PROGRESS ON MULTILOOP CALCULATIONS

(MAINLY TWO EXAMPLES OF SOME DEVELOPMENTS I FIND INTERESTING)

Tools for High Precision LHC Simulations
Ringberg Castle - 1/11/2022

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Technische Universität München

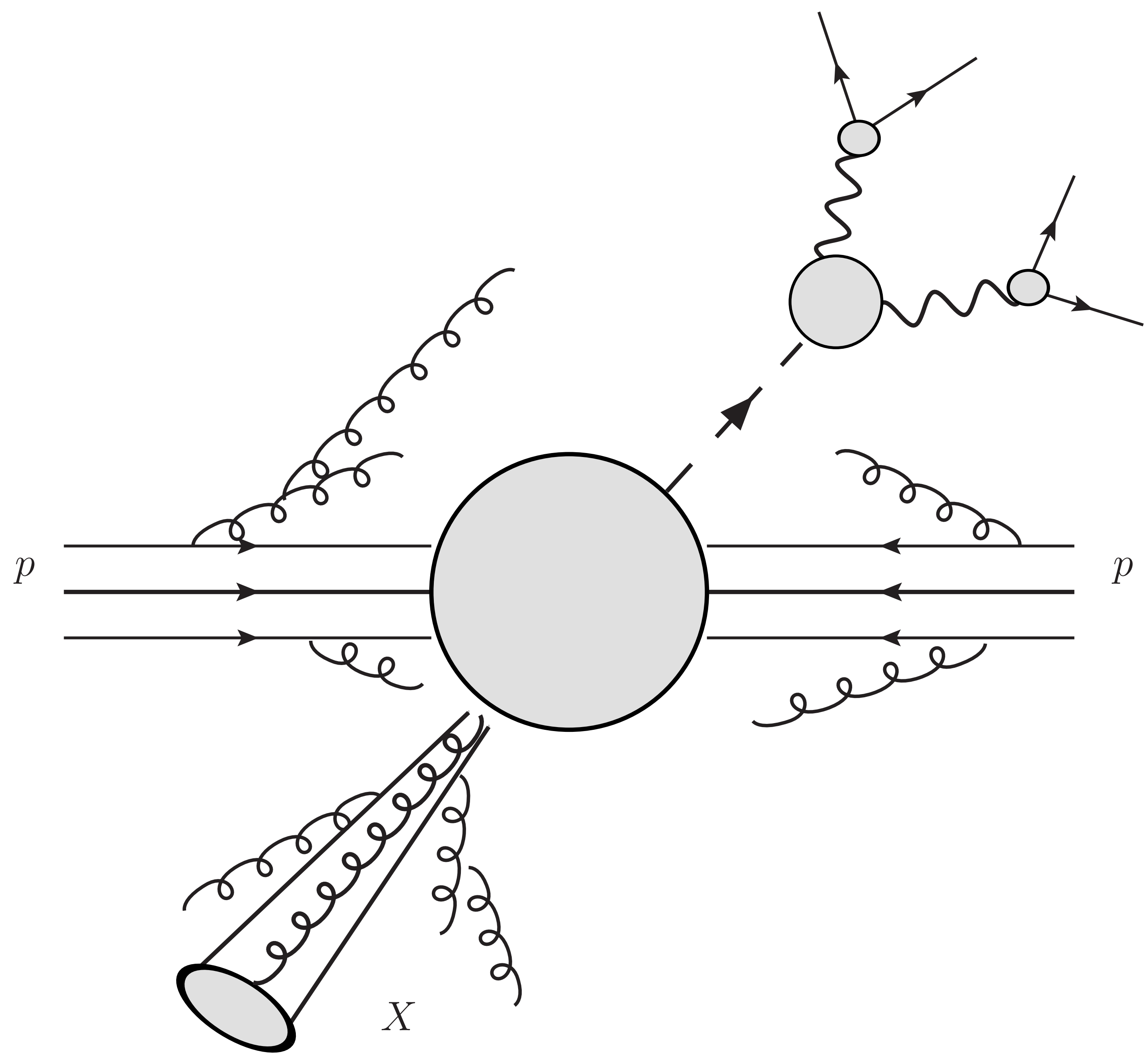
DISCLAIMER

Not trying to be a review, and therefore *not* complete in any way

Just giving an account of some problems I find interesting and that I have personally been working on in the past months

FROM LAGRANGIANS TO CROSS-SECTIONS

From Lagrangian to Cross-Section it's a long way



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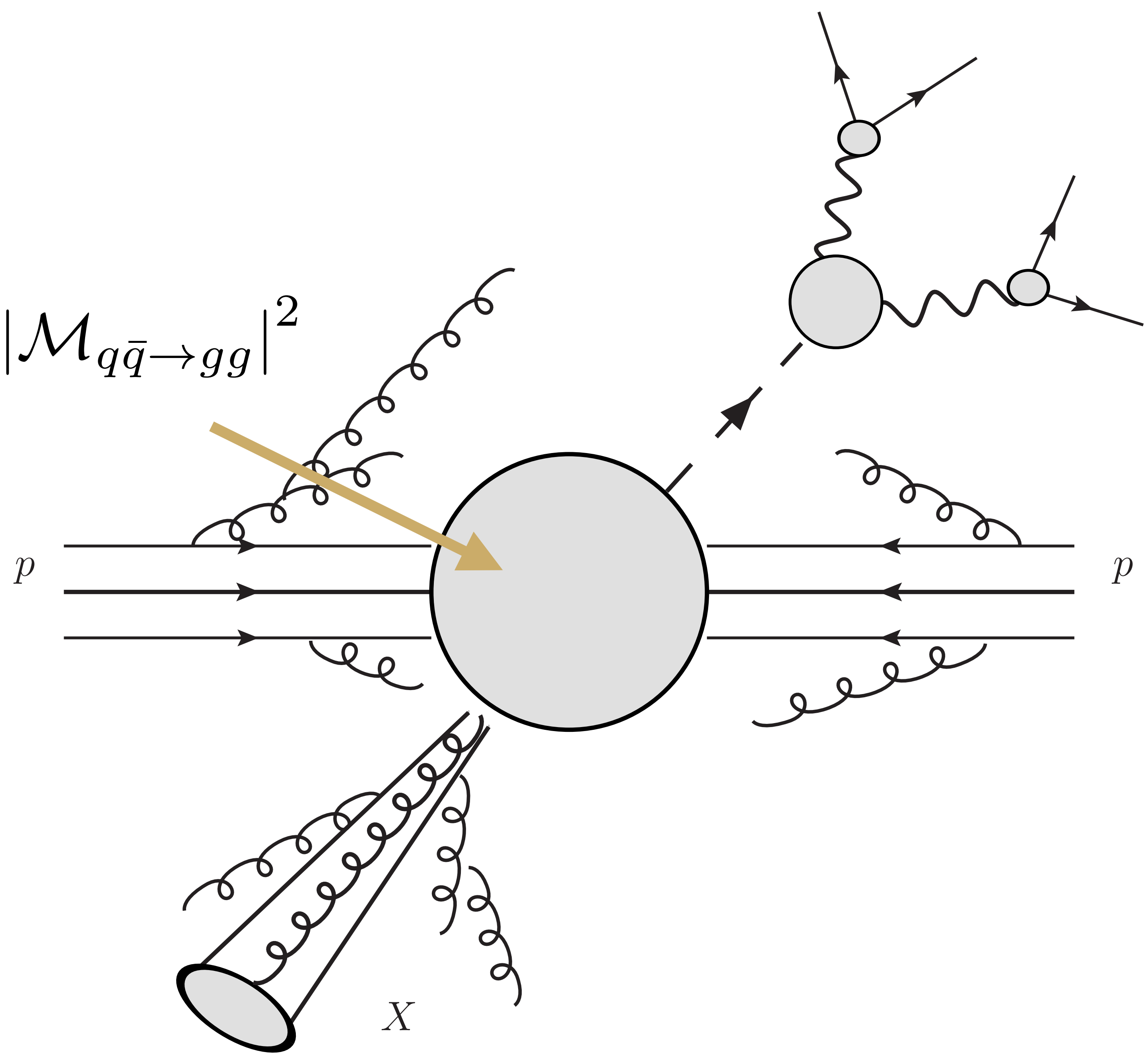
$$\sigma_{q\bar{q}\to gg} = \int [\mathrm{dPS}]\, |\mathcal{M}_{q\bar{q}\to gg}|^2$$



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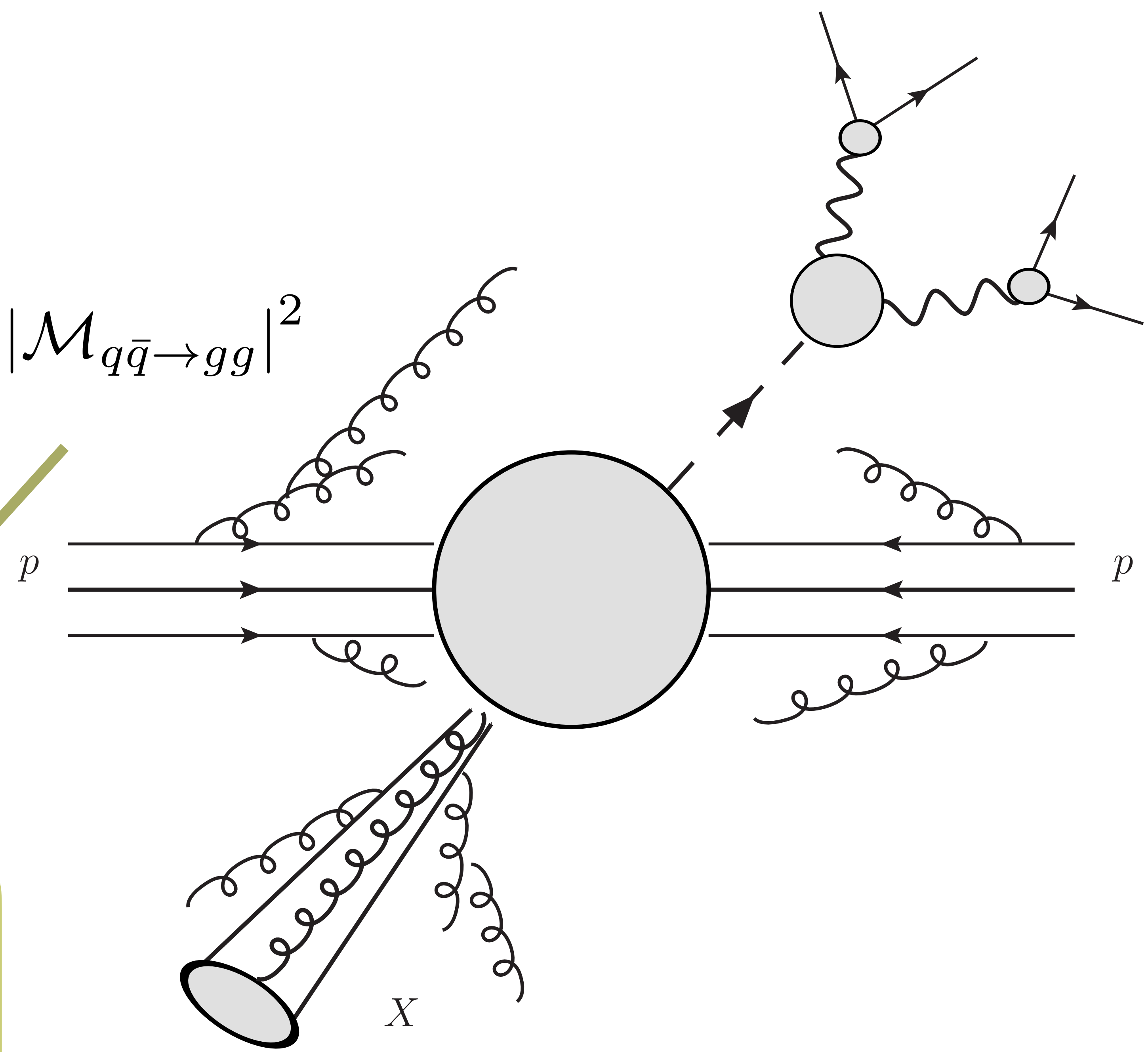
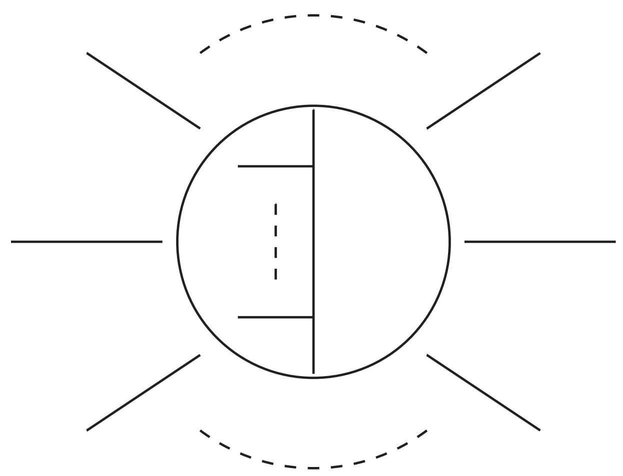


