Power Series Formulas With m-fold Hypergeometric Term Coefficients

Tagung der Fachgruppe Computeralgebra

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Outline

1. Introduction
2. *m*-fold Hypergeometric Term Solutions of Holonomic Recurrence Equations
3. Computing Hypergeometric Type Power Series
4. Live Computations

1. Introduction

Motivation

Let $\mathbb{K}$ be a field of characteristic zero, and

$$ {}_sF_t\left(\begin{matrix} a_1 & \cdots & a_s \\ b_1 & \cdots & b_t \end{matrix} \middle| z\right) := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_s)_n}{(b_1)_n \cdots (b_t)_n n!} z^n = \sum_{n=0}^{\infty} A_n z^n, \tag{1} $$

the generalized hypergeometric series. We have $A_{n+1}/A_n \in \mathbb{K}(n)$.

Some examples of functions related to this family are:

$$ {}_0F_0\left(\begin{matrix} - \\ - \end{matrix} \middle| z\right) = e^z \tag{2} $$

$$ {}_1F_0\left(\begin{matrix} -\alpha \\ - \end{matrix} \middle| -z\right) = (1+z)^{\alpha} \tag{3} $$

$$ z \cdot {}_0F_1\left(\begin{matrix} - \\ 3/2 \end{matrix} \middle| -\frac{z^2}{4}\right) = \sin(z) \tag{4} $$

Motivation

The sine function satisfies the recurrence equation

$$(n+1)(n+2)\cdot a_{n+2}+a_n=0 \quad\Rightarrow\quad a_{n+2}/a_n\in\mathbb{Q}(n). \tag{5}$$

In [Koepf, 1992], such a function is called function of hypergeometric type with type $m=2$.

m-fold hypergeometric term

A term a_n is said to be an m-fold hypergeometric term, for $m\in\mathbb{N}$, if the term ratio a_{n+m}/a_n is a rational function over $\mathbb{K}$.

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Algorithmic approach

1. Compute a holonomic differential equation;
2. Convert the differential equation obtained into a recurrence equation (holonomic) for the power series coefficients;
3. Solve the recurrence equation obtained and use initial values to deduce the power series formula.

Limitations

1. The recurrence equation obtained is not always a two-term recurrence relation. For $f(z) := \sqrt{1+z} + 1/\sqrt{1+z}$, we get the recurrence equation

$$4(n+1)\cdot a_{n+1} + 6n\cdot a_n + (2n-3)\cdot a_{n-1} = 0. \tag{6}$$

Thanks to Petkovsšek's and van Hoeij's algorithms, the hypergeometric case ($m = 1$) could already be investigated.
Using `convert/FormalPowerSeries` in Maple 2021 one gets the left-hand side below

$$\sum_{n=0}^{\infty} \frac{(-1)^n \Gamma\left(-\frac{1}{2}+n\right)(n-1)}{n\sqrt{n}} z^n = \sum_{n=0}^{\infty} \frac{2\,(n-1)\,(-1)^n\,(2n)!\,z^n}{(2n-1)\,4^n\,n!^2}. \tag{7}$$

Limitations

2. When the recurrence equation obtained has many m-fold hypergeometric term solutions, what is the type of the resulting power series? How to compute those m-fold hypergeometric term solutions? (see also [Ryabenko, 2002])

mfoldHyper.

Limitations

2. When the recurrence equation obtained has many *m*-fold hypergeometric term solutions, what is the type of the resulting power series? How to compute those *m*-fold hypergeometric term solutions? (see also [Ryabenko, 2002])

   mfoldHyper.

3. How to find the linear combination of series summations involved in the representation?
4. How to consider Puiseux series?

Definition

Hypergeometric type power series

We call hypergeometric type series a linear combination of series of the form

$$\sum_{n=n_0}^{\infty} a_n \cdot (z - z_0)^{n/p} \; (k \in \mathbb{N}, n_0 \in \mathbb{Z}), \tag{8}$$

where a_n is an m-fold hypergeometric term. That is a linear combination of Laurent-Puiseux series whose coefficients are m-fold hypergeometric terms.

