# *Inequalities and computer algebra*

Veronika Pillwein

Doctoral Program
Computational Mathematics
Numerical Analysis and Symbolic Computation

Der Wissenschaftsfonds.

LAND OBERÖSTERREICH

Tagung der Fachgruppe Computeralgebra, March 9–11, 2022 Munich/online

## Cylindrical Algebraic Decomposition (CAD)

### Input: polynomial formulas

- Everything that can be formed from variables, rational (or real algebraic) numbers, $+, -, \cdot, /, =, \neq, \leq, <, \geq, >, \neg, \wedge, \vee, \Rightarrow, \Leftrightarrow,$ $\mathrm{True}, \mathrm{False}$ according to the usual syntactic rules.

# *Cylindrical Algebraic Decomposition (CAD)*

## Input: polynomial formulas

- Everything that can be formed from variables, rational (or real algebraic) numbers, $+, -, \cdot, /, =, \neq, \leq, <, \geq, >, \neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$, $\mathrm{True}, \mathrm{False}$ according to the usual syntactic rules.
- Examples:
  - $x^2 - (x-1)(y-1) > 0 \ \wedge \ x \geq y^2 \geq 1 \ \Rightarrow \ y^2 - 3x < 0$
  - $xz^2 - \sqrt{2}y + z^3 > 1 \ \wedge \ x > y > z > 1 \ \Rightarrow \ xyz > \frac{1}{2}$

## *Cylindrical Algebraic Decomposition (CAD)*

Input: polynomial formulas

- Everything that can be formed from variables, rational (or real algebraic) numbers, $+, -, \cdot, /, =, \neq, \leq, <, \geq, >, \neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$, $\mathrm{True}, \mathrm{False}$ according to the usual syntactic rules.
- Examples:
  - $x^2 - (x-1)(y-1) > 0 \ \wedge\ x \geq y^2 \geq 1 \ \Rightarrow\ y^2 - 3x < 0$
  - $xz^2 - \sqrt{2}y + z^3 > 1 \ \wedge\ x > y > z > 1 \ \Rightarrow\ xyz > \frac{1}{2}$
- Counterexamples:
  - $x\cos(x) - y\sin(y) > \frac{\pi}{4} \ \Rightarrow \cos(x)\sin(x) < \sin^2(x)$
  - $x\exp(y) - y^2 < xy \ \vee\ \log(y) < x \ \Rightarrow\ x^2 + y^2 < 4$

# *CAD: Quantifier Elimination*

Given any polynomial formula $\Phi$ *involving quantifiers* $(\forall, \exists)$ CAD can compute a polynomial formula $\Psi$ *without quantifiers* that is *equivalent* (over $\mathbb{R}$) to $\Phi$.

# *CAD: Quantifier Elimination*

Given any polynomial formula $\Phi$ *involving quantifiers* $(\forall, \exists)$ CAD can compute a polynomial formula $\Psi$ *without quantifiers* that is *equivalent* (over $\mathbb{R}$) to $\Phi$.

Examples:

$$\forall\, x\colon\ 0 < x < 1 \ \Rightarrow\ x^2 - y + 1 \leq B(xy + 1) \quad \overset{CAD}{\longrightarrow} \quad y > -1 \ \wedge\ B \geq \frac{2-y}{y+1}$$

# *CAD: Quantifier Elimination*

Given any polynomial formula $\Phi$ *involving quantifiers* $(\forall, \exists)$ CAD can compute a polynomial formula $\Psi$ *without quantifiers* that is *equivalent* (over $\mathbb{R}$) to $\Phi$.

Examples:

$$\forall\, x:\ 0 < x < 1 \ \Rightarrow\ x^2 - y + 1 \le B(xy+1) \quad \xrightarrow{CAD} \quad y > -1 \ \wedge\ B \ge \frac{2-y}{y+1}$$

$$\forall\, x\, \exists\, y:\ 0 < x < 1 \ \wedge\ 0 < y < x^2 \Rightarrow\ x^2 - y + 1 \le B(xy+1) \quad \xrightarrow{CAD} \quad B \ge 1$$

## *CAD: Quantifier Elimination*

Given any polynomial formula $\Phi$ *involving quantifiers* $(\forall, \exists)$ CAD can compute a polynomial formula $\Psi$ *without quantifiers* that is *equivalent* (over $\mathbb{R}$) to $\Phi$.

Examples:

$$\forall\, x\colon\ 0 < x < 1 \ \Rightarrow\ x^2 - y + 1 \le B(xy+1) \quad \overset{CAD}{\longrightarrow} \quad y > -1 \ \wedge\ B \ge \frac{2-y}{y+1}$$

$$\forall\, x\, \exists\, y\colon\ 0 < x < 1 \ \wedge\ 0 < y < x^2 \Rightarrow\ x^2 - y + 1 \le B(xy+1) \quad \overset{CAD}{\longrightarrow} \quad B \ge 1$$

$$\forall\, x\, \exists\, y\colon x^2 + y^2 - 4 > 0 \Leftrightarrow (x-1)(y-1) - 1 > 0 \quad \overset{CAD}{\longrightarrow} \quad \text{True}$$

# CAD: Quantifier Elimination

Given any polynomial formula $\Phi$ *involving quantifiers* $(\forall, \exists)$ CAD can compute a polynomial formula $\Psi$ *without quantifiers* that is *equivalent* (over $\mathbb{R}$) to $\Phi$.

Examples:

$$\forall\, x\colon\ 0 < x < 1 \ \Rightarrow\ x^2 - y + 1 \leq B(xy + 1) \quad \xrightarrow{CAD} \quad y > -1 \ \wedge\ B \geq \frac{2 - y}{y + 1}$$

$$\forall\, x\, \exists\, y\colon\ 0 < x < 1 \ \wedge\ 0 < y < x^2 \Rightarrow\ x^2 - y + 1 \leq B(xy + 1) \quad \xrightarrow{CAD} \quad B \geq 1$$

$$\forall\, x\, \exists\, y\colon x^2 + y^2 - 4 > 0 \ \Leftrightarrow\ (x - 1)(y - 1) - 1 > 0 \quad \xrightarrow{CAD} \quad \text{True}$$

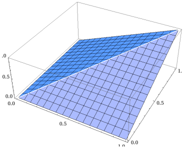
*Note:* The execution of CAD may be *computationally expensive!*

*Implementations:* Mathematica, Maple, QEPCAD, Redlog, etc.

