MATHEMATICS

TECHNISCHE UNIVERSITÄT KAISERSLAUTERN

On programming with algebraic groups

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Algebraic Groups

A quick introduction

Throughout this talk let k be an algebraically closed field.

Example

$\mathrm{GL}_n(\mathrm{k}) \subseteq \mathrm{k}^{n\times n}$ is also a subset of a vector space and the set of roots of the polynomial

$$f : \mathrm{k}^{n\times n} \times \mathrm{k} \longrightarrow \mathrm{k}, \quad (A, y) \mapsto \det(A)y - 1.$$

Zariski topology

Algebraic groups are groups as well as topological spaces equipped with the **Zariski topology**. The closed sets are given by the set of roots of a set of polynomials (for example).

Algebraic Groups

A quick introduction

Algebraic groups

For simplicity we will assume

$$G \leqslant \mathrm{GL}_n(\mathrm{k}),$$

where G is a closed subgroup in $\mathrm{GL}_n(k)$ with respect to the Zariski topology.

Properties

We are especially interested in learning about additional structures in these groups. We will from now on consider *reductive, connected* groups. They include

- maximale tori ?
- Borel subgroups ?
- root systems and root subgroups ?
- Lie algebras ?

Structures in $\mathrm{GL}_n(\mathrm{k})$

- maximal tori are maximal abelian subgroups which are isomorphic to $\mathrm{k}^\times \times \cdots \times \mathrm{k}^\times$.

$$T := \left\{ \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{pmatrix} \middle| \, t_i \in \mathrm{k}^\times \right\} \subseteq \mathrm{GL}_n(\mathrm{k})$$

- Borel subgroups are maximal, closed, connected, and solvable subgroups of an algebraic group G

$$B := \left\{ \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix} \right\} \subseteq \mathrm{GL}_n(\mathrm{k})$$

in particular we for each maximal torus we can find a Borel subgroup such that $T \subseteq B$.

Structures in $\mathrm{GL}_n(\mathrm{k})$

- A root system is given by a subset $\Phi \subseteq \mathrm{Hom}(T, \mathrm{k}^\times)$, in $\mathrm{GL}_n(\mathrm{k})$ we obtain

$$\chi_{ij} : T \longrightarrow \mathrm{k}^\times, \quad \begin{pmatrix} t_1 & & 0 \\ 0 & \ddots & \\ & & t_n \end{pmatrix} \longmapsto t_i t_j^{-1}$$

  with root system $\Phi := \{\chi_{ij} \mid 1 \leqslant i, j \leqslant n,\ i \neq j\}$.
- As for abstract root systems there is a Weyl group,

$$W = {}^{N_G(T)}\!/\!{}_{C_G(T)} \simeq \langle s_\alpha \mid \alpha \in \Phi \rangle.$$

  In $\mathrm{GL}_n(\mathrm{k})$ we have $W \simeq S_n$ generated by permutation matrices.

Structures in $\mathrm{GL}_n(\mathrm{k})$

- For each $\alpha \in \Phi$ we can define root subgroups U_α by taking U_α to be the image of an isomorphism

$$u_\alpha : (\mathrm{k}, +) \longrightarrow U_\alpha \leqslant G,$$

such that $t u_\alpha(c) t^{-1} = u_\alpha(\alpha(t)c)$ for all $t \in T$, $c \in \mathrm{k}$.
In $\mathrm{GL}_n(\mathrm{k})$ we obtain

$$U_{\chi_{ij}} = \left\{ \begin{array}{cc} \begin{matrix} \quad & i \\ \quad & \downarrow \end{matrix} & \\ \begin{pmatrix} 1 & \cdots & a \\ & \ddots & \\ & & 1 \end{pmatrix} & \leftarrow j \end{array} \;\middle|\; a \in \mathrm{k} \right\}.$$