We give an overview of **mfoldHyper** and our algorithmic approach to compute hypergeometric type power series.

Examples

Examples

$$\exp\left(\sqrt{z^3}\right) + \log\left(1 + z^2\right) = \sum_{n=0}^{\infty} \frac{z^{\frac{3\cdot n}{2}}}{n!} + \sum_{n=0}^{\infty} \frac{(-1)^n \cdot z^{2\cdot(1+n)}}{n+1} \tag{9}$$

$$\sqrt{\sqrt{8z^3+1}-1} + \sqrt{13z^4+7} = \sum_{n=0}^{\infty} \frac{2\left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n (-8)^n 4^n z^{3n+\frac{3}{2}}}{(2n+1)!} - \sum_{n=0}^{\infty} \frac{(-13)^n (2n)! z^{4n}}{7^{n-\frac{1}{2}} (2n-1)\, 4^n n!^2} \tag{10}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{1-n} (2n+1)(4n)! z^{3n+\frac{3}{2}}}{(2n+1)!^2} + \sum_{n=0}^{\infty} -\frac{4^{-n} 7^{\frac{1}{2}-n} (-1)^n 13^n (2n)! z^{4n}}{(2n-1)\, n!^2} \tag{11}$$

2. m-fold Hypergeometric Term Solutions of Holonomic Recurrence Equations

Computing *m*-fold Hypergeometric Terms

We consider the holonomic recurrence equation

$$P_d(n) \cdot a_{n+d} + P_{d-1}(n) \cdot a_{n+d-1} + \cdots + P_0(n) \cdot a_n = 0, \tag{12}$$

$P_j(n) \in \mathbb{K}[n], 0 \leqslant j \leqslant d$ such that $P_0(n) \cdot P_d(n) \neq 0$, and $d \in \mathbb{N}$.
The goal is to compute **a basis** of all m-fold hypergeometric term solutions of (12), $m \in \mathbb{N}$.

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m-fold hypergeometric term

If a_n is an m-fold hypergeometric term for $m \in \mathbb{N}$ then there exists $r(n) \in \mathbb{K}(n)$ such that

$$r(m \cdot n + l) = \frac{a_{m\cdot(n+1)+l}}{a_{m\cdot n+l}}, \tag{13}$$

$0 \leqslant l \leqslant m - 1$.

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For $m \in \mathbb{N}$, we consider the case $l = 0$. Note that the other representations can easily be deduced afterwards.

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First we need to bound the value of m (see [Teguia Tabuguia and Koepf, 2022]).

Lemma

Let h_n be an m-fold hypergeometric term, $m \in \mathbb{N}$. Assume

$$\forall u \in \mathbb{N},\ u < m, \quad \text{there is no rational function } r_u(n) \in \mathbb{K}(n) \ :\ h_{u+n} = r_u(n) \cdot h_n. \tag{14}$$

Then there is no holonomic recurrence equation over $\mathbb{K}$ of order less than m satisfied by h_n.

Therefore all m-fold hypergeometric term solutions of (12) satisfy $m \leq d$.

Computing m-fold Hypergeometric Terms

Definition (m-fold holonomic recurrence equation)

A holonomic recurrence equation of the indeterminate sequence (a_n) is said to be m-fold holonomic, $m \in \mathbb{N}$, if it has at least two non-zero polynomial coefficients and the difference between indices of two appearing terms of (a_n) in that equation is a multiple of m. Choosing 0 as the trailing term order gives the general form

$$P_d(n) \cdot a_{n+m\cdot d} + P_{d-1}(n) \cdot a_{n+m\cdot(d-1)} + \cdots + P_0(n) \cdot a_n = 0, \tag{15}$$

with $P_d \cdot P_0 \neq 0$.

