Invariant Varieties and Collision-Freeness for Polynomial Systems

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Computeralgebra Tagung
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Ordinary Differential Equations (ODE)

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Related initial value problem (IVP):

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has a unique solution

$$\varphi(\cdot, x^0) : J(x^0) \to \mathbb{R}^n$$

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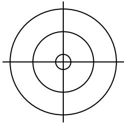

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A set $M \subseteq \mathbb{R}^n$ is called **invariant** for the ODE $\dot{x} = f(x)$ if for all $x^0 \in M$, the solution $\varphi(\cdot, x^0)$ to the IVP $\dot{x} = f(x)$, $x(0) = x^0$ fulfils $\varphi(t, x^0) \in M$ for all $t \in J(x^0)$.

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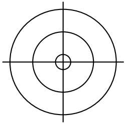

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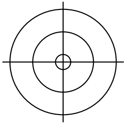

$\rightsquigarrow\ M = \{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = \gamma\},\ \gamma \in \mathbb{R}_{\geq 0}$ is an invariant set.

Invariant Varieties

- $\mathcal{P} := \mathbb{R}[X_1, ..., X_n]$ Polynomial ring in n variables over $\mathbb{R}$
- $\mathcal{V}(I) := \{x \in \mathbb{R}^n \mid p_1(x) = ... = p_k(x) = 0\}$ Variety of $I = \langle p_1, ..., p_k \rangle \subseteq \mathcal{P}$
- $\mathcal{J}(M) := \{p \in \mathcal{P} \mid p(x) = 0 \ \forall x \in M\}$ Vanishing ideal of $M \subseteq \mathbb{R}^n$
- $L_f(p) := \frac{\partial p}{\partial X} \cdot f = \sum_{i=1}^{n} \frac{\partial p}{\partial X_i} \cdot f_i \in \mathcal{P}^n$ Lie-derivative of $p \in \mathcal{P}$ along $f \in \mathcal{P}^n$

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For a variety $V = \mathcal{V}(I) \subseteq \mathbb{R}^n$ and $f \in \mathcal{P}^n$ the following are equivalent:

(1) *V is invariant for $\dot{x} = f(x)$.*

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Example: $n = 2,\ f = (X_2, -X_1)^T \in \mathcal{P}^2,\ V = \mathcal{V}(X_1^2 + X_2^2 - \gamma),\ \gamma \in \mathbb{R}_{\geq 0}$

$$L_f(X_1^2 + X_2^2 - \gamma) = 2X_1 \cdot X_2 + 2X_2 \cdot (-X_1) = 0 \ \in \mathcal{J}(V)$$

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Set of all polynomial vector fields such that V is invariant for $\dot{x} = f(x)$:

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Invariant Varieties – Computation of $\mathcal{M}(V)$

Given: $V = \mathcal{V}(I) \subseteq \mathbb{R}^n$ variety

$\mathcal{M}(V) := \{f \in \mathcal{P}^n \mid V \text{ is invariant for } \dot{x} = f(x)\} = \{f \in \mathcal{P}^n \mid L_f(\mathcal{J}(V)) \subseteq \mathcal{J}(V)\}$

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Invariant Varieties – Irreducible Components

- $\mathcal{V}(I_1 \cap I_2) = \mathcal{V}(I_1) \cup \mathcal{V}(I_2)$ holds for all ideals $I_i \subseteq \mathcal{P}$.
- A variety V is **irreducible**
  $\Leftrightarrow$ V cannot be written as a union $V = V_1 \cup V_2$ with subvarieties $V_i \subsetneq V$
  $\Leftrightarrow$ $\mathcal{J}(V)$ is a prime ideal.
- **Irreducible component** of a variety V: maximal irreducible subvariety of V.
  $\rightsquigarrow$ V has only finitely many irreducible components $V_1, \ldots, V_k$.
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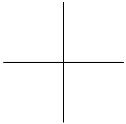

Controlled Invariance

Consider the polynomial control system

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t)$$

with $f \in \mathcal{P}^n$ and $g \in \mathcal{P}^{n \times m}$.

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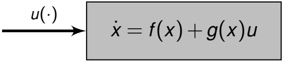

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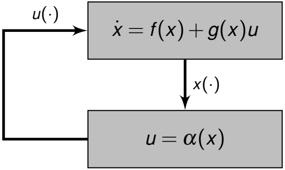

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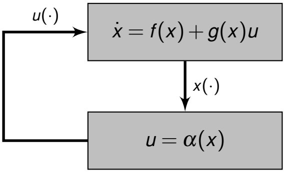
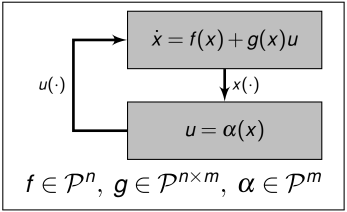


Definition

A subset $M \subseteq \mathbb{R}^n$ is called **controlled invariant** for $\dot{x} = f(x) + g(x)u$ if there exists a state feedback $u(t) = \alpha(x(t))$, $\alpha \in \mathcal{P}^m$, such that M is invariant for $\dot{x} = F(x)$ with $F := f + g\alpha \in \mathcal{P}^n$.