Building blocks are Scattering Amplitudes

FROM AMPLITUDES TO INTEGRALS

Scattering Amplitudes



$$\sim \qquad \mathscr{A} = \epsilon_1^{\mu_1} \cdots \epsilon_n^{\mu_n}\, \bar{v}(q)\, \Gamma_{\mu_1,\ldots,\mu_n}\, u(p)$$

FROM AMPLITUDES TO INTEGRALS

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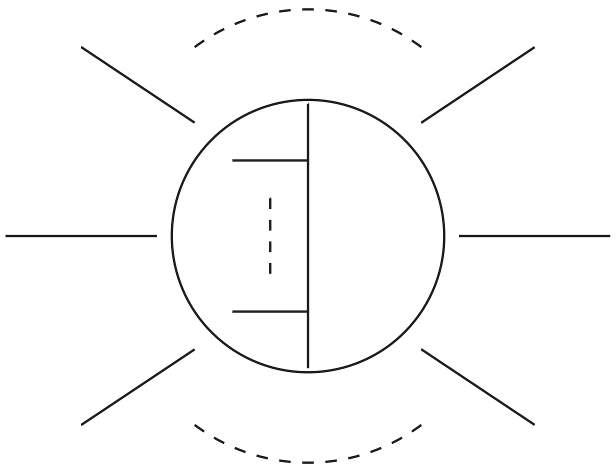

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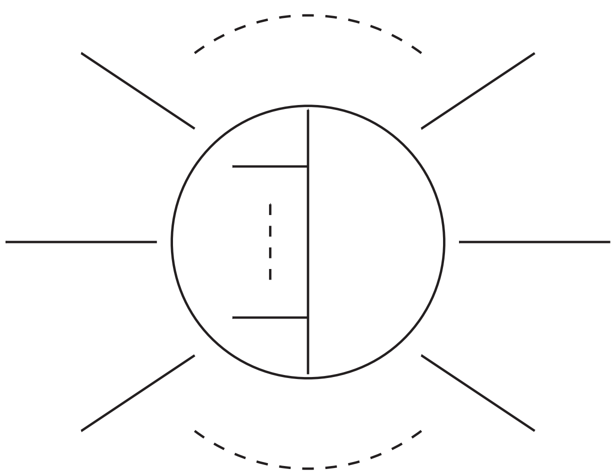
(Scalar) Feynman Integrals

$$\mathscr{I} = \int \prod_{l=1}^{L} \frac{d^D k_l}{(2\pi)^D} \frac{S_1^{b_1} \ldots S_m^{b_m}}{D_1^{a_1} \ldots D_n^{a_n}}$$

with $S_i \in \{k_i \cdot k_j, \ldots, k_i \cdot p_j\}$

FROM AMPLITUDES TO INTEGRALS

Scattering Amplitudes



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IBPs, Finite fields etc

differential equations

Feynman parameters

Numerical methods …

Some analytic or numerical result for the amplitudes

SCALAR FEYNMAN INTEGRALS

From tensor reduction, huge number of scalar integrals ($gg \to gg$ @ 3 loops $\sim 10^7$ integrals!)
Standard Approach: **divide et impera**

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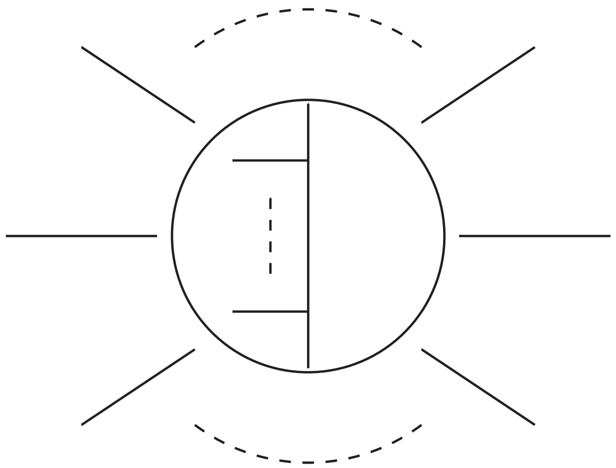

Integration by parts identities → **master integrals**

[Chetyrkin, Tkachov '81] **& many others: most recently finite fields, intersection theory etc**

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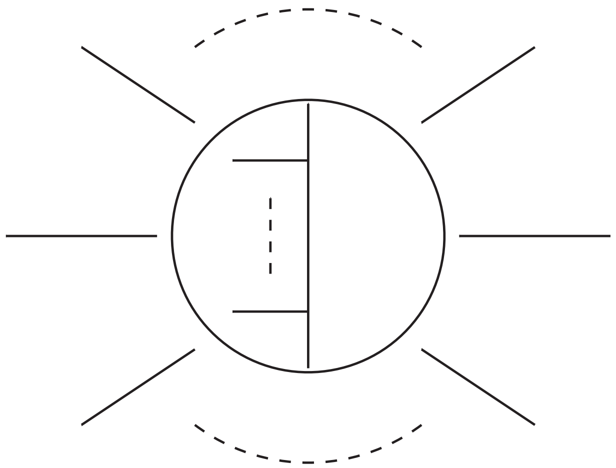

$$= \sum_{i=1}^{N} R_i(x_1, \ldots, x_r)\, \mathcal{I}_i(x_1, \ldots, x_n)$$

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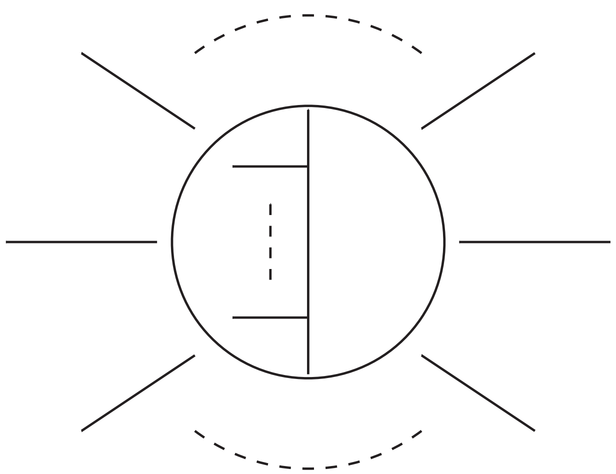

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Coefficients (process-dependent)

Very complicated rational functions, hundreds of **MBs** for complicated processes:

Algebraic Complexity

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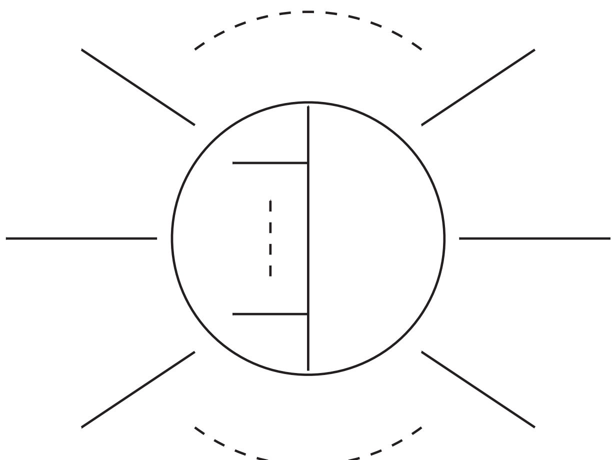

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Algebraic Complexity

Process-independent building blocks: **Master Integrals**

Involved **special functions** with complicated mathematical properties:

Analytic complexity

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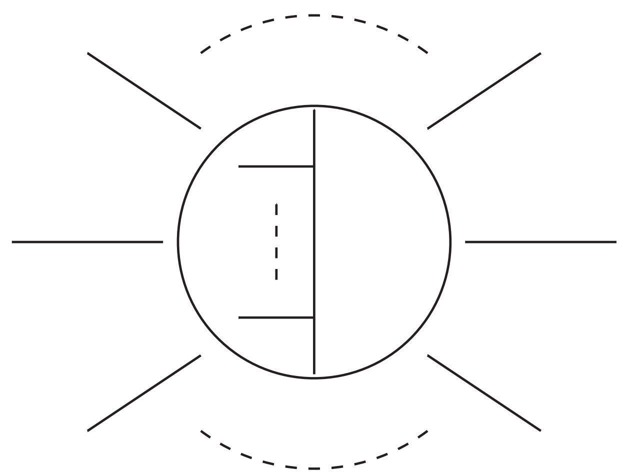
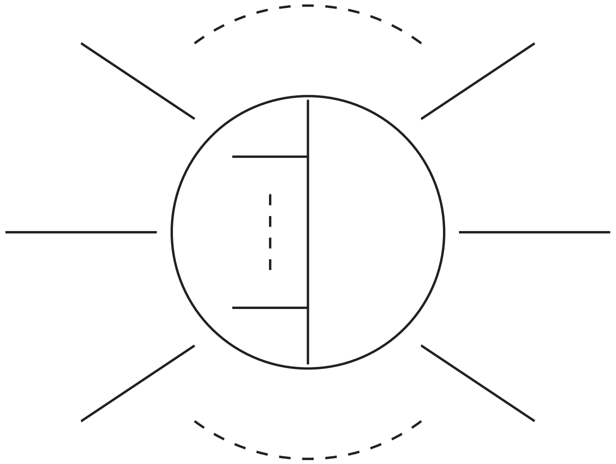


$$= \sum_{i=1}^{N} R_i(x_1, \ldots, x_r) \, \mathcal{I}_i(x_1, \ldots, x_n)$$

Extremely successful strategy: in the past 2 decades it has allowed us to overcome the **two-loop frontier** for $2 \to 2$ and $2 \to 3$ processes, with increasing number of scales (and masses), and recently opened the way to $2 \to 2$ **three loop calculations**

