JT Gravity, Matrix Models, and Applications

A Master Thesis Presentation

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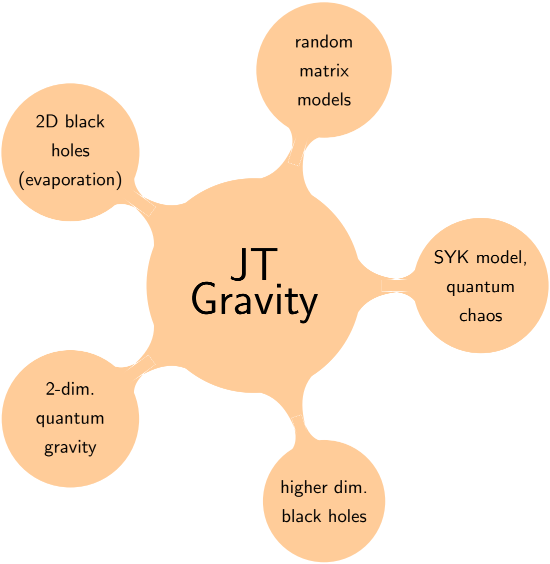
Outline

- JT Gravity & Schwarzian Theory
- Matrix Models
- Constructing (S)JT from Matrix Models (*Johnson '19, '20*)
- A New Family of Matrix Models
- Solving the String Equation
- Results
- Summary & Outlook

Motivation



JT Gravity
random matrix models
2D black holes (evaporation)
SYK model, quantum chaos
2-dim. quantum gravity
higher dim. black holes

Jackiw-Teitelboim (JT) Gravity

action:

$$S = S_0\,\chi(\mathcal{M}) + \int_{\mathcal{M}} d^2x\,\sqrt{g}\,\phi\,\left(R - \Lambda\right) + 2\int_{\partial\mathcal{M}} d\tau\,\sqrt{h}\,\phi_b K$$

with e.o.m.s

$$R = -2 \qquad\qquad g_{\mu\nu}\left(\Box + \frac{\Lambda}{2}\right)\phi - \nabla_\mu\nabla_\nu\phi = 0$$

is AdS_2 (also locally). Counting topologies: $\hbar = e^{-S_0}$

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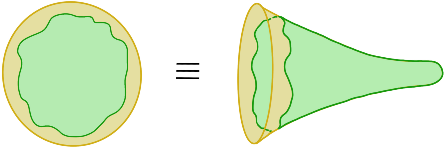

Considering N AdS_2 boundaries one arrives at the Schwarzian action.

The Schwarzian Theory

action:

$$S = -C \int dt \ \{F(t), t\} \qquad \{F(t), t\} = \frac{F'''}{F'} - \frac{3}{2}\left(\frac{F''}{F'}\right)^2$$

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$Z(g)$ of each topology is one-loop exact (*Stanford, Witten '17*), e.g.

$$Z_0(\beta) = \frac{\#}{g^3} \exp\left(\frac{\pi}{g^2}\right) \sim \int_0^\infty \underbrace{\sinh\left(2\pi\sqrt{E}\right)}_{\equiv \rho_0(E)} e^{-\beta E} \qquad \text{with } \beta \sim g^2$$

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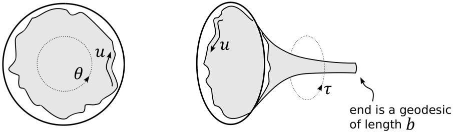
Topological expansion is dual to $\frac{1}{N}$-expansion of random matrices (*Saad, Shenker, Stanford '19*)