## *An example from fuzzy logic [Kauers+P+Saminger-Platz]*

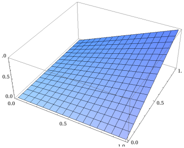
Some basic examples for triangular norms ($t$-norms):

minimum norm



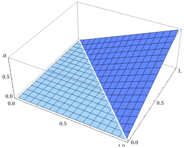
$T_M(x, y) = \min(x, y)$

product norm



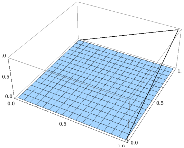
$T_P(x, y) = xy$

Łukasiewicz norm



$T_L(x, y) = \max(0, x + y - 1)$

drastic norm



$T_D(x, 1) = T_D(1, x) = x$

$T_D(x, y) = 0$ else

## *Dominance in the family of Sugeno-Weber $t$-norms*

For all $u, v, x, y \in [0, 1]$:

$$\begin{aligned}
&\max\Big(0, (1-\lambda)\max\big(0, (1-\mu)uv + \mu(u+v-1)\big) \\
&\qquad \cdot \max\big(0, (1-\mu)xy + \mu(x+y-1)\big) \\
&\qquad + \lambda(\max\big(0, (1-\mu)uv + \mu(u+v-1)\big) \\
&\qquad + \max\big(0, (1-\mu)xy + \mu(x+y-1)\big) - 1)\Big) \\
&\geq \max\Big(0, (1-\mu)\max\big(0, (1-\lambda)ux + \lambda(u+x-1)\big) \\
&\qquad \cdot \max\big(0, (1-\lambda)vy + \lambda(v+y-1)\big) \\
&\qquad + \mu\big(\max\big(0, (1-\lambda)ux + \lambda(u+x-1)\big) \\
&\qquad + \max\big(0, (1-\lambda)vy + \lambda(v+y-1)\big) - 1\big)\Big).
\end{aligned}$$

## *Breaking into Smaller Pieces*

1. Handle some special cases by hand
2. Eliminate the outer maxima
3. Eliminate the inner maxima
4. Discard some redundant cases using CAD
5. Apply some logical simplifications using CAD
6. Apply some algebraic simplifications
7. Apply CAD to finish up

## *Putting things together …*

… we arrive at the following equivalent formulation of the initial task:

$$
\begin{aligned}
&\forall\, u, v, x, y\colon \\
&\quad 0 < \lambda < \mu \wedge 0 < x < 1 \wedge 0 < y < 1 \wedge 0 < u < 1 \wedge y < v < 1 + \lambda y \\
&\quad\; \Rightarrow \big(u((\lambda-1)x+1)((\mu-1)v+1) + (\mu-1)vx + v + x - 1 \geq 0 \\
&\qquad \vee\, vx(1-(\lambda-1)(\mu-1)uy) + y((\lambda-1)uy((\mu-1)x+1) + u - x) \geq 0\big).
\end{aligned}
$$

## *Putting things together ...*

... we arrive at the following equivalent formulation of the initial task:

$$\forall\, u, v, x, y\colon$$
$$0<\lambda<\mu \wedge 0<x<1 \wedge 0<y<1 \wedge 0<u<1 \wedge y<v<1+\lambda y$$
$$\Rightarrow \big(u((\lambda-1)x+1)((\mu-1)v+1)+(\mu-1)vx+v+x-1\geq 0$$
$$\vee\, vx(1-(\lambda-1)(\mu-1)uy)+y((\lambda-1)uy((\mu-1)x+1)+u-x)\geq 0\big).$$

Applying CAD yields the final result

$$0<\lambda\leq\mu\leq 17+12\sqrt{2} \quad \vee$$
$$\mu>17+12\sqrt{2}\;\wedge\; 0<\lambda\leq\left(\frac{1-3\sqrt{\mu}}{3-\sqrt{\mu}}\right)^2.$$

## *Holonomic sequences*

Let $\mathbb{K}$ be a computable field. A sequence $(a_n)_{n\geq 0} \in \mathbb{K}^{\mathbb{N}}$ is called *holonomic* (or *P-finite* or *P-recursive*), if there exist polynomials $p_0, \ldots, p_r \in \mathbb{K}[x]$, not all zero, such that,

$$p_r(n)a_{n+r} + \cdots + p_1(n)a_{n+1} + p_0(n)a_n = 0.$$

## *Holonomic sequences*

Let $\mathbb{K}$ be a computable field. A sequence $(a_n)_{n\geq 0} \in \mathbb{K}^{\mathbb{N}}$ is called *holonomic* (or *P-finite* or *P-recursive*), if there exist polynomials $p_0, \ldots, p_r \in \mathbb{K}[x]$, not all zero, such that,

$$p_r(n)a_{n+r} + \cdots + p_1(n)a_{n+1} + p_0(n)a_n = 0.$$

- *Fibonacci numbers*

$$F_{n+2} - F_{n+1} - F_n = 0, \quad F_0 = 0,\ F_1 = 1$$

## *Holonomic sequences*

Let $\mathbb{K}$ be a computable field. A sequence $(a_n)_{n\geq 0} \in \mathbb{K}^{\mathbb{N}}$ is called *holonomic* (or *P-finite* or *P-recursive*), if there exist polynomials $p_0, \ldots, p_r \in \mathbb{K}[x]$, not all zero, such that,

$$p_r(n)a_{n+r} + \cdots + p_1(n)a_{n+1} + p_0(n)a_n = 0.$$

- *Fibonacci numbers*

$$F_{n+2} - F_{n+1} - F_n = 0, \quad F_0 = 0,\ F_1 = 1$$

- *Harmonic numbers* $H_n = \sum_{k=1}^{n} \frac{1}{k}$

$$(n+2)H_{n+2} - (2n+3)H_{n+1} + (n+1)H_n = 0, \quad H_0 = 0,\ H_1 = 1$$

## *Holonomic sequences*

Let $\mathbb{K}$ be a computable field. A sequence $(a_n)_{n\geq 0} \in \mathbb{K}^{\mathbb{N}}$ is called *holonomic* (or *P-finite* or *P-recursive*), if there exist polynomials $p_0, \ldots, p_r \in \mathbb{K}[x]$, not all zero, such that,

$$p_r(n)a_{n+r} + \cdots + p_1(n)a_{n+1} + p_0(n)a_n = 0.$$

- *Fibonacci numbers*

$$F_{n+2} - F_{n+1} - F_n = 0, \quad F_0 = 0,\ F_1 = 1$$

- *Harmonic numbers* $H_n = \sum_{k=1}^{n} \frac{1}{k}$

$$(n+2)H_{n+2} - (2n+3)H_{n+1} + (n+1)H_n = 0, \quad H_0 = 0,\ H_1 = 1$$

- *Legendre polynomials* $P_n(x)$ ($L^2$-orthogonal on $[-1, 1]$),

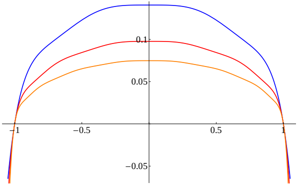
$$(n+2)P_{n+2}(x) - (2n+3)xP_{n+1}(x) + (n+1)P_n(x) = 0,$$