I will not review all these developments, there are way too many :-)

DECOMPOSITION INTO SCALAR INTEGRALS



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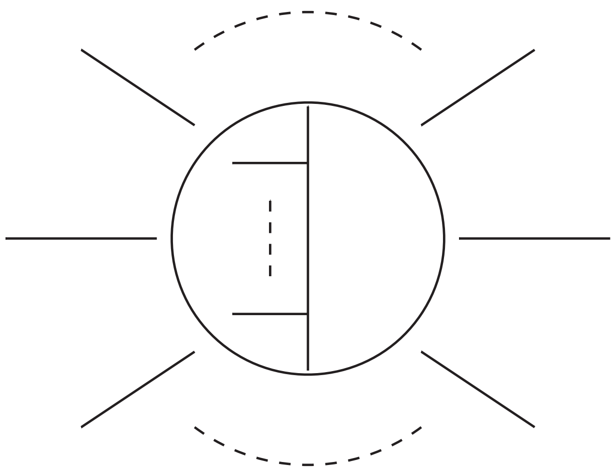
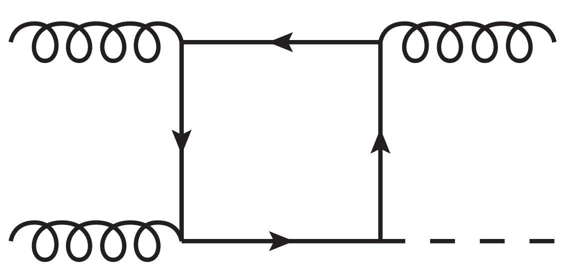


$$= \sum_{i=1}^{N} R_i(x_1, \ldots, x_r)\, \mathcal{I}_i(x_1, \ldots, x_n)$$

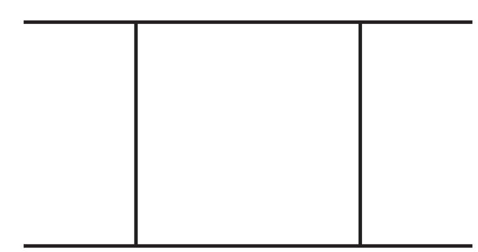
$\mathcal{M}_{gg\to Hg} \sim$



First step:

Strip it of Lorentz and Dirac structures

$$\sim \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 (k-p_2)^2 (k-p_2-p_3)^2 (k-p_1-p_2-p_3)^2} =$$



Scalar Feynman Integrals are what we know how to compute

TENSOR DECOMPOSITION

Projector-Form Factors method in a nutshell

1. Pick your favourite process, for example $q\bar{q} \to Zg$
2. Use **Lorentz + gauge + any symmetry** (parity, Bose etc…) to find minimal set of tensor structures in d space-time dimensions:

$$\mathscr{A} = \sum_j F_j T_j$$

3. Derive **projectors operators** to single out corresponding form factors: $\mathscr{P}_j \mathscr{A} = F_j$

$$M_{ij} = \sum_{pol} T_i^\dagger T_j \qquad \mathscr{P}_j = \sum_k \left(M^{-1}\right)_{jk} T_k^\dagger$$

4. Apply these projectors on Feynman diagrams repr of the scattering amplitude

TENSOR DECOMPOSITION: PROS AND CONS

Problems in d-dimensions

Powerful and very general method

Often used in **CDR**, can become intractable for complicated problems due to **evanescent structures in d=4**

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Typical case 4 quark scattering $q(p_2) + \bar{q}(p_1) \to Q(p_3) + \bar{Q}(p_4)$

$$D_i \sim \bar{u}(p_1)\Gamma^{\mu_1,\ldots,\mu_n} u(p_2)\ \bar{u}(p_3)\Gamma_{\mu_1,\ldots,\mu_n} u(p_4)$$

Infinite number of tensor structures in d dimensions

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Infinite number of tensor structures in d dimensions

$$\begin{aligned}
\mathcal{D}_1 &= \bar{u}(p_1)\gamma_{\mu_1}u(p_2)\ \bar{u}(p_3)\gamma_{\mu_1}u(p_4), \\
\mathcal{D}_2 &= \bar{u}(p_1)\not{p}_3 u(p_2)\ \bar{u}(p_3)\not{p}_1 u(p_4), \\
\mathcal{D}_3 &= \bar{u}(p_1)\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}u(p_2)\ \bar{u}(p_3)\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}u(p_4), \\
\mathcal{D}_4 &= \bar{u}(p_1)\gamma_{\mu_1}\not{p}_3\gamma_{\mu_3}u(p_2)\ \bar{u}(p_3)\gamma_{\mu_1}\not{p}_1\gamma_{\mu_3}u(p_4), \\
\mathcal{D}_5 &= \bar{u}(p_1)\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\gamma_{\mu_4}\gamma_{\mu_5}u(p_2)\ \bar{u}(p_3)\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\gamma_{\mu_4}\gamma_{\mu_5}u(p_4), \\
\mathcal{D}_6 &= \bar{u}(p_1)\gamma_{\mu_1}\gamma_{\mu_2}\not{p}_3\gamma_{\mu_4}\gamma_{\mu_5}u(p_2)\ \bar{u}(p_3)\gamma_{\mu_1}\gamma_{\mu_2}\not{p}_1\gamma_{\mu_4}\gamma_{\mu_5}u(p_4).
\end{aligned}$$

→ up to 2 loops!

TENSOR DECOMPOSITION: UPGRADE IN THV

Improvements in d=4 [Peraro, Tancredi '19,'20]

Only two of these structures are linearly independent if external states are in $d = 4$

$$q(p_2) + \bar{q}(p_1) \to Q(p_3) + \bar{Q}(p_4)$$

$$\begin{aligned} \mathcal{D}_1 &= \bar{u}(p_1)\gamma_{\mu_1} u(p_2)\ \bar{u}(p_3)\gamma_{\mu_1} u(p_4), \\ \mathcal{D}_2 &= \bar{u}(p_1)\not{p}_3 u(p_2)\ \bar{u}(p_3)\not{p}_1 u(p_4), \end{aligned}$$

They are enough to obtain full result in 't Hooft-Veltman scheme

They are also enough for the finite remainder in CDR!

Use to complete $pp \to pp$ @ 3 loops [Caola, Chakraborty, Gambuti, Manteuffel, Tancredi '21,'22]

TENSOR DECOMPOSITION FOR CHIRAL THEORIES

Let's see how this works for **chiral theories** **(new & unpublished)**

Consider the production of a Z-boson and a jet in quark-antiquark annihilation

$$q(p_1) + \bar{q}(p_2) \rightarrow g(p_3) + Z(p_4)$$

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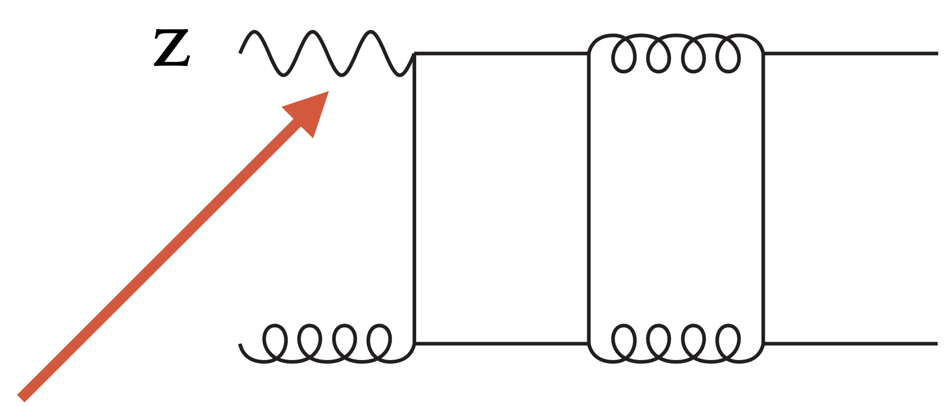
Status:

Pheno @ NNLO including **only vector-like** couplings of singlet type

Amplitudes [Garland, Gerhmann et al '02]

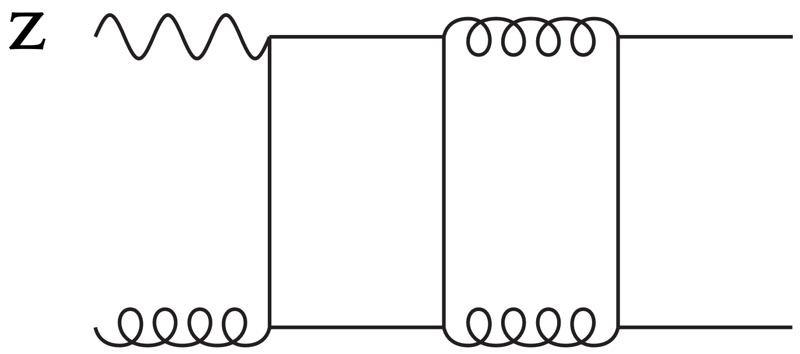
Pheno [Gehrmann-De Ridder et al '17, '18] etc etc



$\gamma^\mu \gamma^5$ — axial coupling neglected in *singlet* contributions —

Need to include top+bottom to get consistent result (anomaly!)