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with $P_d \cdot P_0 \neq 0$.

Example

- 2-fold:

  $$(2+n) \cdot (4+n) \cdot (6+n) \cdot a_{n+6} - 2 \cdot (2+n) \cdot (4+n) \cdot a_{n+4} + 4 \cdot (2+n) \cdot a_{n+2} - 8 \cdot a_n = 0.$$

- 4-fold:

  $$(4+n) \cdot (8+n) \cdot (12+n) \cdot a_{n+12} - 4 \cdot (4+n) \cdot (8+n) \cdot a_{n+8} + 64 \cdot (4+n) \cdot a_{n+4} - 256 \cdot a_n = 0.$$

Computing m-fold Hypergeometric Terms

m-fold hypergeometric term solutions of m-fold holonomic REs

Idea: compute m-fold hypergeometric terms as we do for hypergeometric terms.

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By application of the change of variable

$$\begin{cases} m \cdot k = n \\ s_k = a_{m\cdot k} \end{cases}, \qquad (16)$$

every basis of m-fold hypergeometric term solutions of the m-fold holonomic RE

$$P_d(n) \cdot a_{n+m\cdot d} + P_{d-1}(n) \cdot a_{n+m\cdot(d-1)} + \cdots + P_0(n) \cdot a_n = 0$$

is a basis of hypergeometric term solutions of the 1-fold holonomic RE

$$P_d(m \cdot k) \cdot s_{k+d} + P_{d-1}(m \cdot k) \cdot s_{k+(d-1)} + \cdots + P_0(m \cdot k) \cdot s_k = 0.$$

Computing m-fold Hypergeometric Terms

Example: m-fold hypergeometric term solutions of m-fold holonomic REs

- RE
$$(2+n)\cdot(4+n)\cdot(6+n)\cdot a_{n+6} - 2\cdot(2+n)\cdot(4+n)\cdot a_{n+4} + 4\cdot(2+n)\cdot a_{n+2} - 8\cdot a_n = 0.$$
- Change of variable
$$\begin{cases} 2\cdot k = n \\ s_k = a_{2\cdot k} \end{cases}.$$

Computing m-fold Hypergeometric Terms

Example: m-fold hypergeometric term solutions of m-fold holonomic REs

- RE

$$(2+n)\cdot(4+n)\cdot(6+n)\cdot a_{n+6}-2\cdot(2+n)\cdot(4+n)\cdot a_{n+4}+4\cdot(2+n)\cdot a_{n+2}-8\cdot a_n=0.$$

- Change of variable

$$\begin{cases} 2\cdot k = n \\ s_k = a_{2\cdot k} \end{cases}.$$

- Corresponding 1-fold holonomic RE

$$(2+2\cdot k)\cdot(4+2\cdot k)\cdot(6+2\cdot k)\cdot s_{k+3}-2\cdot(2+2\cdot k)\cdot(4+2\cdot k)\cdot s_{k+2} + 4\cdot(2+2\cdot k)\cdot s_{k+1}-8\cdot s_k=0.$$

- One gets the basis of all 2-fold hypergeometric term solutions

$$\left\{\frac{1}{k!}\right\}, \text{ and replacing } k \text{ by } n \text{ yields } \left\{\frac{1}{n!}\right\} \text{ for } a_{2n}.$$

Computing *m*-fold Hypergeometric Terms

Question: How to generalize these computations to an arbitrary holonomic RE?

Computing m-fold Hypergeometric Terms

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First, we introduce some other definitions.

Definition (m-fold holonomic distinct and m-fold holonomic equivalent REs)

Let $m \in \mathbb{N}$,

$$RE_1 : \; P_{d_1} a_{n+k_1+m\cdot d_1} + P_{d_1-1} a_{n+k_1+m\cdot(d_1-1)} + \cdots + P_{0_1} a_{n+k_1} = 0, \tag{17}$$

and

$$RE_2 : \; P_{d_2} a_{n+k_2+m\cdot d_2} + P_{d_2-1} a_{n+k_2+m\cdot(d_2-1)} + \cdots + P_{0_2} a_{n+k_2} = 0 \tag{18}$$

be two m-fold holonomic recurrence equations.