Controlled Invariant Varieties

$V = \mathcal{V}(I) \subseteq \mathbb{R}^n$ variety



Theorem

The following are equivalent:

(1) *V is controlled invariant for $\dot{x} = f(x) + g(x)u$.*

(2) *$\exists\, \alpha \in \mathcal{P}^m$ such that V is invariant for $\dot{x} = (f + g\alpha)(x)$.*

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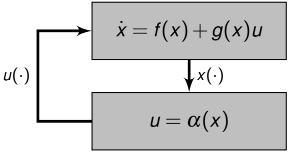
$V = \mathcal{V}(I) \subseteq \mathbb{R}^n$ variety

$\mathcal{M}(V) = \{F \in \mathcal{P}^n \mid V \text{ is invariant for } \dot{x} = F(x)\}$



$f \in \mathcal{P}^n,\ g \in \mathcal{P}^{n\times m},\ \alpha \in \mathcal{P}^m$

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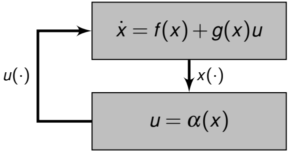

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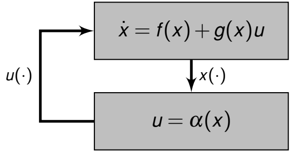

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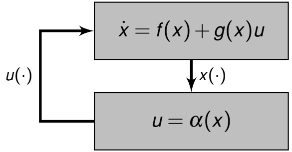

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$$\mathcal{A} = \{\alpha \in \mathcal{P}^m \mid f + g\alpha \in \mathcal{M}(V)\}$$

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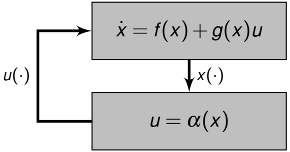

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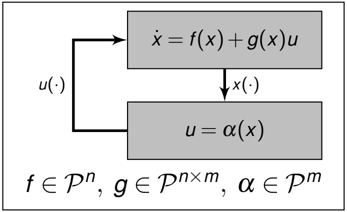
$\rightsquigarrow \mathcal{A} = \emptyset$ or $\mathcal{A} = \alpha + \mathcal{A}_0$ (affine module) with $\alpha \in \mathcal{P}^m$, $\mathcal{A}_0 \subseteq \mathcal{P}^m$ (module)

Controlled Invariant Varieties

$V = \mathcal{V}(I) = \bigcup_{i=1}^{k} V_i \subseteq \mathbb{R}^n$ variety

V_i irreducible component

$\mathcal{M}(V_i) = \{F \in \mathcal{P}^n \mid V_i$ is invariant for $\dot{x} = F(x)\}$



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(4) *$f \in \mathcal{M}(V_i) + \mathrm{im}(g\cdot)$ for all $i \in \{1, \ldots, k\}$ and $\bigcap_{i=1}^{k} \mathcal{A}_i \neq \emptyset$.*

Set of all admissible state feedbacks for V_i:

$$\mathcal{A}_i = \{\alpha \in \mathcal{P}^m \mid f + g\alpha \in \mathcal{M}(V_i)\}$$

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Controlled Invariance – Example

$V = \mathcal{V}(p, q)$ with $p = X_1^2 + X_2^2 - 1$, $q = X_3^2 - 1$

$$\dot{x} = f(x) + g(x)u \text{ with } f = \begin{pmatrix} X_3 \\ X_2 \\ X_1 \end{pmatrix} \text{ and } g = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \\ X_3 & 0 \end{pmatrix}$$

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Set of all admissible state feedbacks: $\mathcal{A} = \alpha + \mathcal{A}_0$ with

$$\mathcal{A}_0 = \pi_2(\ker((g,\ M)\cdot)) = \mathrm{im}(\begin{pmatrix} p & 0 & q & 0 \\ 0 & p & 0 & q \end{pmatrix} \cdot).$$

Collision-Free Systems

$f \in \mathcal{P}^n,\ n \in \mathbb{N},\ \mathcal{P} := \mathbb{R}[X_1, \dots, X_n]$

$$\boxed{\dot{x}(t) = f(x(t))}$$

$\rightsquigarrow$ $n \geq 2$ number of particle

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$\rightsquigarrow$ not collision-free

Collision-Freeness and Invariant Varieties

$$n \geq 2,\ \mathcal{P} := \mathbb{R}[X_1, \dots, X_n]$$
$$\dot{x}(t) = f(x(t)),\ f \in \mathcal{P}^n,$$
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Let

$$V := \bigcup_{1 \leq i < j \leq n} V_{ij}$$

with

$$V_{ij} := \{x \in \mathbb{R}^n \mid x_i = x_j\} = \mathcal{V}(X_i - X_j) \subseteq \mathbb{R}^n.$$

Set of all vectors of $\mathbb{R}^n$ with distinct components: $\mathbb{R}^n \setminus V$