TENSOR DECOMPOSITION FOR CHIRAL THEORIES



One issue for **axial couplings** is
evanescent structures in chiral tensor

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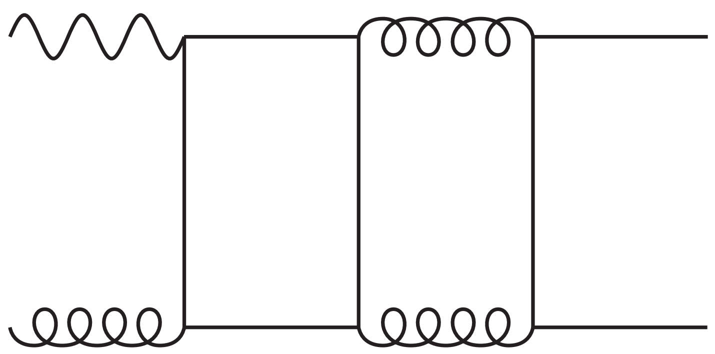

One issue for **axial couplings** is **evanescent structures in chiral tensor**

Our method: only **independent tensors in $d = 4$ are relevant**, we can span it with a basis of vectors in $d = 4$: $p_1^\mu, p_2^\mu, p_3^\mu$, plus the fourth **parity-odd one**

$$\epsilon_{\nu\rho\sigma\mu} p_1^\nu p_2^\rho p_3^\sigma = \epsilon^{p_1 p_2 p_3 \mu} = v_A^\mu$$

With these, a possible basis can be written as: (could be **further optimised** for singlet contributions)

$$\begin{aligned} A_{AV} &= \epsilon_{4,\mu}\epsilon_{3,\nu} A_{AV}^{\mu,\nu} \\ &= \epsilon_{4,\mu}\epsilon_{3,\nu} \Big[\bar{u}(p_2) \not{p}_3 u(p_1) \left(K_1 p_1^\mu p_1^\nu + K_2 p_2^\mu p_1^\nu + K_3 g^{\mu\nu} + R_1 p_1^\mu v_A^\nu + R_2 p_2^\mu v_A^\nu + R_3 v_A^\mu p_1^\nu \right) \\ &\qquad + \bar{u}(p_2)\gamma^\nu u(p_1) \left(K_4 p_1^\mu + K_5 p_2^\mu \right) + \bar{u}(p_2)\gamma^\mu u(p_1) K_6 p_1^\nu \\ &\qquad + \bar{u}(p_2) \not{v}_A u(p_1) \left(R_4 p_1^\mu p_1^\nu + R_5 p_2^\mu p_1^\nu \right) + R_6 \left(\bar{u}(p_2)\gamma^\mu u(p_1) v_A^\nu + \bar{u}(p_2)\gamma^\nu u(p_1) v_A^\mu \right) \Big] \end{aligned}$$

[Gehrmann, Peraro, Tancredi to appear soon]

TENSOR DECOMPOSITION FOR CHIRAL THEORIES

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\end{aligned}
$$

The counting is straightforward:

- **2 helicities** for the $q\bar{q}$ line (massless)
- **2 helicities** for the (physical) gluon
- **3 helicities** for the (physical) Z boson

Gives a total of = **12 helicity amplitudes**

matched by the number of tensors and form factors

Note that manipulations are done in **tHV / Larin scheme**

$$
p_i \cdot v_A = 0 \,, \qquad v_A \cdot v_A = \epsilon^{p_1 p_2 p_3 \mu} \epsilon^{p_1 p_2 p_3 \mu} = \frac{d-3}{4} s_{12} s_{13} s_{23}
$$

[Gehrmann, Peraro, Tancredi to appear soon]

NATURAL SOLUTION FOR GAMMA5 IN LARIN'S SCHEME

$$
\begin{aligned}
A_{AV} &= \epsilon_{4,\mu}\epsilon_{3,\nu} A_{AV}^{\mu,\nu} \\
&= \epsilon_{4,\mu}\epsilon_{3,\nu} \Big[\bar{u}(p_2)\not{p}_3 u(p_1) \left(K_1 p_1^\mu p_1^\nu + K_2 p_2^\mu p_1^\nu + K_3 g^{\mu\nu} + R_1 p_1^\mu v_A^\nu + R_2 p_2^\mu v_A^\nu + R_3 v_A^\mu p_1^\nu\right) \\
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\end{aligned}
$$

[Gehrmann, Peraro, Tancredi to appear soon]

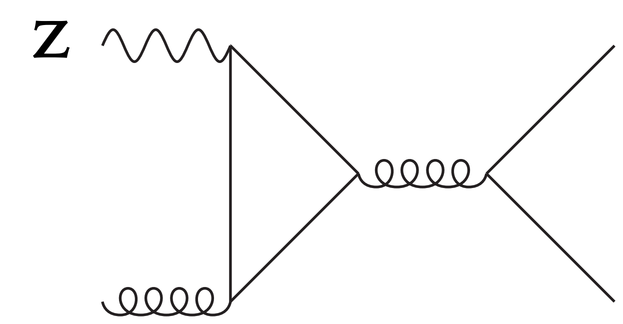
tensors and projectors contain ***at most* one occurrence of** $\epsilon_{\mu\nu\rho\sigma}$

γ_5 **never appears** in the tensor decomposition!

very natural to be applied in **Larin-Scheme**

NATURAL SOLUTION FOR GAMMA5 IN LARIN'S SCHEME

Example: One-loop singlet form factors



At one loop ***there is only 1 form factor*** effectively

$$R_1 = -\frac{16\,i}{(d-2)st(t+u)^2}\left[\text{Bub}(s)\left(\frac{1}{2}(d-2)(t+u)+s\right) - \text{Bub}\left(m^2\right)(s+t+u)\right]$$

$$R_2 = \frac{16\,i}{(d-2)su(t+u)^2}\left[\text{Bub}(s)\left(\frac{1}{2}(d-2)(t+u)+s\right) - \text{Bub}\left(m^2\right)(s+t+u)\right]$$

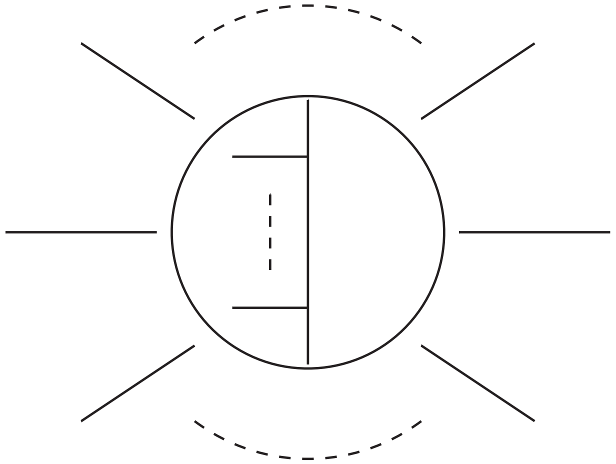
$$R_3 = \frac{16\,i}{(d-2)st(t+u)^2}\left[\text{Bub}(s)\left(\frac{1}{2}(d-2)(t+u)+s\right) - \text{Bub}\left(m^2\right)(s+t+u)\right]$$

$$R_4 = \frac{16\,i}{(d-2)st(t+u)^2}\left[\text{Bub}(s)\left(\frac{1}{2}(d-2)(t+u)+s\right) - \text{Bub}\left(m^2\right)(s+t+u)\right]$$

$$R_5 = \frac{16\,i}{(d-2)st(t+u)^2}\left[\text{Bub}(s)\left(\frac{1}{2}(d-2)(t+u)+s\right) - \text{Bub}\left(m^2\right)(s+t+u)\right]$$

$$R_6 = 0$$

MASTER INTEGRALS: ANALYTIC COMPLEXITY



$$= \sum_{i=1}^{N} R_i(x_1, \ldots, x_r)\, \mathcal{I}_i(x_1, \ldots, x_n)$$

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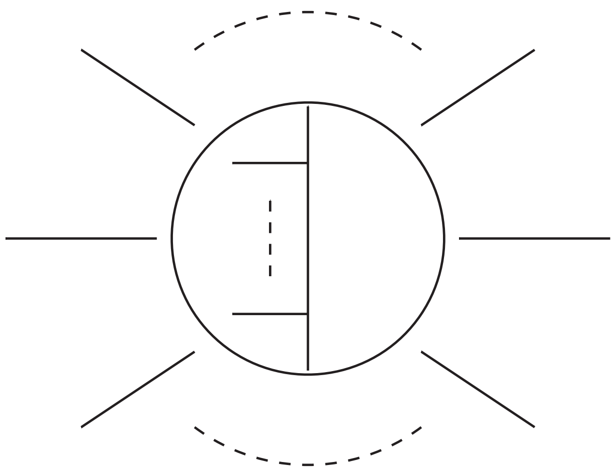

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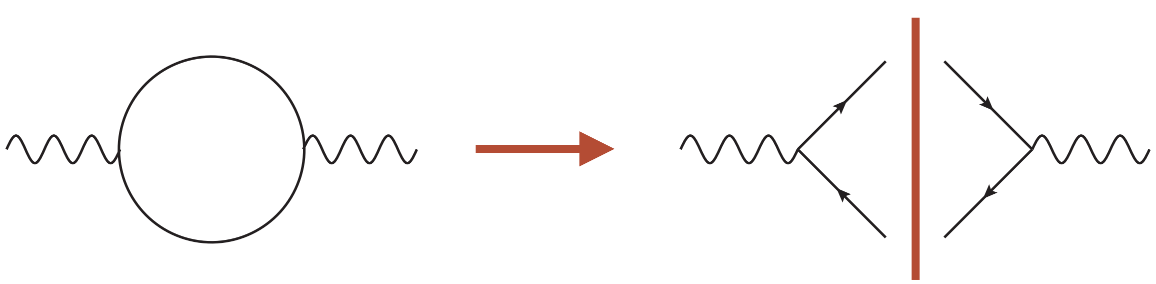


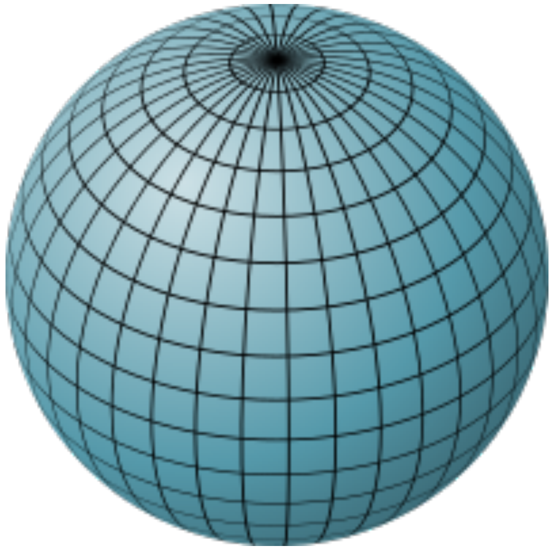
$$\sim \int_{4m^2}^{\infty} \frac{ds'}{s' - s - i\epsilon} \frac{1}{\sqrt{s'(s' - 4m^2)}}$$

$$\sim \frac{1}{\sqrt{s(s-4m^2)}} \ln\left(\frac{\sqrt{s-4m^2} + \sqrt{s}}{\sqrt{s-4m^2} - \sqrt{s}}\right)$$

Scattering amplitude has (poles and) **branch cuts** — encoded in master integrals!