$$P_0(x) = 1,\ P_1(x) = x$$

# *Turán inequality for Legendre polynomials*

For all $n \geq 0$ and all $x \in [-1, 1]$:

$$\Delta_n(x) = P_n(x)^2 - P_{n-1}(x)P_{n+1}(x) \geq 0.$$

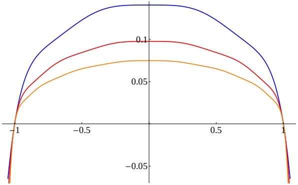



$$\Delta_4(x) = \frac{1-x^2}{64}(35x^6 - 21x^4 + 9x^2 + 9)$$

$$\Delta_6(x) = \frac{1-x^2}{256}(693x^{10} - 1155x^8 + 690x^6 - 150x^4 + 25x^2 + 25)$$

$$\Delta_8(x) = \frac{1-x^2}{16384}(306735x^{14} - 825825x^{12} + 867867x^{10} - \dots)$$

# *Turán inequality for Legendre polynomials*




For all $n \geq 0$ and all $x \in [-1, 1]$:

$$\Delta_n(x) = P_n(x)^2 - P_{n-1}(x)P_{n+1}(x) \geq 0.$$

$$\Delta_4(x) = \frac{1-x^2}{64}(35x^6 - 21x^4 + 9x^2 + 9)$$

$$\Delta_6(x) = \frac{1-x^2}{256}(693x^{10} - 1155x^8 + 690x^6 - 150x^4 + 25x^2 + 25)$$

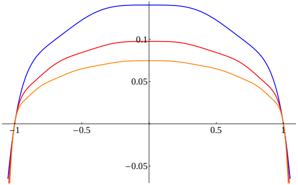
$$\Delta_8(x) = \frac{1-x^2}{16384}(306735x^{14} - 825825x^{12} + 867867x^{10} - \dots)$$

- *On the zeros of the polynomials of Legendre*, Turán, 1950

## *Turán inequality for Legendre polynomials*

For all $n \geq 0$ and all $x \in [-1, 1]$:

$$\Delta_n(x) = P_n(x)^2 - P_{n-1}(x)P_{n+1}(x) \geq 0.$$




$$\Delta_4(x) = \frac{1-x^2}{64}(35x^6 - 21x^4 + 9x^2 + 9)$$

$$\Delta_6(x) = \frac{1-x^2}{256}(693x^{10} - 1155x^8 + 690x^6 - 150x^4 + 25x^2 + 25)$$

$$\Delta_8(x) = \frac{1-x^2}{16384}(306735x^{14} - 825825x^{12} + 867867x^{10} - \dots)$$

- *On the zeros of the polynomials of Legendre*, Turán, 1950
- *On an inequality of P. Turán concerning Legendre polynomials*, Szegő, 1948: 4 different proofs

## Szegő, 1948: first proof

**1. Proof.** The following arrangement is somewhat similar to that of Turán. By using the classical recursion

$$P_{n+1}(x) = \frac{2n+1}{n+1} xP_n(x) - \frac{n}{n+1} P_{n-1}(x) \tag{2}$$

we find for the polynomial $\Delta_n(x)$ the representation

$$P_n^2 + \frac{n}{n+1} P_{n-1}^2 - \frac{2n+1}{n+1} xP_nP_{n-1}. \tag{3}$$

This is a quadratic form in $P_n$ and $P_{n-1}$ which is positive provided

$$\frac{n}{n+1} > \left(\frac{n+1/2}{n+1} x\right)^2, \quad \text{or} \quad |x| < \frac{(n(n+1))^{1/2}}{n+1/2} = \cos\theta_0. \tag{4}$$

For these $x$ the theorem is already proved. For the remaining $x = \cos\theta$, that is, for $0 < \theta \leqq \theta_0$, we use Mehler's formula

$$P_n(\cos\theta) = \frac{2}{\pi}\int_0^\theta \frac{\cos(n+1/2)u}{(2(\cos u - \cos\theta))^{1/2}} du \tag{5}$$

## *Gerhold-Kauers method*

*Given:* $p_0, p_1, p_2 \in K[x]$ and initial values $f(0), f(1)$, s.t.,

$$p_2(n)f(n+2) + p_1(n)f(n+1) + p_0(n)f(n) = 0, \quad n \geq 0.$$

*Prove:* $f(n) \geq 0$ for all $n \geq 0$.

*Proof by induction:* Show that

$$f(n) \geq 0 \wedge f(n+1) \geq 0 \Rightarrow -\frac{p_0(n)}{p_2(n)}f(n) - \frac{p_1(n)}{p_2(n)}f(n+1) \geq 0$$

## *Gerhold-Kauers method*

*Given:* $p_0, p_1, p_2 \in K[x]$ and initial values $f(0), f(1)$, s.t.,

$$p_2(n)f(n+2) + p_1(n)f(n+1) + p_0(n)f(n) = 0, \quad n \geq 0.$$

*Prove:* $f(n) \geq 0$ for all $n \geq 0$.

*Proof by induction:* Show that

$$f(n) \geq 0 \wedge f(n+1) \geq 0 \Rightarrow -\frac{p_0(n)}{p_2(n)}f(n) - \frac{p_1(n)}{p_2(n)}f(n+1) \geq 0$$

*Generalize:* To prove positivity of $f(n)$ it is sufficient to show that

$$\forall y_0, y_1 \in \mathbb{R} \forall x \in \mathbb{R} : y_0 \geq 0 \wedge y_1 \geq 0 \Rightarrow -\frac{p_0(x)}{p_2(x)}y_0 - \frac{p_1(x)}{p_2(x)}y_1 \geq 0$$

## *Gerhold-Kauers method*

*Given:* $p_0, p_1, p_2 \in K[x]$ and initial values $f(0), f(1)$, s.t.,

$$p_2(n)f(n+2) + p_1(n)f(n+1) + p_0(n)f(n) = 0, \quad n \geq 0.$$

*Prove:* $f(n) \geq 0$ for all $n \geq 0$.