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The following are equivalent:

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4.) *$L_f(X_i - X_j) = f_i - f_j \in \mathcal{J}(V_{ij}) = \langle X_i - X_j \rangle$ for all $1 \le i < j \le n$.*

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Example: $n = 2,\ V = V_{12} = \mathcal{V}(X_1 - X_2)$

$$\mathcal{M}(\mathcal{V}(X_1 - X_2)) = \{f \in \mathcal{P}^2 \mid \dot{x} = f(x) \text{ is collision-free}\} = \operatorname{im}\left(\begin{pmatrix} 1 & X_1 - X_2 \\ 1 & 0 \end{pmatrix} \cdot\right)$$

Collision-Freeness – Nonlinear Example

$f \in \mathcal{P}^n,\ n \geq 2,\ \mathcal{P} := \mathbb{R}[X_1, \dots, X_n]$

$$\boxed{\dot{x}(t) = f(x(t))}$$

Recall: $\dot{x} = f(x)$ is collision-free

$\Leftrightarrow \quad f_i - f_j \in \langle X_i - X_j \rangle$ for all $1 \leq i < j \leq n$.

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Example:

1) $f_i = \sum_{k=1}^{n} p(X_k - X_i), \quad p \in \mathbb{R}[X], i \in \{1, \ldots, n\}$

2) $f_i = p(X_i) + \sum_{k=1, k \neq i}^{n} q(X_k, X_i), \quad p \in \mathbb{R}[X], q \in \mathbb{R}[X, Y], i \in \{1, \ldots, n\}$

Collision-Freeness – Nonlinear Example

$f \in \mathcal{P}^n,\ n \geq 2,\ \mathcal{P} := \mathbb{R}[X_1, \dots, X_n]$

$$\boxed{\dot{x}(t) = f(x(t))}$$

Recall: $\dot{x} = f(x)$ is collision-free

$\Leftrightarrow \quad f_i - f_j \in \langle X_i - X_j \rangle$ for all $1 \leq i < j \leq n$.

Example:

1) $f_i = \sum_{k=1}^{n} p(X_k - X_i), \quad p \in \mathbb{R}[X], i \in \{1, \dots, n\}$

2) $f_i = p(X_i) + \sum_{k=1, k \neq i}^{n} q(X_k, X_i), \quad p \in \mathbb{R}[X], q \in \mathbb{R}[X, Y], i \in \{1, \dots, n\}$

$\rightsquigarrow \quad f_i - f_j$ vanishes on $\mathcal{V}(X_i - X_j)$, i.e. $f_i - f_j \in \mathcal{J}(\mathcal{V}(X_i - X_j)) = \langle X_i - X_j \rangle$

for all $1 \leq i < j \leq n$.

Linear Systems

$n \geq 2, \ A \in \mathbb{R}^{n \times n}$

$$\boxed{\dot{x}(t) = Ax(t)}$$

Recall: $\dot{x} = Ax$ is collision-free

$$\Leftrightarrow \quad f_i - f_j = \sum_{k=1}^{n} (A_{ik} - A_{jk}) X_k \in \langle X_i - X_j \rangle \text{ for all } 1 \leq i < j \leq n.$$

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Theorem

The linear ODE $\dot{x} = Ax$ is collision-free if and only if

$$A_{ik} = A_{jk} \quad \textit{and} \quad A_{ii} + A_{ij} = A_{jj} + A_{ji} \quad \forall i,j,k \in \{1,\ldots,n\}, \; i \neq k \neq j.$$

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$\rightsquigarrow$ $\mathcal{R}$ is a subring of $\mathbb{R}^{n \times n}$ and a real vector space of dimension $n+1$

Linear Systems with Symmetry

$n \geq 2, \;\; A \in \mathbb{R}^{n \times n}$

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Assumption: A is symmetric, i.e. $A_{ij} = A_{ji}$ for all $i, j \in \{1, ..., n\}$

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$$\Leftrightarrow \quad A = \begin{pmatrix} b & c & \cdots & c \\ c & b & \ddots & \vdots \\ \vdots & \ddots & \ddots & c \\ c & \cdots & c & b \end{pmatrix} \quad \text{for } \; b, c \in \mathbb{R} \; \text{ arbitrary.}$$

Linear Systems – Example

$n = 2$:

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)\\ \dot{x}_2(t) &= x_2(t)\end{aligned}$$

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- Generalisation to d-dimensional states of particles ($d \in \mathbb{N}$):

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with $x_i(t) = (x_{i1}(t), \ldots, x_{id}(t)) \in \mathbb{R}^d$ for $i \in \{1, \ldots, n\}$.

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  - Varieties: $V = \bigcup_{1 \leq i < j \leq n} V_{ij}$ with $V_{ij} = \mathcal{V}(X_{i1} - X_{j1}, \ldots, X_{id} - X_{jd})$

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- Analogue characterisation of collision-freeness for general $\mathcal{C}^1$-systems

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$\rightsquigarrow$ Tools: GB-methods