Spectral Form Factor

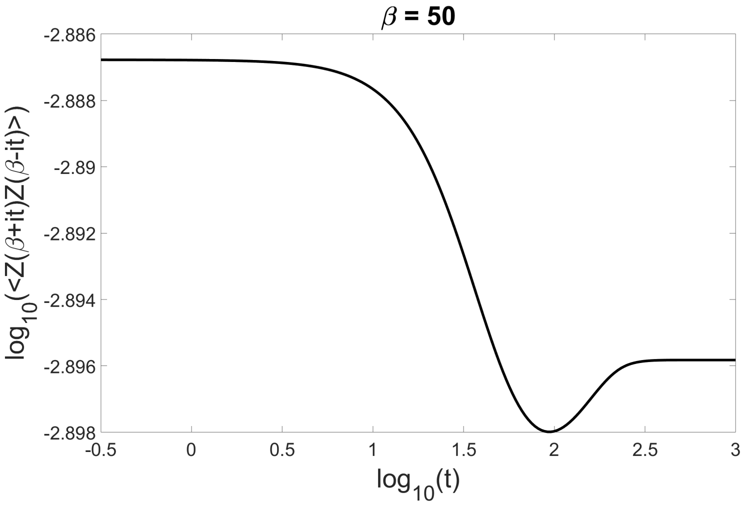


Figure: SFF for JT Gravity with $\Gamma = 0$ and $\beta = 50$

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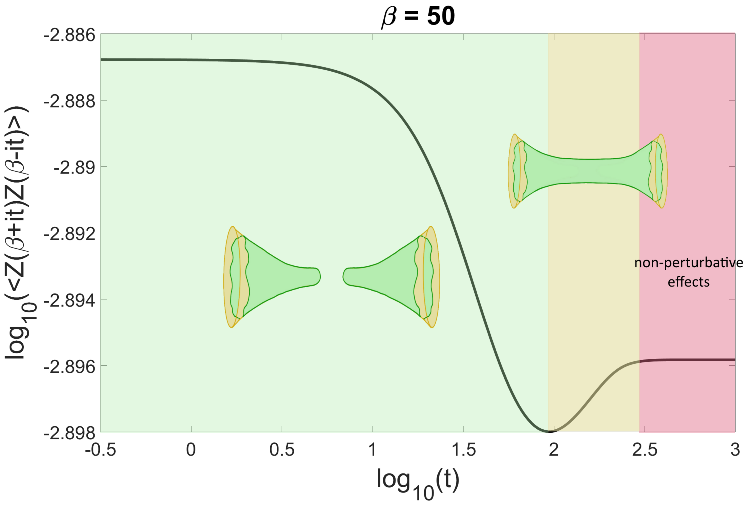

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Matrix Models

Matrix Models describe triangulations of 2-dim. surfaces. The dynamical degrees of freedom are random $N \times N$ matrices M.

$$Z = \int \mathcal{D}M \ \exp\left(-N \ \mathrm{Tr}\left[V(M)\right]\right)$$

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Partition function of a triangulation is loop of length β (*Banks et al. '90*):

$$Z(\beta) = \int_{-\infty}^{\mu} dx \; \langle x|e^{-\beta\mathcal{H}}|x\rangle = \int_0^{\infty} dE \; \rho(E) e^{-\beta E}$$

$$\rho(E) = \int_{-\infty}^{\mu} dx \; \psi(x,E)\psi^*(x,E)$$

Lax equation:

$$\mathcal{H}\psi(x,E) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + u(x)\right]\psi(x,E) = E\psi$$

and $u(x)$ satisfies the **string equation**.

The String Equation

The string equation:

$$u\mathcal{R}^2 - \frac{\hbar^2}{2}\mathcal{R}\mathcal{R}'' + \frac{\hbar^4}{4}\left(\mathcal{R}'\right)^2 = \hbar^2\Gamma^2 \qquad \mathcal{R} = R_k[u(x)] + x$$

with $R_k\left[u(x)\right] = u^k(x) + \ldots + \hbar^{2k-2}u^{(2k-2)}(x)$ the Gelfand-Dikii polynomials.

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Named after the $(p = 2, q = 2k - 1)$ class of minimal string theories which consist of two components:

- a (p, q) minimal CFT with central charge $c = 1 - \frac{6(p-q)^2}{pq} < 1$
- Liouville theory

(S)JT Gravity from Matrix Models (I)

String equation at classical level ($\hbar = 0$):

$$u(x)\,\mathcal{R}^2 = 0 \qquad\qquad u(x) = \begin{cases} (-x)^{\frac{1}{k}} + \dots & x < 0 \\ 0 + \dots & x > 0 \end{cases}$$

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"Couple" minimal strings together via potential:

$$-x = \sum_{k=1}^{\infty} t_k u_0^k \qquad\qquad f(u_0) = -\frac{\partial x}{\partial u_0} = \sum_{k=0}^{\infty} k t_k u_0^{k-1}$$

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Then the spectral density

$$\rho_0(E) = \frac{1}{2\pi\hbar}\int_0^E du_0 \sum_{k=0}^{\infty} \frac{k t_k u_0^{k-1}}{\sqrt{E - u_0}} = \frac{1}{2\pi\hbar}\int_0^E du_0 \frac{f(u_0)}{\sqrt{E - u_0}}$$

(S)JT Gravity from Minimal Strings/Matrix Models (II)