*Proof by induction:* Show that

$$f(n) \geq 0 \wedge f(n+1) \geq 0 \Rightarrow -\frac{p_0(n)}{p_2(n)}f(n) - \frac{p_1(n)}{p_2(n)}f(n+1) \geq 0$$

*Generalize:* To prove positivity of $f(n)$ it is sufficient to show that

$$\forall y_0, y_1 \in \mathbb{R} \forall x \in \mathbb{R} : y_0 \geq 0 \wedge y_1 \geq 0 \Rightarrow -\frac{p_0(x)}{p_2(x)}y_0 - \frac{p_1(x)}{p_2(x)}y_1 \geq 0$$

This can be decided by a quantifier elimination algorithm!

## *Gerhold-Kauers method*

*Given:* $p_0, p_1, p_2 \in K[x]$ and initial values $f(0), f(1)$, s.t.,

$$p_2(n)f(n+2) + p_1(n)f(n+1) + p_0(n)f(n) = 0, \quad n \geq 0.$$

*Prove:* $f(n) \geq 0$ for all $n \geq 0$.

*Proof by induction:* Show that

$$f(n) \geq 0 \wedge f(n+1) \geq 0 \Rightarrow -\frac{p_0(n)}{p_2(n)}f(n) - \frac{p_1(n)}{p_2(n)}f(n+1) \geq 0$$

*Generalize:* To prove positivity of $f(n)$ it is sufficient to show that

$$\forall y_0, y_1 \in \mathbb{R} \forall x \in \mathbb{R} : y_0 \geq 0 \wedge y_1 \geq 0 \Rightarrow -\frac{p_0(x)}{p_2(x)}y_0 - \frac{p_1(x)}{p_2(x)}y_1 \geq 0$$

But it might be false even if $f(n) \geq 0$!

## *Gerhold-Kauers method*

*Given:* $p_0, p_1, p_2 \in K[x]$ and initial values $f(0), f(1)$, s.t.,

$$p_2(n)f(n+2) + p_1(n)f(n+1) + p_0(n)f(n) = 0, \quad n \geq 0.$$

*Prove:* $f(n) \geq 0$ for all $n \geq 0$.

*Proof by induction:* Show that

$$f(n) \geq 0 \wedge f(n+1) \geq 0 \Rightarrow -\frac{p_0(n)}{p_2(n)}f(n) - \frac{p_1(n)}{p_2(n)}f(n+1) \geq 0$$

*Refined induction step formulas:* Extend the induction hypothesis and try to show that

$$\forall\ y_0, y_1, x \in \mathbb{R} : y_0 \geq 0 \wedge y_1 \geq 0 \wedge -\frac{p_0(x)}{p_2(x)}y_0 - \frac{p_1(x)}{p_2(x)}y_1 \geq 0$$
$$\Rightarrow\ \frac{p_0(x)p_1(x+1)}{p_2(x)p_2(x+1)}y_0 + \frac{p_1(x)p_1(x+1) - p_0(x+1)p_2(x)}{p_2(x)p_2(x+1)}y_1 \geq 0$$

## *Turán inequality for Legendre polynomials II*

- Three term recurrence for *Legendre polynomials* $P_n(x)$,

$$(n+2)P_{n+2}(x) - (2n+3)xP_{n+1}(x) + (n+1)P_n(x) = 0$$

- For all $n \geq 0$ and all $x \in [-1, 1]$:

$$\Delta_n(x) = P_n(x)^2 - P_{n-1}(x)P_{n+1}(x) \geq 0.$$

## *Turán inequality for Legendre polynomials II*

- Three term recurrence for *Legendre polynomials* $P_n(x)$,

$$(n+2)P_{n+2}(x) - (2n+3)xP_{n+1}(x) + (n+1)P_n(x) = 0$$

- For all $n \geq 0$ and all $x \in [-1, 1]$:

$$\Delta_n(x) = P_n(x)^2 - P_{n-1}(x)P_{n+1}(x) \geq 0.$$

- Let $y_0 \leftarrow P_n(x), y_1 \leftarrow P_{n+1}(x), \ldots$

$$\forall\, x, -1 \leq x \leq 1 \forall z, z \geq 0 \forall y_0, y_1 : \\ y_1^2 - y_0 y_2 \geq 0 \Rightarrow y_2^2 - y_1 y_3 \geq 0,$$

with

$$y_2 = \frac{2z+3}{z+2} x y_1 - \frac{z+1}{z+2} y_0, \ldots$$

## *Proof by SumCracker (Kauers)*

In[1]:= $\mathbf{ProveInequality}[P_n(x)^2 - P_{n-1}(x)P_{n+1}(x) \geq 0,$
$\mathbf{Using} \to \{-1 \leq x \leq 1\}, \mathbf{Variable} \to n, \mathbf{Infolevel} \to 2]$

## *Proof by SumCracker (Kauers)*

In[1]:= $\mathbf{ProveInequality}[P_n(x)^2 - P_{n-1}(x)P_{n+1}(x) \geq 0,$
$\mathbf{Using} \to \{-1 \leq x \leq 1\}, \mathbf{Variable} \to n, \mathbf{Infolevel} \to 2]$

Collecting terms from given inequality...
Creating difference ring and homomorphism...
...
searching for induction step...
checking initial value...
extending induction hypothesis...
selecting and sorting variables...
checking induction step...
checking initial value...
extending induction hypothesis...
selecting and sorting variables...
checking induction step...

Out[1]= True

## *Related Results*

- When can we detect that a P-finite sequence is positive?
  [Kauers, P] ISSAC'10
- Termination Conditions for Positivity Proving Procedures
  ISSAC'13

## *Related Results*

- When can we detect that a P-finite sequence is positive? [Kauers, P] ISSAC'10
- Termination Conditions for Positivity Proving Procedures ISSAC'13
- On positivity and minimality for second-order holonomic sequences [Kenison, Klurman, Lefaucheux, Luca, Moree, Ouaknine, Whiteland, Worrell] subm. 2021

## *Related Results*

- When can we detect that a P-finite sequence is positive? [Kauers, P] ISSAC'10
- Termination Conditions for Positivity Proving Procedures ISSAC'13
- On positivity and minimality for second-order holonomic sequences [Kenison, Klurman, Lefaucheux, Luca, Moree, Ouaknine, Whiteland, Worrell] subm. 2021
- Log-concavity of P-recursive sequences [Hou, Li], JSC 2021