- We say that RE_1 and RE_2 are m-fold distinct holonomic equations if $k_2 - k_1$ is not divisible by m.
- We say that RE_1 and RE_2 are m-fold equivalent holonomic equations if $k_2 - k_1$ is divisible by m.

Computing m-fold Hypergeometric Terms

Example (3-fold holonomic distinct and 3-fold holonomic equivalent REs)

$$\begin{aligned} RE_1 &: \; P_{1,3} \cdot a_{n+7} + P_{1,2} \cdot a_{n+4} + P_{1,1} \cdot a_{n+1} = 0, \\ RE_2 &: \; P_{2,4} \cdot a_{n+11} + P_{2,3} \cdot a_{n+8} + P_{2,2} \cdot a_{n+5} + P_{2,1} \cdot a_{n+2} = 0, \\ RE_3 &: \; P_{3,4} \cdot a_{n+13} + P_{3,3} \cdot a_{n+10} + P_{3,2} \cdot a_{n+7} + P_{3,1} \cdot a_{n+4} = 0. \end{aligned} \tag{19}$$

RE_1 and RE_2 are 3-fold holonomic distinct whereas RE_1 and RE_3 are 3-fold equivalent.

Computing m-fold Hypergeometric Terms

The major result of this talk is (see [Teguia Tabuguia and Koepf, 2022]) stated below.

Theorem (Structure of holonomic REs having m-fold hypergeometric term solutions)

Let $m \in \mathbb{N}$, $\mathbb{K}$ a field of characteristic zero, and h_n be an m-fold hypergeometric term which is not u-fold hypergeometric over $\mathbb{K}$ for all positive integers $u < m$. Then h_n is a solution of a given holonomic recurrence equation, if that equation is a linear combination of m-fold holonomic recurrence equations, such that h_n is a solution of each m-fold holonomic distinct recurrence equation of that linear combination.

Computing *m*-fold Hypergeometric Terms

The major result of this talk is (see [Teguia Tabuguia and Koepf, 2022]) stated below.

Theorem (Structure of holonomic REs having *m*-fold hypergeometric term solutions)

Let $m \in \mathbb{N}$, $\mathbb{K}$ a field of characteristic zero, and h_n be an m-fold hypergeometric term which is not u-fold hypergeometric over $\mathbb{K}$ for all positive integers $u < m$. Then h_n is a solution of a given holonomic recurrence equation, if that equation is a linear combination of m-fold holonomic recurrence equations, such that h_n is a solution of each m-fold holonomic distinct recurrence equation of that linear combination.

A consequence of this theorem is our algorithm for computing a basis of the subspace of all *m*-fold hypergeometric term solutions of an arbitrary holonomic recurrence equation: mfoldHyper.

Computing m-fold Hypergeometric Terms

Algorithm mfoldHyper

Let

$$P_d(n) \cdot a_{n+d} + P_{d-1}(n) \cdot a_{n+d-1} + \cdots + P_0(n) \cdot a_n = 0, \tag{20}$$

- Set $m = 1$.
- Repeat
  1. If (20) is a linear combination of m-fold holonomic REs then go to step 2. Otherwise go to step 4.
  2. Compute bases of m-fold hypergeometric term solutions of each m-fold distinct holonomic RE in the linear combination found in step 1.
  3. Collect all m-fold hypergeometric terms that are linearly dependent to an element of each basis of m-fold hypergeometric term solutions computed in step 2.
  4. Increment m and go back to step 1.
- Until $m = d$.