GEOMETRY AND FEYNMAN INTEGRALS

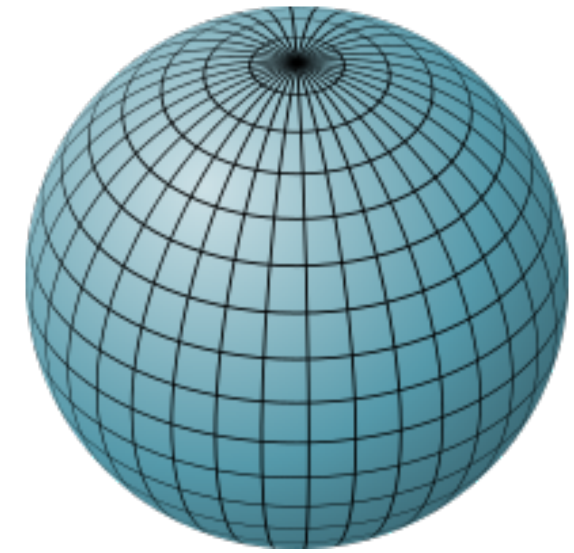
Iterated integrals on the **Riemann Sphere** ~ multiple polylogarithms



$$G(c_1, ..., c_k; x) = \int_0^x dt\, r(c_1, t) G(c_2, ..., c_k; t)$$

GEOMETRY AND FEYNMAN INTEGRALS

Iterated integrals on the **Riemann Sphere** ~ **multiple polylogarithms**



$$G(c_1, \ldots, c_k; x) = \int_0^x dt\, r(c_1, t) G(c_2, \ldots, c_k; t)$$

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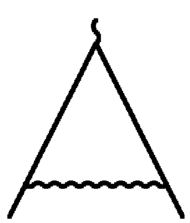
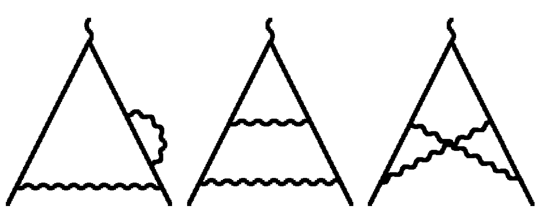
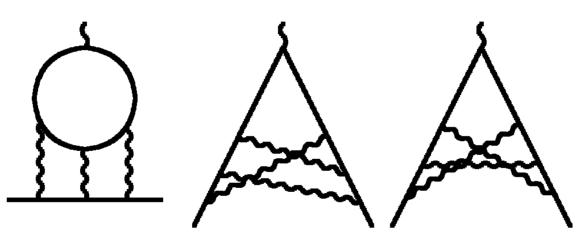
$C_1 =$  $= +0.50000000\ldots$

$C_2 =$  $= -0.328478965\ldots$

$C_3 =$  $= +1.181241456\ldots$

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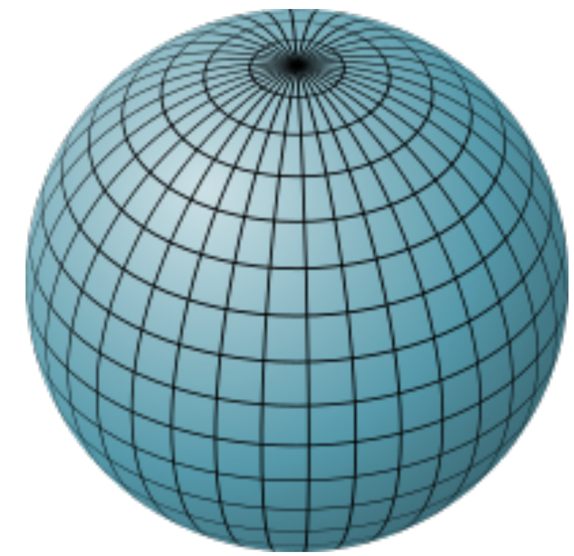

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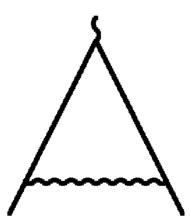
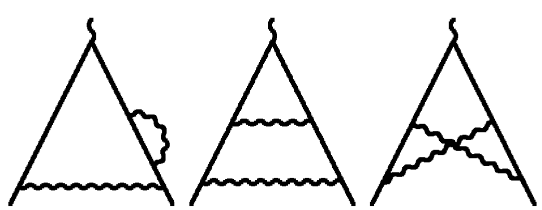
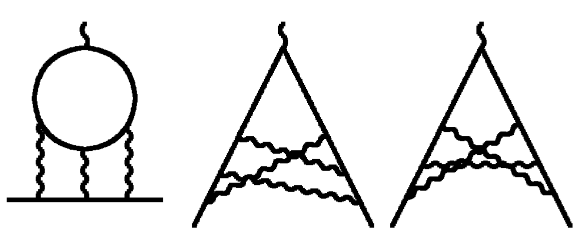
[Kinoshita et al]

$C_1 =$  $= \dfrac{1}{2}$ [Schwinger '48]

$C_2 =$  $= \dfrac{197}{144} + \dfrac{1}{12}\pi^2 - \dfrac{1}{2}\pi^2 \ln 2 + \dfrac{3}{4}\zeta(3)$ [Petermann, Sommerfield '57]

$C_3 =$ 

$$= \frac{83}{72}\pi^2\zeta(3) - \frac{215}{24}\zeta(5) + \frac{100}{3}\left[\left(\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{\ln^4 2}{24}\right) - \frac{\pi^2 \ln^2 2}{24}\right]$$

$$- \frac{239}{2160}\pi^4 + \frac{139}{18}\zeta(3) - \frac{298}{9}\pi^2 \ln 2 + \frac{17101}{810}\pi^2 + \frac{28259}{5184}$$

[Laporta, Remiddi '97]

GEOMETRY AND FEYNMAN INTEGRALS

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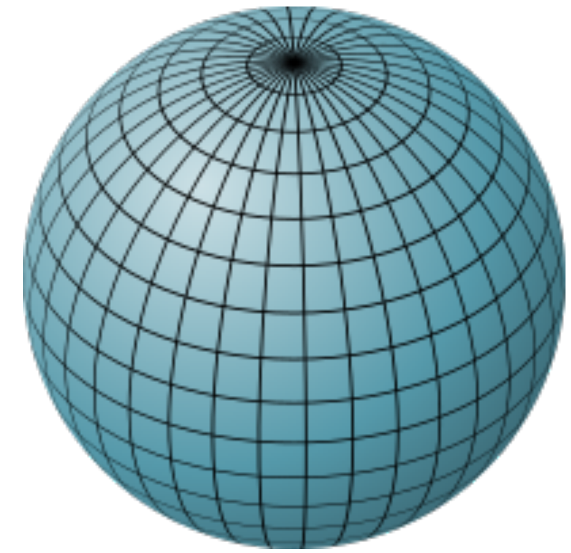

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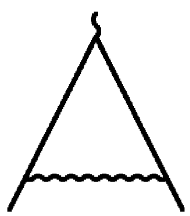
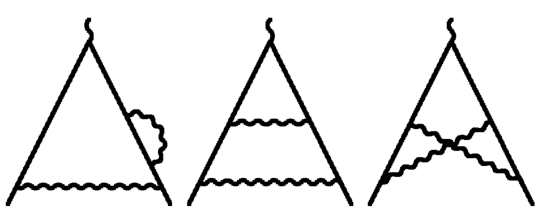
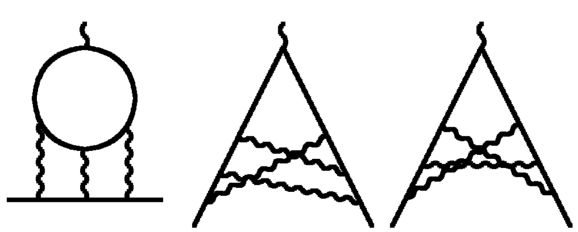
$C_1 =$  $= \frac{1}{2}$ [Schwinger '48]

$C_2 =$  $= \frac{197}{144} + \frac{1}{12}\pi^2 - \frac{1}{2}\pi^2 \ln 2 + \frac{3}{4}\zeta(3)$ [Petermann, Sommerfield '57]

$C_3 =$ 

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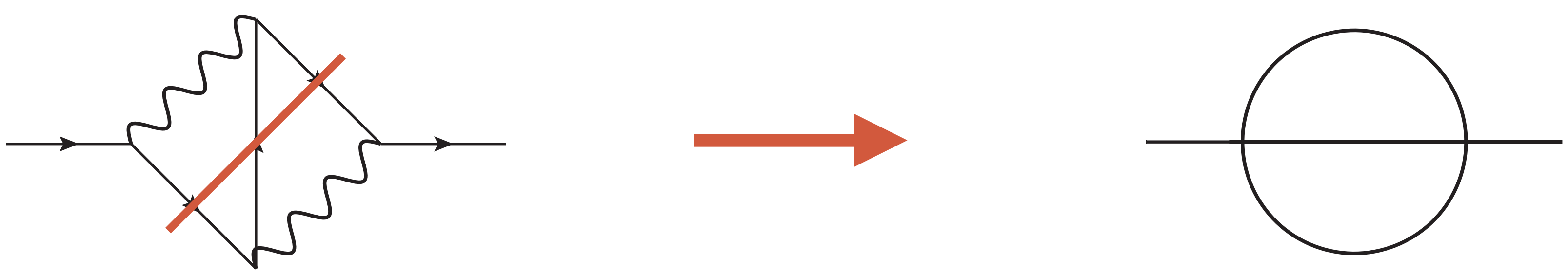
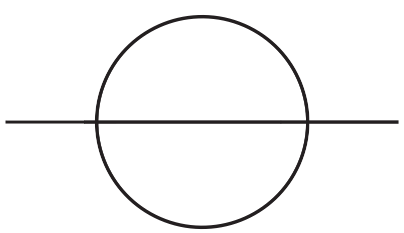
BEYOND GENUS 0

Riemann sphere too simple, Feynman integrals involve more interesting geometries

First non-trivial case with famous sunrise graph received a lot of attention in past decade



The sunrise integral



$$= \frac{1}{\sqrt{(3m-\sqrt{s})(\sqrt{s}+m)^3}}\,\mathrm{K}\left(\frac{16m^3\sqrt{s}}{(3m-\sqrt{s})(\sqrt{s}+m)^3}\right)$$

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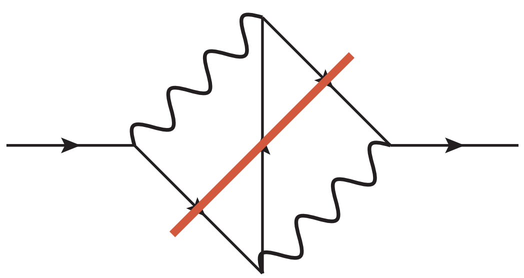

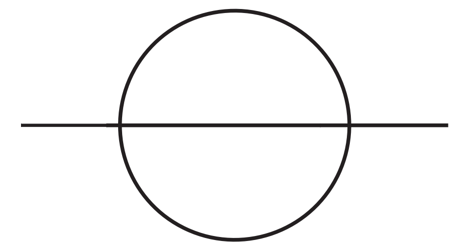

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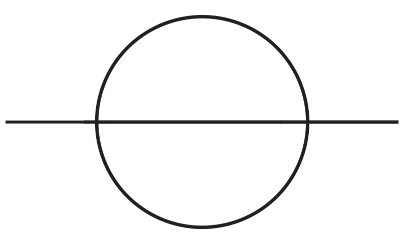

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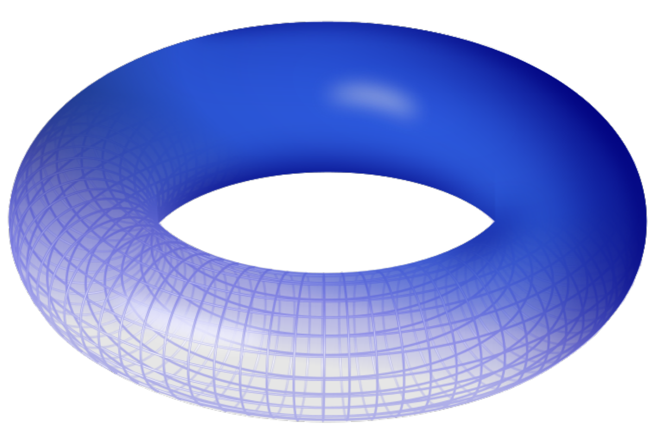
One dimensional surfaces of **genus 1** $\longrightarrow$ **elliptic curves**