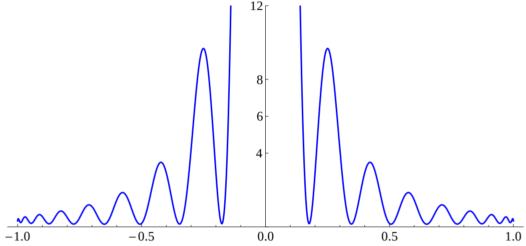
## *Related Results*

- When can we detect that a P-finite sequence is positive? [Kauers, P] ISSAC'10
- Termination Conditions for Positivity Proving Procedures ISSAC'13
- On positivity and minimality for second-order holonomic sequences [Kenison, Klurman, Lefaucheux, Luca, Moree, Ouaknine, Whiteland, Worrell] subm. 2021
- Log-concavity of P-recursive sequences [Hou, Li], JSC 2021
- Positivity of Turán determinants for orthogonal polynomials [Szwarc] 1998
- Positivity of Turán determinants for orthogonal polynomials II [Szwarc] J. Appr. Th. (accepted 2021)

## *Schöberl's inequality*

If $-1 \le x \le 1,\ n \ge 0$, then

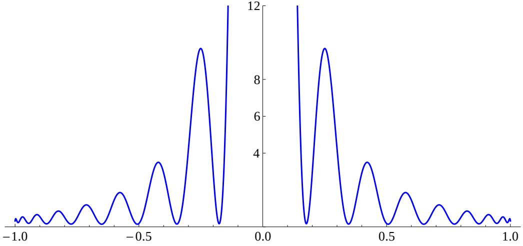
$$S(n,x) = \sum_{j=0}^{n} \tfrac{1}{2}(4j+1)(2n-2j+1)P_{2j}(0)P_{2j}(x) \ge 0.$$



## *Schöberl's inequality*

If $-1 \leq x \leq 1,\ n \geq 0,\ -\frac{1}{2} \leq \alpha \leq \frac{1}{2}$, then

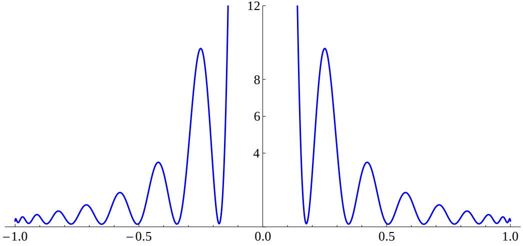
$$S^{\alpha}(n,x) = \sum_{j=0}^{2n} \frac{c_j^{\alpha}}{x} \left( P_{j+1}^{(\alpha,\alpha)}(x) P_j^{(\alpha,\alpha)}(0) - P_j^{(\alpha,\alpha)}(x) P_{j+1}^{(\alpha,\alpha)}(0) \right) \geq 0.$$



## *Schöberl's inequality*

If $-1 \leq x \leq 1,\ n \geq 0$, then

$$S(n,x) = \sum_{j=0}^{n} \tfrac{1}{2}(4j+1)(2n-2j+1)P_{2j}(0)P_{2j}(x) \geq 0.$$



## *Recurrence relation for $S(n,x)$*

The sums $S(n,x)$ satisfy a five term recurrence

$$\begin{aligned} 4(4n+7)(n+4)^2 S(n+4,x) &= (2n+3)^2(4n+15)S(n,x) \\ +(4n+15)(16n^2x^2-8n^2+48nx^2-12n+35x^2+3)S(n+1,x) \\ -(-192n^2x^2+144n^2-1056nx^2+792n-1260x^2+943)S(n+2,x) \\ -(4n+7)(16n^2x^2-8n^2+128nx^2-76n+255x^2-173)S(n+3,x) \end{aligned}$$

- this representation makes it reasonable to try an application of the Gerhold-Kauers method
- the procedure, however, does not terminate
- a reformulation is needed!

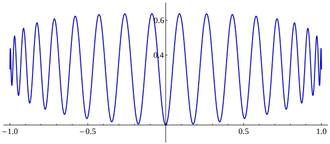
## *First: decompose* $S(n,x)$

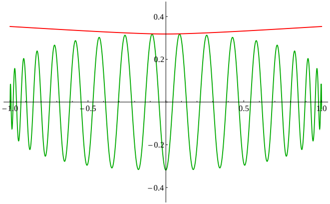
$$x^2 S(n,x) = g(n,x) + f(2n,x,0),$$

where

$$g(n,x) = \frac{2n+1}{2}\left(xP_{2n+1}(x) - \frac{4n+2}{4n+3}P_{2n}(x)\right)P_{2n}(0),$$

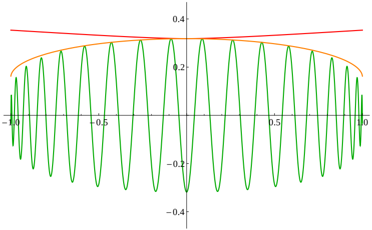
$$f(n,x,y) = -\sum_{j=0}^{n} \frac{1}{(2j-1)(2j+3)} P_j(x)P_j(y)$$



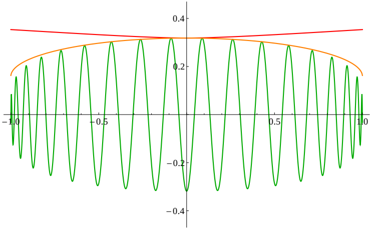
# *Estimate $f$ from below and show positivity*

$$
\begin{aligned}
x^2 S(n,x) &= g(n,x) + f(2n,x,0) \\
&\geq g(n,x) + \tfrac{1}{2}\left(f(2n,x,x) + f(2n,0,0)\right)
\end{aligned}
$$



## *Estimate $f$ from below and show positivity*

$$\begin{aligned} x^2 S(n,x) &= g(n,x) + f(2n,x,0) \\ &\geq g(n,x) + \tfrac{1}{2}\left(f(2n,x,x) + f(2n,0,0)\right) \end{aligned}$$