Computing m-fold Hypergeometric Terms

Example

It turns out that our previous example also has 4-fold hypergeometric term solutions.

$$(2+n)\cdot(4+n)\cdot(6+n)\cdot a_{n+6}-2\cdot(2+n)\cdot(4+n)\cdot a_{n+4}+4\cdot(2+n)\cdot a_{n+2}-8\cdot a_n=0.$$

- Two 4-fold holonomic REs:

$$-2\cdot(2+n)\cdot(4+n)\cdot a_{n+4}-8\cdot a_n=0,$$

  and

$$(2+n)\cdot(4+n)\cdot(6+n)\cdot a_{n+6}+4\cdot(2+n)\cdot a_{n+2}=0.$$

- The corresponding bases of 4-fold hypergeometric term solutions are, respectively,

$$\left\{\frac{(-1)^n}{(2\cdot n)!}\right\}, \quad \text{and} \quad \left\{\frac{(-1)^n}{(2\cdot n)!}\right\}$$

  and therefore identical.

Computing m-fold Hypergeometric Terms

Example

- Therefore since no other linear combination holds for this example, our algorithm finds the following basis of all m-fold hypergeometric term solutions

$$\left[\left[2, \left\{\frac{1}{n!}\right\}\right], \left[4, \left\{\frac{(-1)^n}{(2 \cdot n)!}\right\}\right]\right].$$

Computing *m*-fold Hypergeometric Terms

Example

- Therefore since no other linear combination holds for this example, our algorithm finds the following basis of all *m*-fold hypergeometric term solutions

$$\left[\left[2, \left\{\frac{1}{n!}\right\}\right], \left[4, \left\{\frac{(-1)^n}{(2\cdot n)!}\right\}\right]\right].$$

For our second example

$$(4+n)\cdot(8+n)\cdot(12+n)\cdot a_{n+12} - 4\cdot(4+n)\cdot(8+n)\cdot a_{n+8} + 64\cdot(4+n)\cdot a_{n+4} - 256\cdot a_n = 0,$$

mfoldHyper finds the basis of *m*-fold hypergeometric term solutions

$$\left[\left[4, \left\{\frac{1}{n!}\right\}\right], \left[8, \left\{\frac{(-1)^n\cdot 4^n}{(2\cdot n)!}\right\}\right]\right].$$

These are the needed bases, respectively, to determine the coefficients of the power series of $\exp(z^2)+\cos(z^2)$ and $\exp(z^2)+\cos(z^4)\cdot\sin(z^4)$.

3. Computing Hypergeometric Type Power Series

Computing Hypergeometric Type Power Series

As a first step, we generalize the algorithm in [Koepf, 1992] by the use of mfoldHyper. This is possible thanks to the following theorem.

Theorem ([Koepf, 1992])

Every hypergeometric type function is holonomic.

The goal is to compute power series of the form

$$f(z) = T(z) + F(z), \tag{21}$$

where $T(z) \in \mathbb{K}[\log(z)][z, \frac{1}{z}]$ and $F(z)$ is a hypergeometric type power series.

Computing Hypergeometric Type Power Series

Consider $f(z) = \arctan(z^3) + \log(1 + z^2)$.

Computing Hypergeometric Type Power Series

Consider $f(z) = \operatorname{arctan}(z^3) + \log(1 + z^2)$.

The computed recurrence equation is

$$2\cdot(n-1)\cdot(1+n)\cdot a_{n+1} + 3\cdot(n-3)\cdot n\cdot a_n - 2\cdot(n-5)\cdot(n-1)\cdot a_{n-1} + 2\cdot(n-9)\cdot(n-3)\cdot a_{n-3} + 2\cdot (n-1)\cdot a_{n-5} + 3\cdot(n-6)\cdot(n-3)\cdot a_{n-6} - 2\cdot(n-7)\cdot(n-5)\cdot a_{n-7} + 2\cdot(n-9)^2\cdot a_{n-9} = 0, \tag{22}$$

converted from the differential equation

$$z\cdot\left(1+z^2\right)\cdot\left(1-z^2+z^4\right)\cdot\left(2+3\cdot z-2\cdot z^2+2\cdot z^4\right)\cdot\left(\frac{d^2}{d\,z^2}\mathrm{F}(z)\right) + 2\cdot\left(-1-3\cdot z+3\cdot z^2-5\cdot z^4+5\cdot z^6+6\cdot z^7-3\cdot z^8+z^{10}\right)\cdot\left(\frac{d}{d\,z}\mathrm{F}(z)\right) = 0. \tag{23}$$