What we have:

	t_k	$\rho_0(E)$	$f(u_0)$
JT	$\frac{\pi^{2k-2}}{2k!(k-1)!}$	$\sim \sinh(2\pi\sqrt{E})$	$\sim I_0(2\pi\sqrt{u_0})$
SJT	$\frac{2\pi^{2k}}{(k!)^2}$	$\sim \frac{1}{\sqrt{E}}\cosh(2\pi\sqrt{E})$	$\sim \frac{I_1(2\pi\sqrt{u_0})}{\sqrt{u_0}}$

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Idea: generalize the kernel $f(u_0)$ (and thus t_k and $\rho_0(E)$) to

	t_k	$\rho_0(E)$	$f(u_0)$
n	$\sim \frac{\pi^{2k+n-2}}{k!(k+n-1)!}$	$\sim \frac{L_{n-\frac{1}{2}}(2\pi\sqrt{E})}{(\sqrt{E})^{n-\frac{1}{2}}}$	$\sim \frac{I_n(2\pi\sqrt{u_0})}{(\sqrt{u_0})^n}$

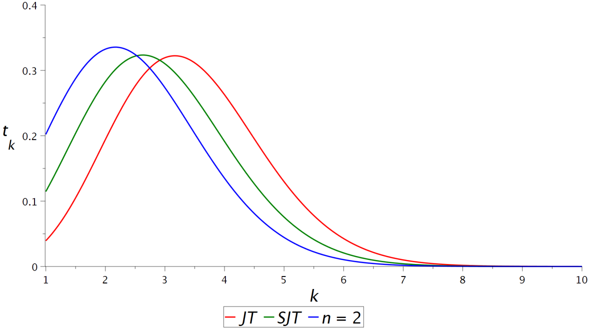
with I_n the n-th Bessel function and L_n Struve functions.

Solving the String Equation

Solve string equation as BVP; to write boundary conditions, make perturbative ansatz:

$$u(x) = u_0(x) + \hbar u_1(x) + \hbar^2 u_2(x) + \ldots$$

Turning on all t_k's yields infinite order differential equation $\Rightarrow$ truncate t_k up to some $k_{\text{max}} = 6$.



Results for Potentials

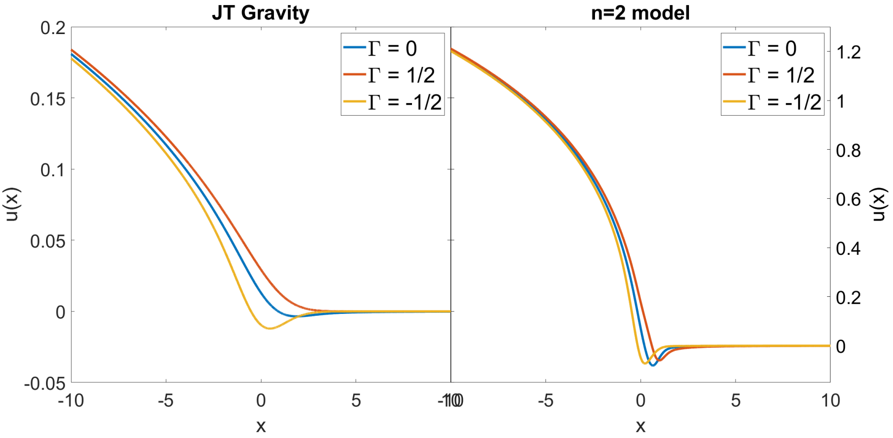


Figure: Comparison of the potentials for JT Gravity and the $n = 2$ model for three different values of $\Gamma = 0, \frac{1}{2}, -\frac{1}{2}$

Matrix Numerov Method

Then numerically solve Schrödinger equation

$$\mathcal{H}\psi(x) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + u(x)\right]\psi(x) = E\psi(x)$$

with fully quantum potentials to obtain wavefunctions $\psi(x)$

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$\Rightarrow$ use Matrix Numerov method (*Pillai, Goglio, Walker '12*)

