal{K}^b(\mathcal{E}^{\oplus})$ with lower bound m and upper bound n. Then $F(U)$ can be computed as follows:

$$\cdots \xrightarrow{\partial^{n-4}} U^{n-3} \xrightarrow{\partial^{n-3}} U^{n-2} \xrightarrow{\partial^{n-2}} U^{n-1} \xrightarrow{\partial^{n-1}} U^{n} \longrightarrow 0$$

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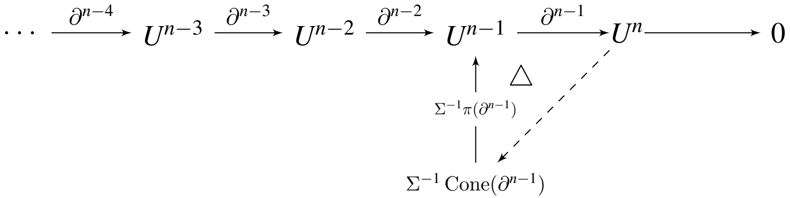

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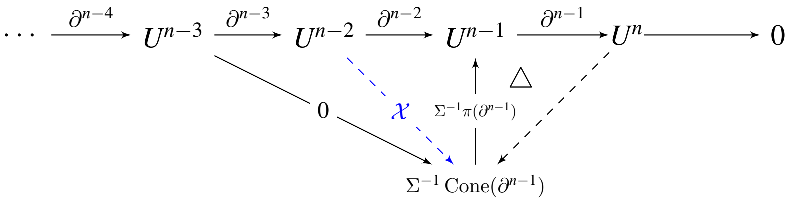

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- Compute $\mathcal{X}$ with $\mathcal{X} \bullet \Sigma^{-1}\pi(\partial^{n-1}) = \partial^{n-2}$ and $\partial^{n-3} \bullet \mathcal{X} = 0$.
- Repeat the procedure for the following new object U:

$$\cdots \xrightarrow{\partial^{n-4}} U^{n-3} \xrightarrow{\partial^{n-3}} U^{n-2} \xrightarrow{\mathcal{X}} \Sigma^{-1}\operatorname{Cone}(\partial^{n-1}) \longrightarrow 0 \longrightarrow 0$$

The convolution functor in one slide

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Convolution functor also depends on solving two-sided linear systems.

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Thank you for your attention