Computing Hypergeometric Type Power Series

Using algorithm mfoldHyper one finds the following basis of m-fold hypergeometric term solutions:

$$\left[\left[2,\left\{\frac{(-1)^n}{n}\right\}\right],\left[6,\left\{\frac{(-1)^n}{n}\right\}\right]\right]. \tag{24}$$

Computing Hypergeometric Type Power Series

Using algorithm mfoldHyper one finds the following basis of m-fold hypergeometric term solutions:

$$\left[\left[2, \left\{\frac{(-1)^n}{n}\right\}\right], \left[6, \left\{\frac{(-1)^n}{n}\right\}\right]\right]. \tag{24}$$

Therefore for the power series representation we need a linear combination of the series coefficients

$$\left[\left[2, \left[\frac{(-1)^n \cdot z^{2\cdot n}}{n}, \frac{(-1)^n \cdot z^{1+2\cdot n}}{2\cdot n+1}\right]\right], \left[6, \left[\frac{(-1)^n \cdot z^{6\cdot n}}{n}, \frac{(-1)^n \cdot z^{1+6\cdot n}}{6\cdot n+1}, \frac{(-1)^n \cdot z^{2+6\cdot n}}{3\cdot n+1}, \frac{(-1)^n \cdot z^{3+6\cdot n}}{2\cdot n+1}, \frac{(-1)^n \cdot z^{4+6\cdot n}}{3\cdot n+2}, \frac{(-1)^n \cdot z^{5+6\cdot n}}{6\cdot n+5}\right]\right]\right]. \tag{25}$$

Computing Hypergeometric Type Power Series

We evaluate these coefficients for a certain number of initial values and build a linear system by equating the evaluated coefficients with those of the Taylor expansion of the **minimal order** that leads to the linear combination sought if it exists.

Power series of $\arctan(z^3) + \log(1 + z^2)$

$$f(z) := \arctan(z^3) + \log(1 + z^2) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot z^{3+6\cdot n}}{2 \cdot n + 1} + \sum_{n=0}^{\infty} \frac{(-1)^n \cdot z^{2\cdot(1+n)}}{n+1}. \tag{26}$$

Computing Hypergeometric Type Power Series

For a more general case let us take $f(z) := (1+z^2)\cdot\exp(z) + \text{arcsech}(\sqrt{z})$. We find the RE

$$4\cdot(1+n)\cdot(2+n)^2\cdot a_{n+2} + 2\cdot(1+n)^2\cdot(5\cdot n-12)\cdot a_{n+1} - 9\cdot(n-1)\cdot n\cdot(1+2\cdot n)\cdot a_n - (n-1)\cdot\left(95-112\cdot n+14\cdot n^2\right)\cdot a_{n-1} + (n-2)\cdot\left(49-67\cdot n+14\cdot n^2\right)\cdot a_{n-2} + (n-3)\cdot\left(103-52\cdot n+4\cdot n^2\right)\cdot a_{n-3} - 2\cdot(n-4)\cdot(2\cdot n-7)\cdot a_{n-4} = 0. \quad (27)$$

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$$-(n-1)\cdot\left(95-112\cdot n+14\cdot n^2\right)\cdot a_{n-1}+(n-2)\cdot\left(49-67\cdot n+14\cdot n^2\right)\cdot a_{n-2}+(n-3)$$
$$\cdot\left(103-52\cdot n+4\cdot n^2\right)\cdot a_{n-3}-2\cdot(n-4)\cdot(2\cdot n-7)\cdot a_{n-4}=0. \quad (27)$$

Puiseux number and starting point

We have established that the roots of the leading and trailing polynomial coefficients indicate

- the Puiseux number: least common multiple of rational root denominators;
- starting point: the maximum integer root incremented by one.