For $u_a = \begin{pmatrix} 1 & \cdots & a \\ & \ddots & \\ & & 1 \end{pmatrix} \in U_{\chi_{ij}}$ and $t \in T$ we get

$$t u_a t^{-1} = u_{\chi_{ij}(t)a} = \begin{pmatrix} 1 & \cdots & \chi_{ij}(t)a \\ & \ddots & \\ & & 1 \end{pmatrix}.$$

Structures in $\mathrm{GL}_n(\mathrm{k})$

- For each algebraic group, we can define a Lie algebra (given by the tangent space), $\mathrm{Lie}(\mathrm{GL}_n(\mathrm{k})) = \mathfrak{gl}_n(\mathrm{k}) = \mathrm{k}^{n \times n}$.
- We have an action of G on $\mathrm{Lie}(G)$:

  $$\mathrm{Ad} : G \longrightarrow \mathrm{GL}(\mathrm{Lie}(G)).$$

  In $\mathrm{GL}_n(\mathrm{k})$ this is given by conjugation: $\mathrm{Ad}(A)(B) = ABA^{-1}$, $A \in \mathrm{GL}_n(\mathrm{k})$ and $B \in \mathrm{k}^{n \times n}$.
- every element in G is given as a multiplication of four group elements, also called Bruhat decomposition:

elements in $\prod_{\alpha \in \Phi^+} U_\alpha$

$$g_w = u' \cdot t \cdot n_w \cdot u$$

element in T

representative of $w \in W$ in $N_G(T)$

Properties

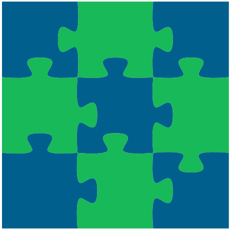
We are especially interested in learning about additional structures in these groups. They include

- maximale tori $\rightsquigarrow$ is something like this: $\begin{pmatrix} t_1 & & 0 \\ 0 & \ddots & \\ & & t_n \end{pmatrix}$
- Borel subgroups $\rightsquigarrow$ looks something like this: $\begin{pmatrix} * & \cdots & * \\ 0 & \ddots & \vdots \\ & & * \end{pmatrix}$
- root systems and root subgroups $\rightsquigarrow \Phi \subseteq \mathrm{Hom}(T, \mathrm{k}^\times)$, with $\chi_{ij} : \begin{pmatrix} t_1 & & 0 \\ 0 & \ddots & \\ & & t_n \end{pmatrix} \longmapsto t_i t_j^{-1}$ and $u_a = \begin{pmatrix} 1 & \ddots & a \\ & \ddots & \\ & & 1 \end{pmatrix} \in U_{\chi_{ij}}$.
- Lie algebras $\rightsquigarrow$ linearisation of problems in algebraic groups, for example the vector space $\mathrm{k}^{n \times n}$

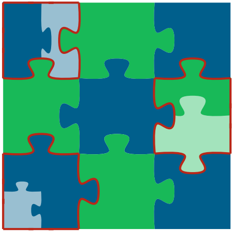
How can we use this for computing?

- The multiplication of group elements is well understood
  - ⇝ In some algebra computer programmes it is even possible to define algebraic groups directly and compute the multiplication with elements
- Action by group elements on the Lie algebra leads to linear combinations in Lie algebra depending on parameters in k
- However, not every computer programme does allow us to do so over algebraic closures
  - ⇝ in Magma, we can work over algebraic closures. We can define Lie algebras and the action of G on $\mathrm{Lie}(G)$ is given by linear maps that can easily be implemented