In[2]:= **ProveInequality[ + ≥ 0,**
**Using → {−1 ≤ *x* ≤ 1}, Variable → *n*]**

Out[2]= True

# *CAD-input for Schöberl's inequality (general case)*

$$\forall n, \alpha, x, y, z, w\big((n \geq 0 \wedge -1 \leq x \leq 1 \wedge -1 \leq 2\alpha \leq 1 \wedge (2\alpha + 4n + 1)(y^2 + z^2)(\alpha + 2n + 1)^2 - (2\alpha + 4n + 1)(2\alpha + 4n + 3)wxz(\alpha + 2n + 1) + (2n + 1)(2\alpha + 2n + 1)(2\alpha + 4n + 3)w^2 \geq 0) \Rightarrow (2n + 3)(\alpha + 2n + 1)^2(\alpha + 2n + 3)^2(2\alpha + 2n + 3)(2\alpha + 4n + 5)y^2(\alpha + 2n + 2)^2 + (\alpha + 2n + 1)^2(64n^5 - 256x^2n^4 + 160\alpha n^4 + 464n^4 + 144\alpha^2n^3 - 512\alpha x^2n^3 - 1184x^2n^3 + 928\alpha n^3 + 1344n^3 + 56\alpha^3n^2 + 628\alpha^2n^2 - 384\alpha^2x^2n^2 - 1776\alpha x^2n^2 - 1984x^2n^2 + 2016\alpha n^2 + 1944n^2 + 8\alpha^4n + 164\alpha^3n + 912\alpha^2n - 128\alpha^3x^2n - 888\alpha^2x^2n - 1984\alpha x^2n - 1434x^2n + 1944\alpha n + 1404n + 12\alpha^4 + 120\alpha^3 + 441\alpha^2 - 16\alpha^4x^2 - 148\alpha^3x^2 - 496\alpha^2x^2 - 717\alpha x^2 - 378x^2 + 702\alpha + 405)z^2(\alpha + 2n + 2)^2 - w^2(-256n^7 + 4096x^4n^6 - 3072x^2n^6 - 896\alpha n^6 - 1728n^6 + 12288\alpha x^4n^5 + 25088x^4n^5 - 1216\alpha^2n^5 - 9216\alpha x^2n^5 - 19968x^2n^5 - 5184\alpha n^5 - 4864n^5 + 15360\alpha^2x^4n^4 + 62720\alpha x^4n^4 + 62464x^4n^4 - 800\alpha^3n^4 - 5872\alpha^2n^4 - 11008\alpha^2x^2n^4 - 49920\alpha x^2n^4 - 53120x^2n^4 - 12160\alpha n^4 - 7408n^4 - 256\alpha^4n^3 + 10240\alpha^3x^4n^3 + 62720\alpha^2x^4n^3 + 124928\alpha x^4n^3 + 81216x^4n^3 - 3104\alpha^3n^3 - 11072\alpha^2n^3 - 6656\alpha^3x^2n^3 - 47744\alpha^2x^2n^3 - 106240\alpha x^2n^3 - 74176x^2n^3 - 14816\alpha n^3 - 6592n^3 - 32\alpha^5n^2 - 752\alpha^4n^2 + 3840\alpha^4x^4n^2 + 31360\alpha^3x^4n^2 + 93696\alpha^2x^4n^2 + 121824\alpha x^4n^2 + 58320x^4n^2 - 4448\alpha^3n^2 - 10192\alpha^2n^2 - 2112\alpha^4x^2n^2 - 21696\alpha^3x^2n^2 - 76416\alpha^2x^2n^2 - 111264\alpha x^2n^2 - 57396x^2n^2 - 9888\alpha n^2 - 3424n^2 - 64\alpha^5n - 736\alpha^4n + 768\alpha^5x^4n + 7840\alpha^4x^4n + 31232\alpha^3x^4n + 60912\alpha^2x^4n + 58320\alpha x^4n + 21978x^4n - 2784\alpha^3n - 4576\alpha^2n - 320\alpha^5x^2n - 4608\alpha^4x^2n - 23296\alpha^3x^2n - 53568\alpha^2x^2n - 57396\alpha x^2n - 23340x^2n - 3424\alpha n - 960n - 32\alpha^5 - 240\alpha^4 + 64\alpha^6x^4 + 784\alpha^5x^4 + 3904\alpha^4x^4 + 10152\alpha^3x^4 + 14580\alpha^2x^4 + 10989\alpha x^4 + 3402x^4 - 640\alpha^3 - 800\alpha^2 - 16\alpha^6x^2 - 352\alpha^5x^2 - 2504\alpha^4x^2 - 8240\alpha^3x^2 - 13881\alpha^2x^2 - 11670\alpha x^2 - 3897x^2 - 480\alpha - 112)(\alpha + 2n + 2)^2 - 2(\alpha + 2n + 1)wx(128n^6 - 1024x^2n^5 + 384\alpha n^5 + 1408n^5 + 448\alpha^2n^4 - 2560\alpha x^2n^4 - 5504x^2n^4 + 3520\alpha n^4 + 5592n^4 + 256\alpha^3n^3 + 3296\alpha^2n^3 - 2560\alpha^2x^2n^3 - 11008\alpha x^2n^3 - 11488x^2n^3 + 11184\alpha n^3 + 10888n^3 + 72\alpha^4n^2 + 1424\alpha^3n^2 + 7870\alpha^2n^2 - 1280\alpha^3x^2n^2 - 8256\alpha^2x^2n^2 - 17232\alpha x^2n^2 - 11688x^2n^2 + 16332\alpha n^2 + 11258n^2 + 8\alpha^5n + 272\alpha^4n + 2278\alpha^3n + 7692\alpha^2n - 320\alpha^4x^2n - 2752\alpha^3x^2n - 8616\alpha^2x^2n - 11688\alpha x^2n - 5814x^2n + 11258\alpha n + 5940n + 16\alpha^5 + 220\alpha^4 + 1124\alpha^3 + 2669\alpha^2 - 32\alpha^5x^2 - 344\alpha^4x^2 - 1436\alpha^3x^2 - 2922\alpha^2x^2 - 2907\alpha x^2 - 1134x^2 + 2970\alpha + 1257)z(\alpha + 2n + 2)^2 \geq 0\big)$$

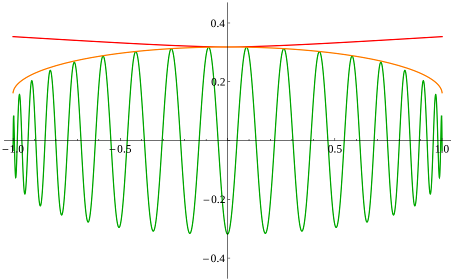
## *A different approach*

$$x^2 S(n,x) = g(n,x) + f(2n,x,0),$$

where

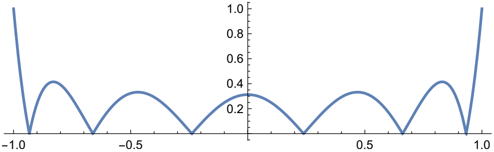
$$g(n,x) = \frac{2n+1}{2}\left(xP_{2n+1}(x) - \frac{4n+2}{4n+3}P_{2n}(x)\right)P_{2n}(0),$$

$$f(n,x,y) = -\sum_{j=0}^{n}\frac{1}{(2j-1)(2j+3)}P_j(x)P_j(y)$$



## *Boundedness of Legendre polynomials*

For $n \geq 2$ the successive relative maxima of $|P_n(x)|$, when $x$ decreases from 1 to 0, form a decreasing sequence.