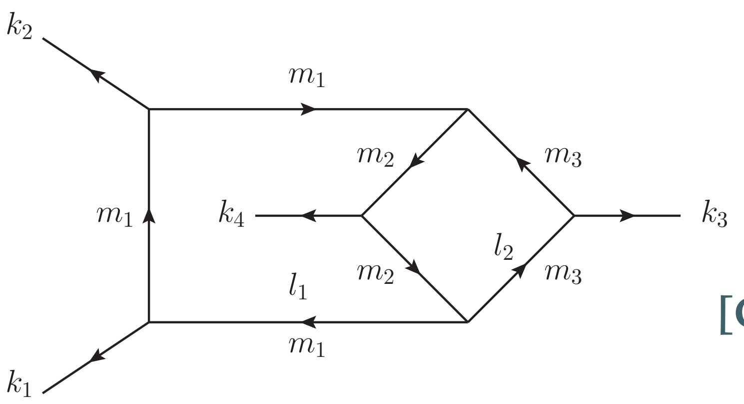
$$\mathcal{E}_4\left(\begin{smallmatrix} n_1 & \cdots & n_k \\ c_1 & \cdots & c_k \end{smallmatrix}; x, \vec{a}\right) = \int_0^x dt\, \Psi_{n_1}(c_1, t, \vec{a})\, \mathcal{E}_4\left(\begin{smallmatrix} n_2 & \cdots & n_k \\ c_2 & \cdots & c_k \end{smallmatrix}; t, \vec{a}\right)$$

[Brown, Levin ’11; Adams, Weinzierl ’13,’15; Broedel, Duhr, Dulat, Penante, Tancredi ’17,’18,’19; Broedel, Mafra, Matthes, Schlotterer ’15,’16]

HIGHER GENERA AND HIGHER DIMENSIONS

Even genus 1 is not enough, even at two loops…

Examples of Higher genera



Genus 3

[Georgoudis, Zhang '15]

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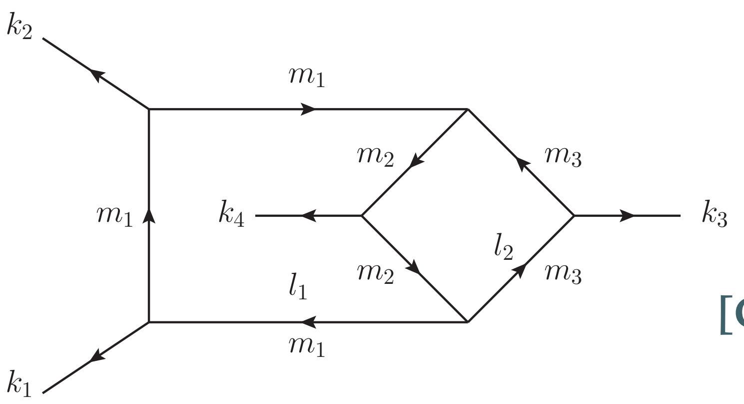

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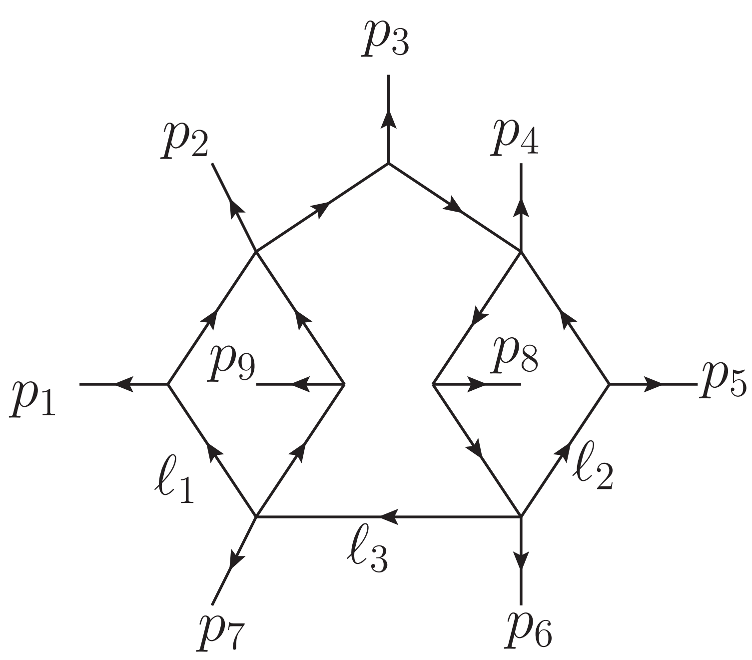

[Huang, Zhang '13]

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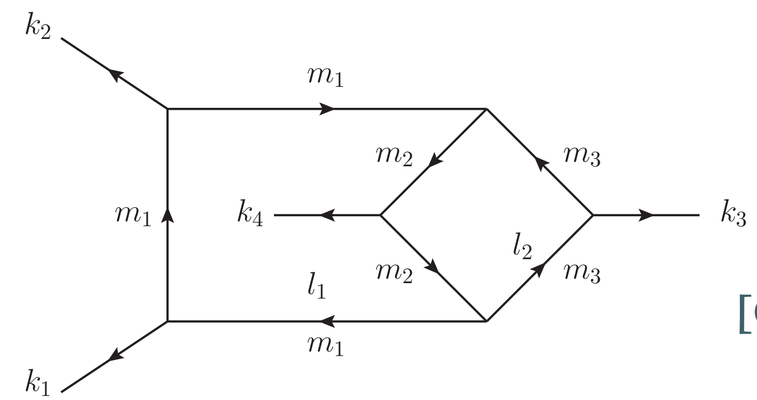

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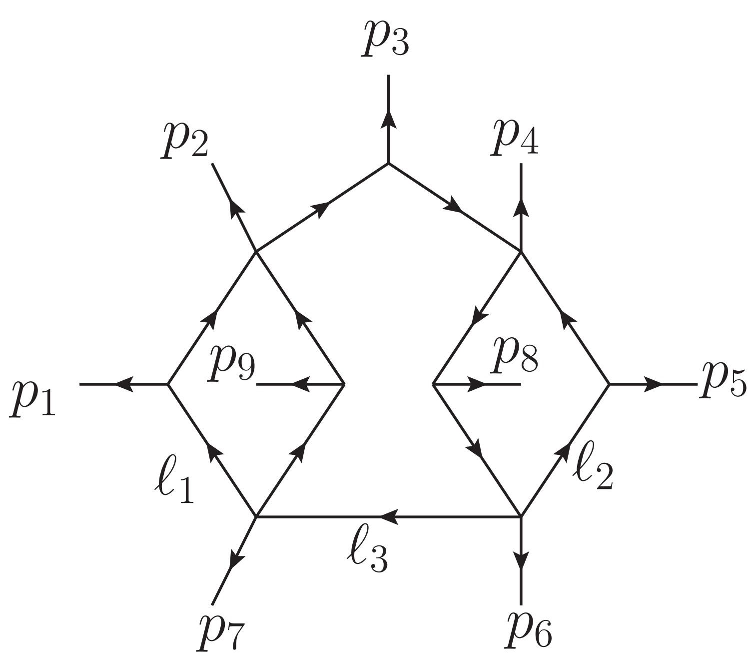

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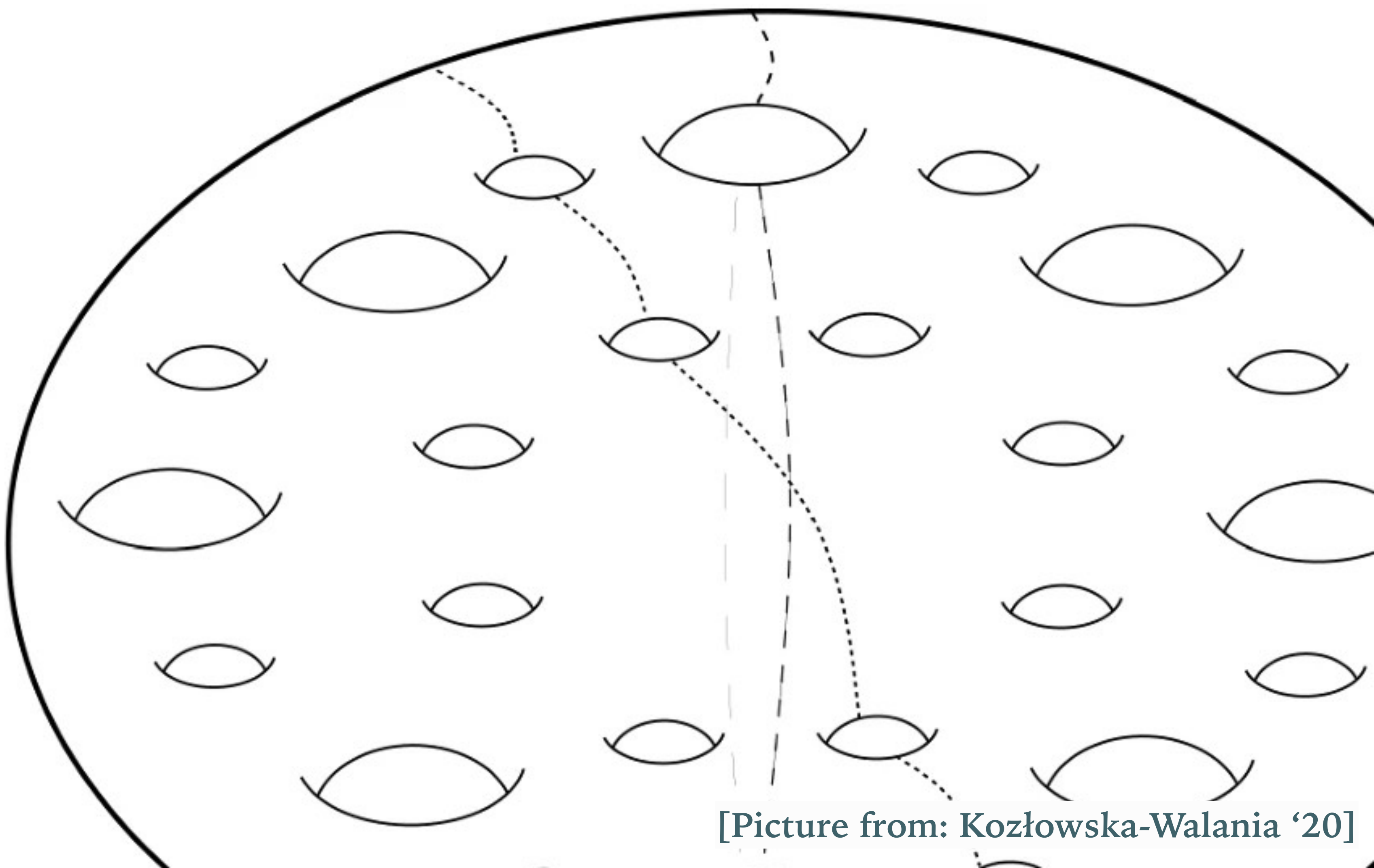