Results for Spectral Densities

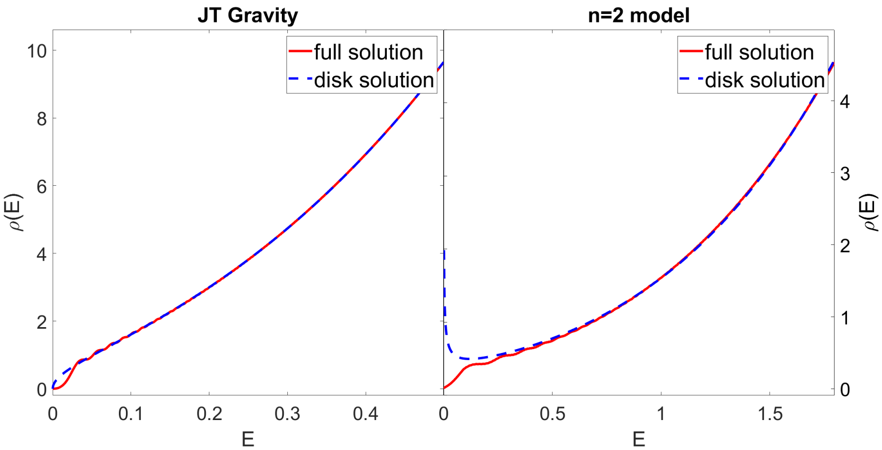

Figure: Comparison of the disk contribution and full spectral densities for JT Gravity and the $n = 2$ model for $\Gamma = 0$

Spectral Form Factor

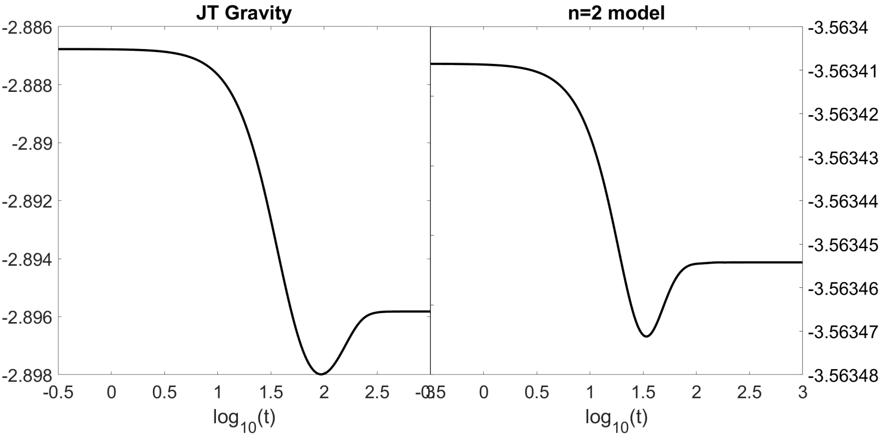

Figure: $\log_{10}$ of the temporal shape of the SFF $\langle Z(\beta + it)Z(\beta - it)\rangle$ with $\beta = 50$ comparing JT Gravity with the $n = 2$ model for $\Gamma = 0$

Summary & Outlook

- Discussion of matrix models and JT gravity
- New family of matrix models with interesting features
- Successful implementation and numerical results

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- Discussion of matrix models and JT gravity
- New family of matrix models with interesting features
- Successful implementation and numerical results
- Connection to other Schwarzian theories ($\mathcal{N} = 2$ SJT)
- More fundamental insights through MST
- Discussion for more values of n and Γ
- Improve numerical precision (truncation)

Thank you for your attention!

Sachdev-Ye-Kitaev (SYK) Model

ensemble of 1D quantum systems with N Majorana fermions

$$H_{\text{SYK, int}} = \sum_{1 \leq i < j < k < l \leq N} J_{ijkl}\, \psi_i\, \psi_j\, \psi_k\, \psi_l$$

with J_{ijkl} random numbers drawn from a Gaussian distribution.

Single models exhibit chaos at late times

$$Z(\beta) = \int \mathcal{D}\psi\ \exp\left[-\int_0^\beta d\tau\ \left(\psi_i \partial_\tau \psi_i + J_{ijkl}\psi_i\psi_j\psi_k\psi_l\right)\right]$$

Interesting physics appear in ensemble average $\Rightarrow$ arrive at Schwarzian action for $1 \ll \beta J$

$$\langle Z(\beta)\rangle_J = \int \mathcal{D}J_{ijkl}\ \exp\left(-\sum_{1 \leq i < j < k < l \leq N} \frac{J_{ijkl}^2}{2\sigma^2}\right) Z(\beta)$$

The Double Scaling Limit (DSL) (I)