The problem with the orbits



In “good” characteristic



In small characteristic

Nilpotent Pieces

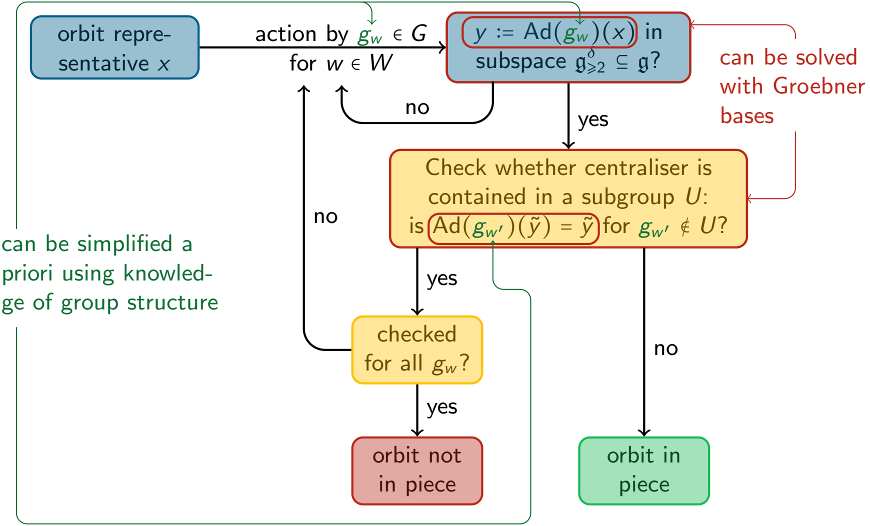
Definition (Nilpotent Pieces)

Let

1. $\delta : \Phi \longrightarrow \mathbb{Z}$ be a linear map describing a nilpotent orbit in good characteristic
2. R be a set of the representatives of nilpotent orbits
3. $R_\delta \subseteq R$ such that $x \in R_\delta \Leftrightarrow \exists g \in G$ such that
   - i. $g.x \in \mathfrak{g}^\delta_{\geqslant 2}$ ← specific subset of Lie(G)
   - ii. $C_G(\,[g.x]_{\mathfrak{g}^\delta_2}) \subseteq \langle T, U_\alpha \mid \delta(\alpha) \geqslant 0\rangle$ set linear coefficients of elements in $g.x$ not in $\mathfrak{g}^\delta_2$ to 0

Then $\mathcal{N}^\delta_\mathfrak{g} = \bigsqcup_{x\in R_\delta} \mathcal{O}_x$. ⟵ Union of nilpotent orbits

A schematic representation of the algorithm



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Example

Let $\Phi(G_2) := \{\pm\alpha, \pm\beta, \pm(\alpha+\beta), \pm(2\alpha+\beta), \pm(3\alpha+\beta), \pm(3\alpha+2\beta)\}$, $\mathrm{char}(\mathrm{k}) = 3$, and $x = e_\beta + e_{\alpha+\beta} \in \mathrm{Lie}(G)$.

1. Elements like

$$u_\beta(c_\beta)u_{\alpha+\beta}(c_{\alpha+\beta})u_{2\alpha+\beta}(c_{2\alpha+\beta})u_{3\alpha+2\beta}(c_{3\alpha+2\beta})$$

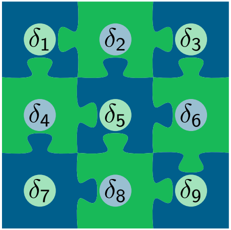
   for all $c_\beta, c_{\alpha+\beta}, c_{2\alpha+\beta}, c_{3\alpha+2\beta} \in \mathrm{k}$ fix x
   $\rightsquigarrow$ enough to compute the action of $u' t n_w u_\alpha(c_\alpha) u_{3\alpha+\beta}(c_{3\alpha+\beta})$ on x.

2. for $w = n_{s_\beta s_\alpha}$ compute the coefficients of linear combination of $g_w.x$
   $\rightsquigarrow$ obtain an equation system:

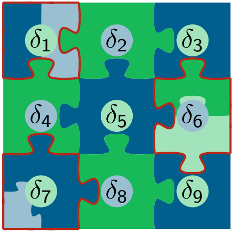
$$\begin{aligned} 2\frac{d_1^5}{d_2^2}c_\alpha^3 &= 0 \\ 2d_1^2 d_2 c_\alpha^2 + d_1^2 d_2 c_\alpha &= 0 \end{aligned} \qquad \text{where } d_1,\ d_2 \in \mathrm{k}^\times,\ c_\alpha \in \mathrm{k}.$$

   with solution $c_\alpha = 0$.

Results



In “good” characteristic



In small characteristic