[Picture from: Kozłowska-Walania '20]

HIGHER GENERA AND HIGHER DIMENSIONS

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A Calabi-Yau surface can be thought as an elliptic curve in more dimensions

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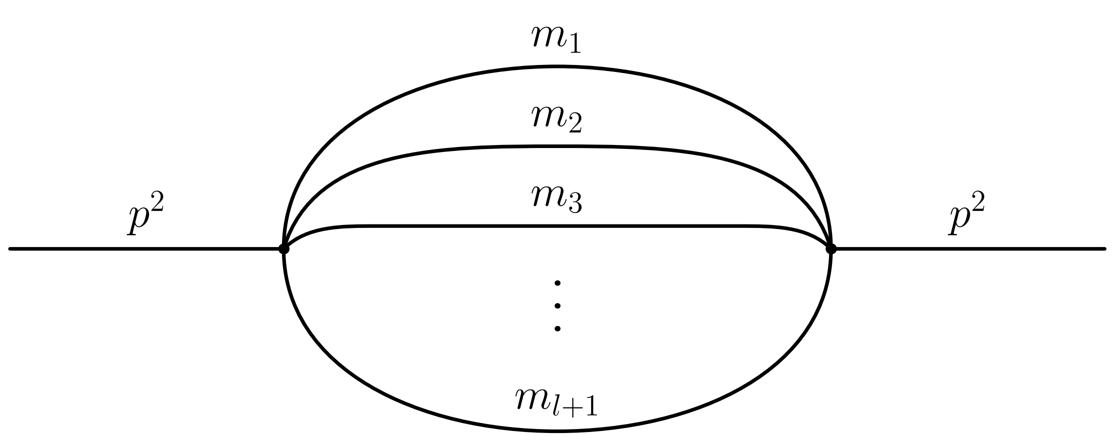
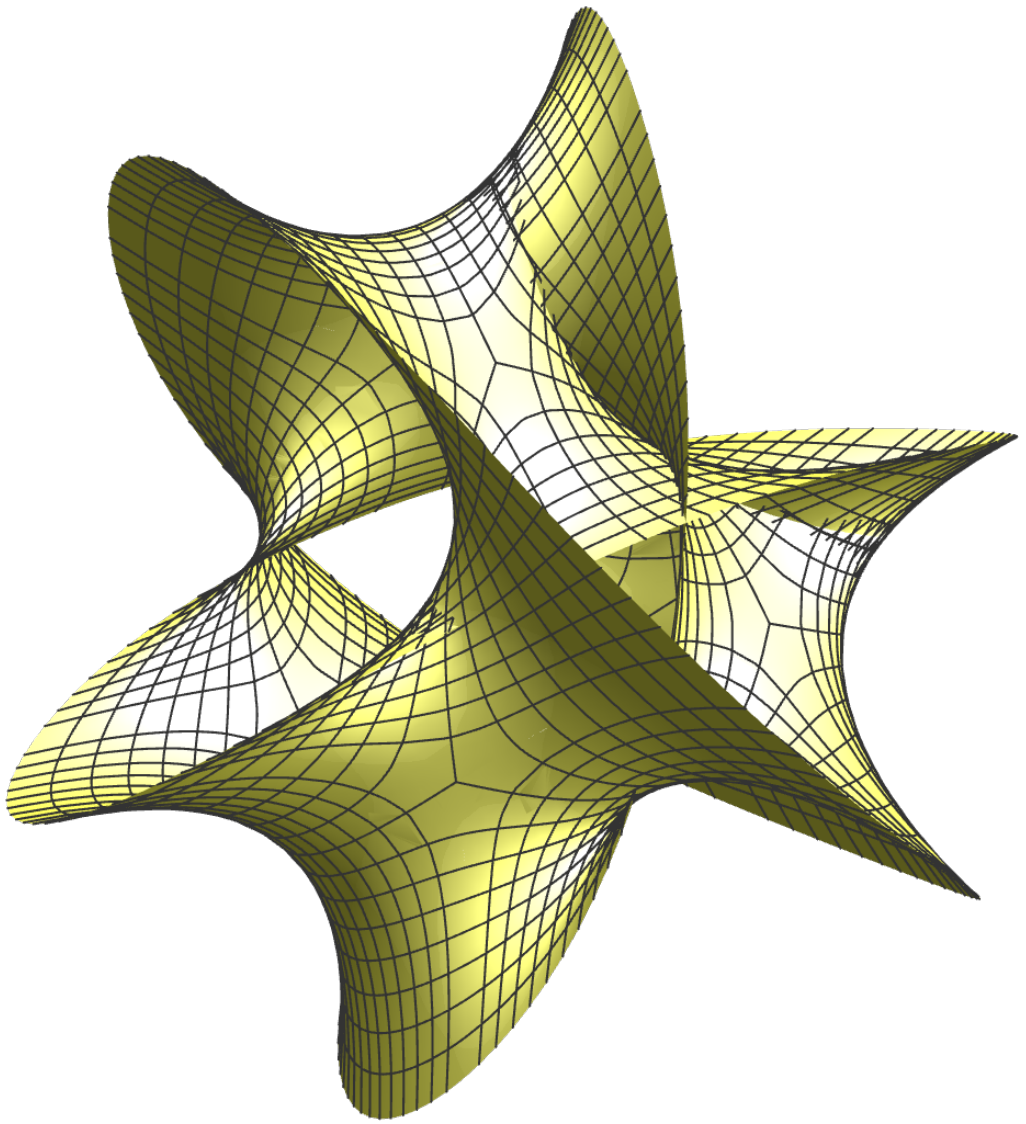
l-loop “banana graphs” generate (l-1)-fold CYs



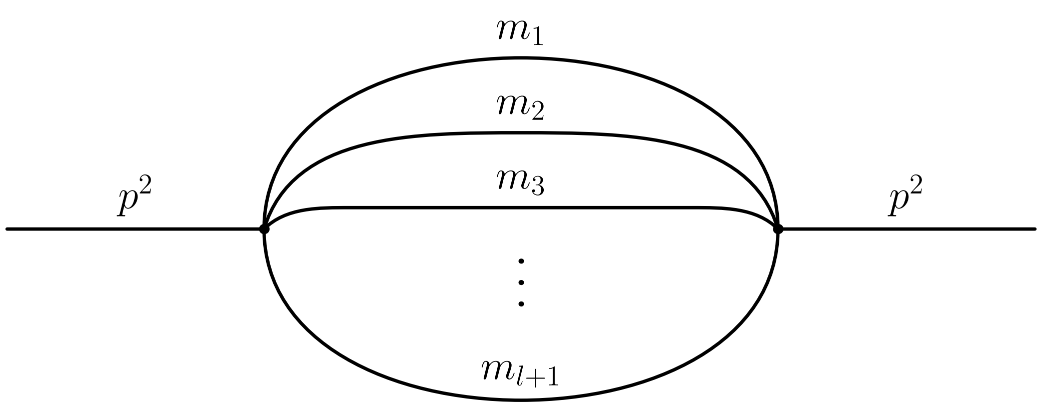


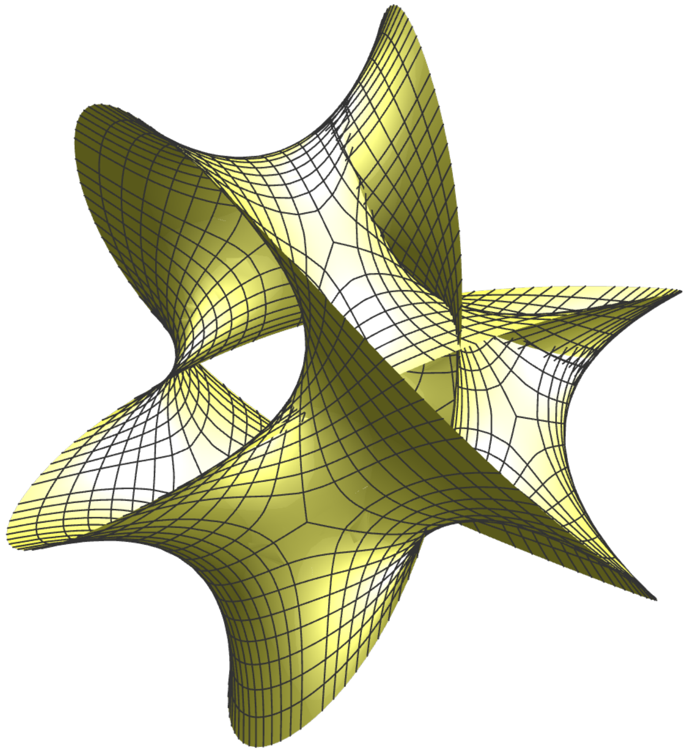
[Primo, Tancredi ’17], [Brödel, Duhr, Dulat, Marzucca, Penante, Tancredi ‘19][Bönisch, Duhr, Fischbach, Klemm, Nega ’21] [Pögel, Wang, Weinzierl ’22] …. MANY OTHERS…

HIGHER GENERA AND HIGHER DIMENSIONS

l-loop "banana graphs" generate **(l-1)-fold CYs**

[Bönisch, Duhr, Fischbach, Klemm, Nega '21]





$$\mathrm{d}\underline{J}_r(\underline{z};\epsilon) = \mathbf{B}_r(\underline{z};\epsilon)\,\underline{J}_r(\underline{z};\epsilon) + \underline{N}_r(\underline{z};\epsilon)$$

Solution obtained integrating N_r over the inverse Wronskian

The "Wronskian"

$$\mathbf{W}_l(z) := \begin{pmatrix} \varpi_{l,0}(z) & \varpi_{l,1}(z) & \dots & \varpi_{l,l-1}(z) \\ \partial_z \varpi_{l,0}(z) & \partial_z \varpi_{l,1}(z) & \dots & \partial_z \varpi_{l,l-1}(z) \\ \vdots & \vdots & & \vdots \\ \partial_z^{l-1} \varpi_{l,0}(z) & \partial_z^{l-1} \varpi_{l,1}(z) & \dots & \partial_z^{l-1} \varpi_{l,l-1}(z) \end{pmatrix}$$

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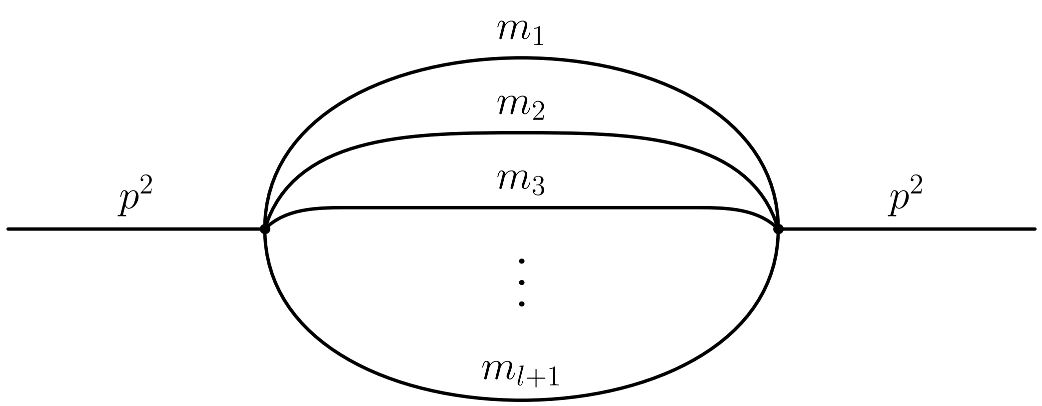