Partition function

$$Z(g) = \int \mathcal{D}M \; \exp\left(-N \, \mathrm{Tr}\left[V(M)\right]\right)$$

$$\overset{M=U\Lambda U^{-1}}{=} \int \prod_i d\lambda_i \; \Delta(\lambda_i) \; \exp\left[-\sum_i V(\lambda_i)\right]$$

with $\Delta(\lambda)$ the v.d.M. det. This can be rewritten in terms of orthonormal polynomials $P_n(\lambda_i)$ (*Zuber et al. '78*) which satisfy recursion relation

$$\lambda P_n(\lambda) = P_{n+1}(\lambda) + A_n P_{n-1}(\lambda)$$

Introduce free fermions

$$\psi_n(\lambda) = \frac{1}{\sqrt{h_n}} P_n(\lambda) \; \exp\left(-\frac{1}{2} N V(\lambda)\right)$$

The Double Scaling Limit (DSL) (II)

Second quantization and Fermi sea $|N\rangle$:

$$\Psi(\lambda) = \sum_{n=0}^{\infty} a_n \psi_n(\lambda) \qquad \left\{a_n, a_m^\dagger\right\} = \delta_{n,m}$$

$$a_n |N\rangle = 0 \qquad n \geq N \qquad a_n^\dagger |N\rangle = 0 \qquad n < N$$

recursion relation translates to

$$\lambda \psi_n = \sqrt{r_{n+1}} \psi_{n+1} + \sqrt{r_n} \psi_{n-1}$$

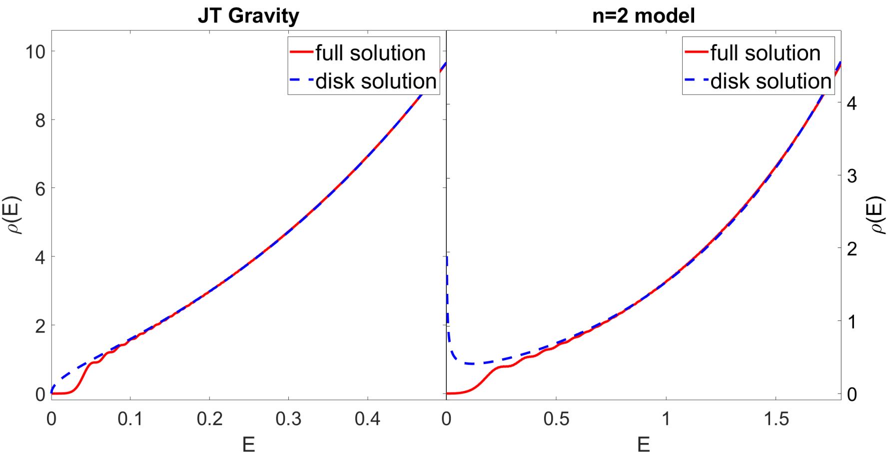
Wick's theorem: can compute all amplitudes via 2-pt function

$$K_N(\lambda_1, \lambda_2) \equiv \langle N | \Psi^\dagger(\lambda_1) \Psi(\lambda_2) | N \rangle \qquad \rho(\lambda) = K_N(\lambda, \lambda)$$

spectral density is (in general) multicritical, DSL:

$$\frac{n}{N} \to \infty \qquad \lambda \to \lambda_c \qquad r_n[V] \to r_c + \epsilon\, u(x)$$

Results for Spectral Densities



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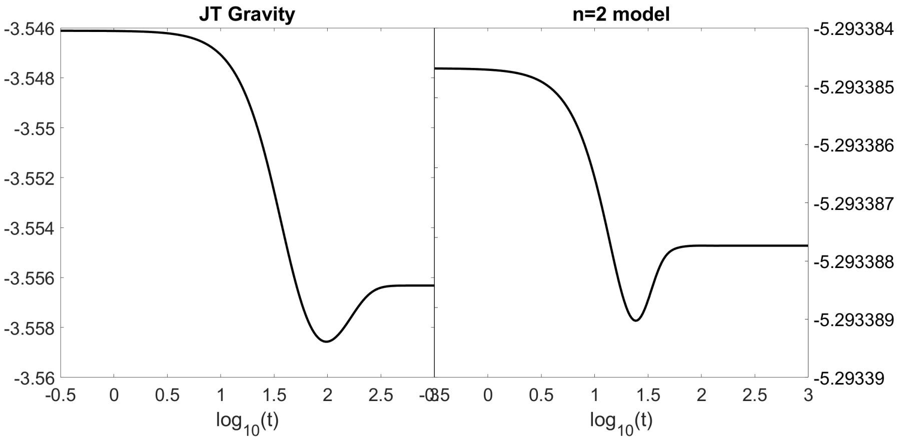

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