Computing Hypergeometric Type Power Series

For this example, we find the Puiseux number 2 and we use the following RE of $f(z^2)$

$$2\cdot(n-1)\cdot(1+n)^2\cdot a_{n+1}+(n-1)^2\cdot(5\cdot n-39)\cdot a_{n-1}-9\cdot(n-5)\cdot(n-3)\cdot(n-2)\cdot a_{n-3}$$
$$-(n-5)\cdot\left(589-154\cdot n+7\cdot n^2\right)\cdot a_{n-5}+(n-7)\cdot\left(362-109\cdot n+7\cdot n^2\right)\cdot a_{n-7}+2\cdot(n-9)$$
$$\cdot\left(190-32\cdot n+n^2\right)\cdot a_{n-9}-4\cdot(n-11)\cdot(n-10)\cdot a_{n-11}=0, \quad (28)$$

whose starting point is 1.

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$$2\cdot(n-1)\cdot(1+n)^2\cdot a_{n+1}+(n-1)^2\cdot(5\cdot n-39)\cdot a_{n-1}-9\cdot(n-5)\cdot(n-3)\cdot(n-2)\cdot a_{n-3}$$
$$-(n-5)\cdot\left(589-154\cdot n+7\cdot n^2\right)\cdot a_{n-5}+(n-7)\cdot\left(362-109\cdot n+7\cdot n^2\right)\cdot a_{n-7}+2\cdot(n-9)$$
$$\cdot\left(190-32\cdot n+n^2\right)\cdot a_{n-9}-4\cdot(n-11)\cdot(n-10)\cdot a_{n-11}=0, \quad (28)$$

whose starting point is 1.

Power series of $f(z) := (1+z^2)\cdot\exp(z)+\operatorname{arcsech}(\sqrt{z})$

Therefore by the same process we find the power series sought after substituting z by $z^{1/2}$ and add the generalized Taylor expansion of order 0.

$$f(z)=\sum_{n=0}^{\infty}\left(\frac{\left((1+n)^2-n\right)}{(1+n)!}-\frac{4^{-1-n}\cdot(1+2\cdot n)!}{(1+n)^2\cdot n!^2}\right)\cdot z^{1+n}-\frac{\log(z)}{2}+\log(2)+1 \quad (29)$$

4. Live Computations

Computations with Maple 2021

Our algorithms are implemented in Maxima and Maple (see [Teguia Tabuguia and Koepf, 2021]). The Maple version will appear this year as the new FormalPowerSeries package. We present some live computations with Maple 2021.

The package is available at:
`http://www.mathematik.uni-kassel.de/~bteguia/FPS_webpage/FPS.htm`

mfoldHyper and Hypergeometric type power series

$$\arcsin(z) + \arctan(3\,z^3) = \sum_{n=0}^{\infty} \frac{(2n)!4^{-n} z^{2n+1}}{(2n+1)\,n!^2} + \sum_{n=0}^{\infty} \frac{3\,(-1)^n\,9^n z^{6n+3}}{2n+1}$$

$$\log\left(1 + \sqrt{z} + z + z^{3/2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n\,z^{n+1}}{n+1} + \sum_{n=0}^{\infty} \frac{(-1)^n\,z^{\frac{n+1}{2}}}{n+1}$$

$$(\sin(z) + \cos(z))^5 = \sum_{n=0}^{\infty} -\frac{(-1)^n\,(5\,9^n + 25^n - 10)\,z^{2n}}{4\,(2n)!} + \sum_{n=0}^{\infty} \frac{5\,(-1)^n\,(3\,9^n - 25^n + 2)\,z^{2n+1}}{4\,(2n+1)!}$$

Thank You!