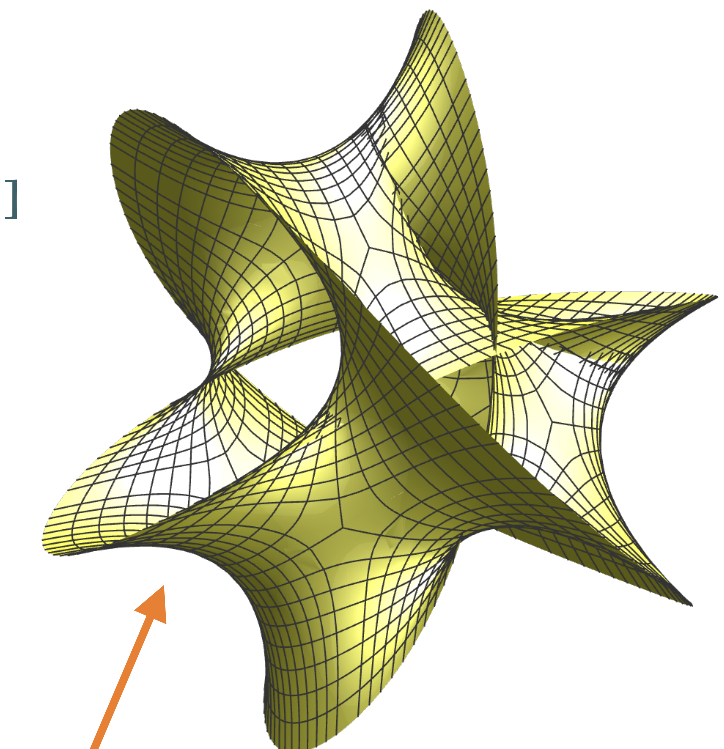

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“Periods” of the CY

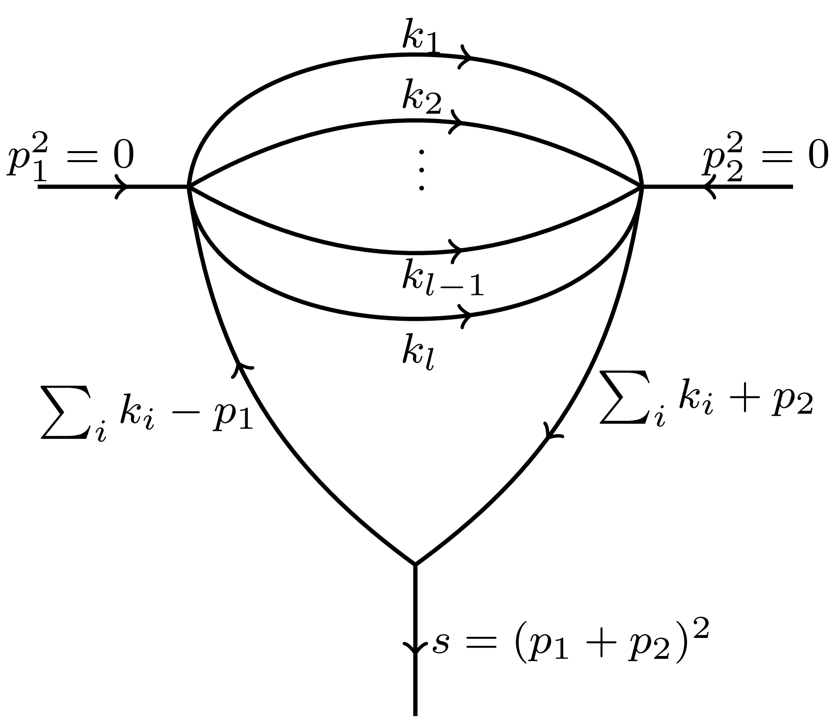
Independent “ways” how you can move along the surface

$$\mathrm{d}\mathbf{W}_r(\underline{z}) = \mathbf{B}_{0,r}(\underline{z})\,\mathbf{W}_r(\underline{z})$$

HIGHER GENERA AND HIGHER DIMENSIONS

A curious generalisation   The Ice (cream) cone graphs

[Duhr, Klemm, Nega, Tancredi, to appear soon]



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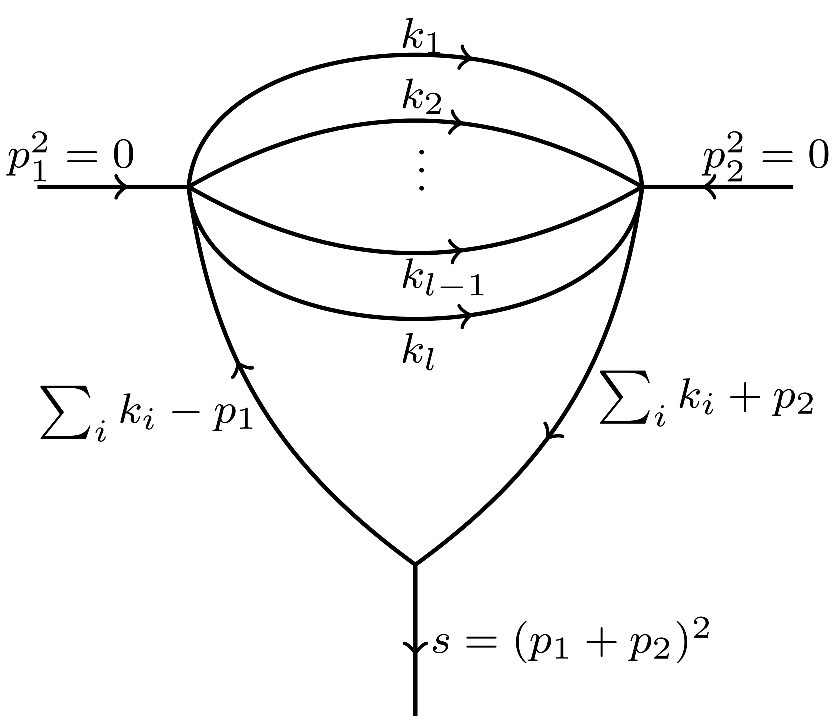


We can prove that @ l loops its finite part contains two **equivalent copies** of the banana graph and nothing else

2l-1 master integrals = **2(l-1) + 1**

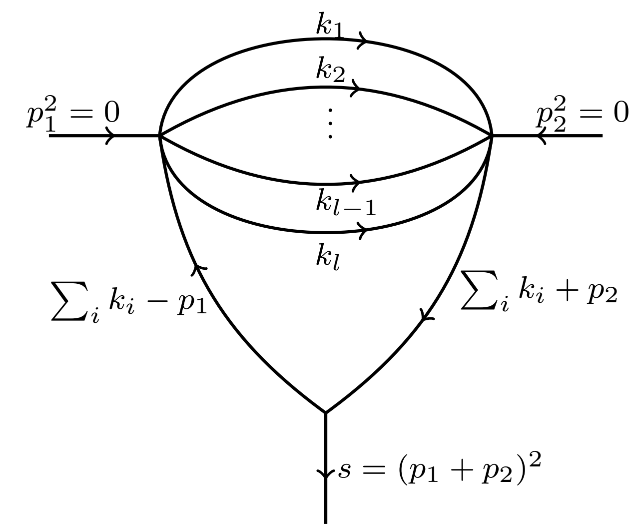
$$\mathrm{Cut}\left[I_{1,\ldots,1;0,0}(s;2)\right] \propto \oint_{\mathcal{C}} \frac{dz_3}{(z_3 + m^2 x)\left(z_3 + \frac{m^2}{x}\right)} \mathrm{Cut}\left[\mathrm{Ban}^{(l-1)}(z_3)\right]$$

based on [Primo, Tancredi '16,'17]

Two residues in d=2 → two copies of the Banana graph evaluates at different points $s = m^2 \frac{(1+x)^2}{x}$

HIGHER GENERA AND HIGHER DIMENSIONS

The Ice (cream) cone graphs @ 3 loops = 5 MIs



Relevant, for example, for 3 loop
$gg \to H$ with massive quarks

$$\frac{\mathrm{d}}{\mathrm{d}x}\underline{I}^{(3)} =
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{2}{(1-x)(1+x)} & 0 & \frac{1+x^2}{x(1-x)(1+x)} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & \frac{3(1-x)}{x^2(1+9x)} & -\frac{1+3x}{x^2(1+x)(1+9x)} & \frac{(1-3x)(1+3x)}{x(1+x)(1+9x)} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & -\frac{3(1-x)}{x^2(9+x)} & 0 & 0 & -\frac{3+x}{x(1+x)(9+x)} & -\frac{9+20x+3x^2}{x(1+x)(9+x)} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\underline{I}^{(3)}$$

Last integral not independent in d=2, can be chosen to be zero, it decouples

[Remiddi, Tancredi '13]

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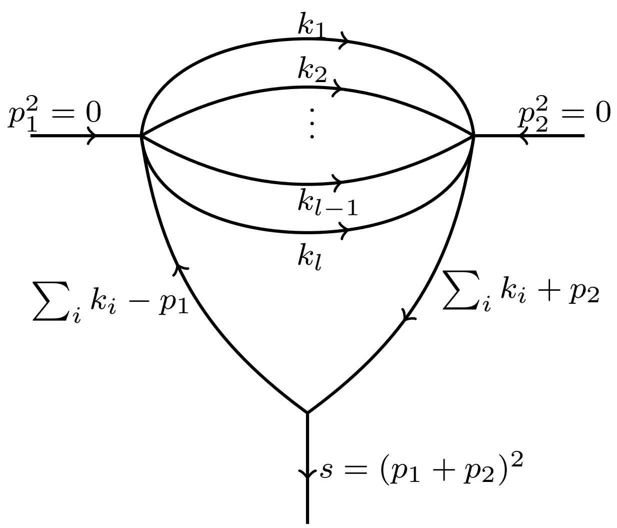

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CONCLUSIONS

A lot has been happening in multi-loop calculations

A LOT of beautiful results are (almost continuously) being released: $2 \to 3$ massless and now massive, recent first studies for massless $2 \to 4$, etc etc

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I am truly sorry I did not review any of it :-)

I tried to show two developments I am involved with, to give a glimpse of some of **the structures** that appear in high precision calculations for the LCH

With a message for (mainly) young people: LHC physics requires messy calculations, *we cannot avoid that*, but there is **a lot of "beauty"** in pQFT, and it is a lot of fun to be looking for it, while ***"crunching numbers for cross sections"***

THANK YOU VERY MUCH!