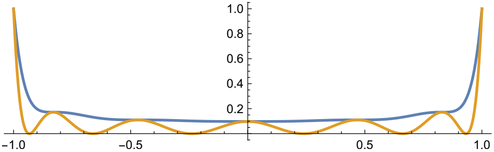


## *Boundedness of Legendre polynomials*

For $n \geq 2$ the successive relative maxima of $|P_n(x)|$, when $x$ decreases from 1 to 0, form a decreasing sequence.

Define

$$n(n+1)h_n(x) = n(n+1)P_n(x)^2 + (1-x^2)P_n'(x)^2.$$



## *Boundedness of Legendre polynomials*

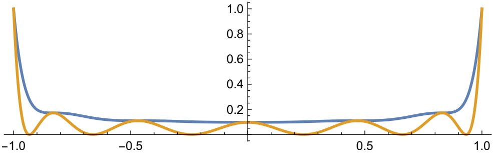
| |
|---|
| For $n \geq 2$ the successive relative maxima of $\lvert P_n(x)\rvert$, when $x$ decreases from 1 to 0, form a decreasing sequence. |

Define

$$n(n+1)h_n(x) = n(n+1)P_n(x)^2 + (1-x^2)P_n'(x)^2.$$

Using the Legendre differential equation one obtains

$$n(n+1)h_n'(x) = 2xP_n'(x)^2$$



## *Boundedness of Legendre polynomials*

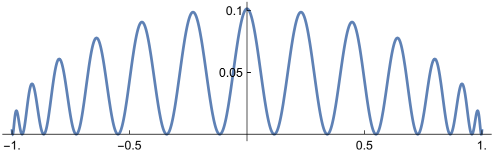
> For $n \geq 2$ the successive relative maxima of $|P_n(x)|$, when $x$ decreases from 1 to 0, form a decreasing sequence.

Define

$$n(n+1)h_n(x) = n(n+1)P_n(x)^2 + (1-x^2)P_n'(x)^2.$$

Using the Legendre differential equation one obtains

$$n(n+1)h_n'(x) = 2xP_n'(x)^2$$



## *Constructing the envelope for $g$*

Recall that

$$g(n,x) = \frac{2n+1}{2}\left(xP_{2n+1}(x) - \frac{4n+2}{4n+3}P_{2n}(x)\right)P_{2n}(0)$$

## *Constructing the envelope for $g$*

Recall that

$$g(n,x) = \frac{2n+1}{2}\left(xP_{2n+1}(x) - \frac{4n+2}{4n+3}P_{2n}(x)\right)P_{2n}(0)$$

We will work with the simplified version

$$G_n(x) = (4n+3)xP_{2n+1}(x) - 2(2n+1)P_{2n}(x).$$

## *Constructing the envelope for $g$*

Recall that

$$g(n,x) = \frac{2n+1}{2}\left(xP_{2n+1}(x) - \frac{4n+2}{4n+3}P_{2n}(x)\right)P_{2n}(0)$$

We will work with the simplified version

$$G_n(x) = (4n+3)xP_{2n+1}(x) - 2(2n+1)P_{2n}(x).$$

Ansatz for the envelope $h_n(x)$ with normalization $h_n(0) = G_n(0)^2$:

$$h_n(x) = (a_0 + a_2x^2)G_n(x)^2 + (1-x^2)(b_0 + b_2x^2)G_n'(x)^2$$
$$h_n^{der}(x) = c_1xG_n(x)^2 + x(d_0 + d_2x^2)G_n'(x)^2$$

## *The envelope for $G_n^2$*

The envelope for $G_n^2(x)$ is given by

$$H_n(x) = \frac{1}{1 + a_2 x^2} h_n(x)$$

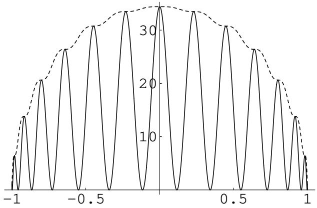
and its first derivative satisfies

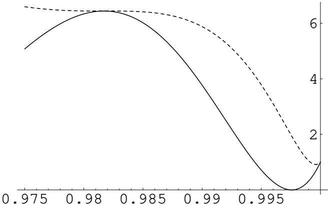
$$(1 + a_2 x^2)^2 H_n'(x) = x\, G_n'(x)^2 \big(-2a_2(1 - x^2)(b_0 + b_2 x^2) + (1 + a_2 x^2)(d_0 + d_2 x^2)\big).$$

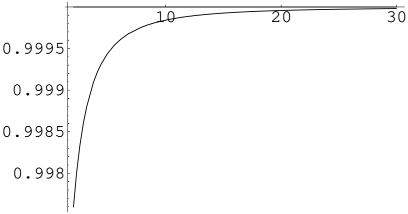
This yields the desired result for $x \in [-\rho(n), \rho(n)]$, where

$$\rho^2(n) = \Big(\left(4n^2 + 5n + 2\right)\left(8n^2 + 14n + 7\right) - \sqrt{84n^4 + 252n^3 + 291n^2 + 153n + 31}\Big) \cdot 1/\left((n+1)(2n+1)(4n+3)^2\right)$$

# *Envelope and* $\rho(n)$ *for* $n = 6$



## *A Theorem of Alexander Alexandrov and Geno Nikolov*

Let $f$ be a polynomial of degree at most $n$ having only real zeros and define

$$L_k(f;x) = \sum_{j=0}^{2k} (-1)^{k-j} \frac{f^{(j)}(x)}{j!} \, \frac{f^{(2k-j)}(x)}{(2k-j)!}, \quad 1 \le k \le n.$$

## *A Theorem of Alexander Alexandrov and Geno Nikolov*

Let $f$ be a polynomial of degree at most $n$ having only real zeros and define

$$L_k(f;x) = \sum_{j=0}^{2k} (-1)^{k-j} \frac{f^{(j)}(x)}{j!} \, \frac{f^{(2k-j)}(x)}{(2k-j)!}, \quad 1 \le k \le n.$$

For the choice $f(x) = H_n(x)$ (the $n$th Hermite polynomial) the function $L_k(H_n;\cdot)$ is

- *monotonically decreasing* in $(-\infty, 0]$, and
- *monotonically increasing* in $[0, \infty)$

for all $1 \le k \le n$.

## *The functional for Hermite polynomials*

Using that derivatives of Hermite polynomials satisfy the relations

$$H_n'(x) = 2nH_{n-1}(x) \quad \text{and} \quad D_x^m H_n(x) = \frac{2^m n!}{(n-m)!} H_{n-m}(x)$$

yields

$$L_k(H_n; x) = \sum_{j=0}^{2k} (-1)^{k-j} \binom{n}{j} \binom{n}{2k-j} 4^k H_{n-j}(x) H_{n-2k+j}(x).$$

## *The functional for Hermite polynomials*

Using that derivatives of Hermite polynomials satisfy the relations

$$H_n'(x) = 2nH_{n-1}(x) \quad \text{and} \quad D_x^m H_n(x) = \frac{2^m n!}{(n-m)!} H_{n-m}(x)$$

yields

$$L_k(H_n; x) = \sum_{j=0}^{2k} (-1)^{k-j} \binom{n}{j} \binom{n}{2k-j} 4^k H_{n-j}(x) H_{n-2k+j}(x).$$

- Compute a set of annihilating operators.

## *The functional for Hermite polynomials*

Using that derivatives of Hermite polynomials satisfy the relations

$$H_n'(x) = 2nH_{n-1}(x) \quad \text{and} \quad D_x^m H_n(x) = \frac{2^m n!}{(n-m)!} H_{n-m}(x)$$

yields

$$L_k(H_n; x) = \sum_{j=0}^{2k} (-1)^{k-j} \binom{n}{j} \binom{n}{2k-j} 4^k H_{n-j}(x) H_{n-2k+j}(x).$$

- Compute a set of annihilating operators.
- The goal is to show that $L_k'(H_n; x) \geq 0$ for $x \geq 0$.

## *The functional for Hermite polynomials*

Using that derivatives of Hermite polynomials satisfy the relations

$$H_n'(x) = 2nH_{n-1}(x) \quad \text{and} \quad D_x^m H_n(x) = \frac{2^m n!}{(n-m)!} H_{n-m}(x)$$

yields

$$L_k(H_n; x) = \sum_{j=0}^{2k} (-1)^{k-j} \binom{n}{j} \binom{n}{2k-j} 4^k H_{n-j}(x) H_{n-2k+j}(x).$$

- Compute a set of annihilating operators.
- The goal is to show that $L_k'(H_n; x) \geq 0$ for $x \geq 0$.
- Idea: search for a relation in the ideal where the coefficients do *not* depend on $x$.

## *The functional for Hermite polynomials*

Using that derivatives of Hermite polynomials satisfy the relations

$$H_n'(x) = 2nH_{n-1}(x) \quad \text{and} \quad D_x^m H_n(x) = \frac{2^m n!}{(n-m)!} H_{n-m}(x)$$

yields

$$L_k(H_n; x) = \sum_{j=0}^{2k} (-1)^{k-j} \binom{n}{j} \binom{n}{2k-j} 4^k H_{n-j}(x) H_{n-2k+j}(x).$$

- Compute a set of annihilating operators.
- The goal is to show that $L_k'(H_n; x) \geq 0$ for $x \geq 0$.
- Idea: search for a relation in the ideal where the coefficients do *not* depend on $x$.
- Using HolonomicFunctions [Koutschan] we find for $y_{k,n}(x) = L_k(H_n; x)$

  $(k+1) y_{k+1,n+1}(x) = 4(n+1) y_{k,n}(x) + 2(k+1)(n+1) y_{k+1,n}(x)$

## *The functional for Hermite polynomials*

Using that derivatives of Hermite polynomials satisfy the relations

$$H_n'(x) = 2nH_{n-1}(x) \quad \text{and} \quad D_x^m H_n(x) = \frac{2^m n!}{(n-m)!} H_{n-m}(x)$$

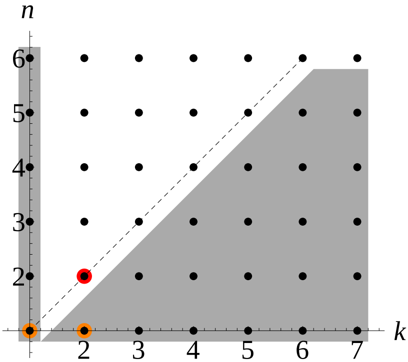
yields

$$L_k(H_n;x) = \sum_{j=0}^{2k} (-1)^{k-j} \binom{n}{j} \binom{n}{2k-j} 4^k H_{n-j}(x) H_{n-2k+j}(x).$$

- Compute a set of annihilating operators.
- The goal is to show that $L_k'(H_n;x) \geq 0$ for $x \geq 0$.
- Idea: search for a relation in the ideal where the coefficients do *not* depend on $x$.
- Using HolonomicFunctions [Koutschan] we find for $y_{k,n}(x) = L_k(H_n;x)$ or for the derivative

  $$(k+1){y'}_{k+1,n+1}(x) = 4(n+1){y'}_{k,n}(x) + 2(k+1)(n+1){y'}_{k+1,n}(x)$$

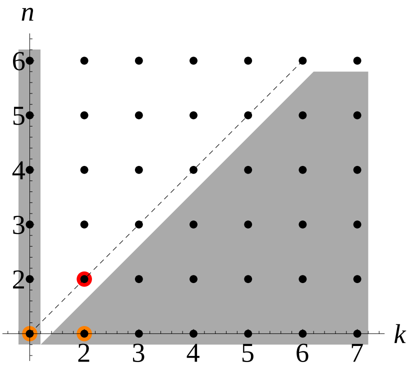
## The recurrence




$$(k+1)y'_{k+1,n+1}(x) = 4(n+1)y'_{k,n}(x) + 2(k+1)(n+1)y'_{k+1,n}(x)$$

$$L_k(f;x) = \sum_{j=0}^{2k}(-1)^{k-j}\,\frac{f^{(j)}(x)}{j!}\;\frac{f^{(2k-j)}(x)}{(2k-j)!}$$

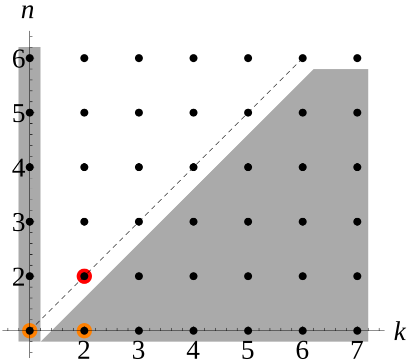
## *The recurrence*




$$(k+1)y'_{k+1,n+1}(x) = 4(n+1)y'_{k,n}(x) + 2(k+1)(n+1)y'_{k+1,n}(x)$$

$$L_1(H_n;x) = 4n\left(nH_{n-1}(x)^2 - (n-1)H_{n-2}(x)H_n(x)\right)$$

## The recurrence




$$(k+1)y'_{k+1,n+1}(x) = 4(n+1)y'_{k,n}(x) + 2(k+1)(n+1)y'_{k+1,n}(x)$$

$$L'_1(H_n;x) = 8(n^2-n)\left(nH_{n-2}(x)H_{n-1}(x)-(n-2)H_{n-3}(x)H_n(x)\right)$$


Thank you!