Soft Skill Workshop @ MPI

Scattering amplitudes with the pure spinor formalism

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Outline

I. Basics of the Pure Spinor Formalism

II. Vertex operators for the gauge multiplet

III. Tree level amplitudes for open string states

IV. Higher point couplings in closed string sector

V. Conclusion

I. Basics of the Pure Spinor Formalism

The Pure Spinor Formalism (PS) interpolates between the two traditional approaches to Superstring Theory avioding their drawbacks:

Ramond Neveu–Schwarz (RNS)	Green Schwarz (GS)
no spacetime spinors on worldsheet,	spacetime spinors θ present ...
thus lack of manifest spacetime SUSY	... but WS action quartic in θ

I. Basics of the Pure Spinor Formalism

The Pure Spinor Formalism (PS) interpolates between the two traditional approaches to Superstring Theory avioding their drawbacks:

Ramond Neveu–Schwarz (RNS)	Green Schwarz (GS)
no spacetime spinors on worldsheet, thus lack of manifest spacetime SUSY	spacetime spinors θ present ... ... but WS action quartic in θ
worldsheet spinors introduce different spin structures	quantization only in light cone gauge

Worldsheet action in PS formalism ≡ Siegel's modification of GS action:

$$\mathcal{S}_{\mathrm{mat}} = \frac{1}{2\pi}\int \mathrm{d}^2 z \, \left\{ \frac{1}{2}\partial X^m \, \bar\partial X_m + p_\alpha \, \bar\partial\theta^\alpha + \bar p_{\hat\alpha} \, \partial\bar\theta^{\hat\alpha} \right\}$$

$p, \theta \equiv$ Grassmann odd spinors of SO(1,9)

More convenient to use SUSY variables $\partial\theta^\alpha$ as well as

$$\Pi^m := \partial X^m + \frac{1}{2}\,\theta\,\gamma^m\,\partial\theta$$

$$d_\alpha := p_\alpha - \frac{1}{2}\left(\partial X^m + \frac{1}{4}(\theta\,\gamma^m\,\partial\theta)\right)(\gamma_m\,\theta)_\alpha$$

Note that GS formalism imposes constraint $d_\alpha = 0$.

Ghost sector of PS formalism involves Grassmann even spinors λ, w:

$$\mathcal{S}_{\text{ghost}} \;=\; -\frac{1}{2\pi}\int \mathrm{d}^2 z\,\left\{ w_\alpha\,\bar\partial\lambda^\alpha \;+\; \bar w_{\hat\alpha}\,\partial\bar\lambda^{\hat\alpha} \right\}$$

Assume BRST operator[1] ($\exists$ no covariant derivation yet)

$$Q_{\text{BRST}} \;=\; \oint \frac{\mathrm{d}z}{2\pi i}\,\lambda^\alpha(z)\,d_\alpha(z) \;+\; \text{right mover}$$

To ensure nilpotency $Q^2_{\text{BRST}} = 0$ and central charge cancellation $c = 0$,

the ghost spinor λ has to satisfy the pure spinor constraint

$$\lambda^\alpha\,\gamma^m_{\alpha\beta}\,\lambda^\beta \;=\; 0 \quad\Rightarrow\quad \text{5 out of 16 d.o.f. killed}$$

[1] Recall the definition $d_\alpha := p_\alpha \,-\, \frac{1}{2}\,\left(\partial X^m \,+\, \frac{1}{4}(\theta\,\gamma^m\,\partial\theta)\right)\,(\gamma_m\,\theta)_\alpha$

II. Vertex operators for the gauge multiplet

Physical states are generated by vertex operators $V(z)$, conformal fields in the Q_{BRST} cohomology.

Vertex operators for gluon g and gluino χ in RNS language are

$$\begin{aligned} V_g(z,\xi,k) &= \xi_m \, \psi^m(z) \, e^{-\phi(z)} \, e^{ik_p X^p(z)} \\ V_\chi(z,u,k) &= u^\alpha \, \Xi_\alpha(z) \, e^{-\phi(z)/2} \, e^{ik_p X^p(z)} \end{aligned}$$

with momenta k and polarizations ξ, u (subject to $\xi^m k_m = \not{k} u = 0$). The NS fermion ψ^m and R spin field Ξ_α define an interacting CFT.

Note that V_g and V_χ are related by spacetime SUSY $q_\alpha := \oint \frac{\mathrm{d}z}{2\pi i} e^{-\phi(z)/2} \Xi_\alpha(z)$.

In PS laguage, $\exists$ manifestly supersymmetric vertex for the (g, χ) multiplet

$$V(z, \xi, u, k) \;=\; \lambda^\alpha(z)\, A_\alpha\big(X(z), \theta(z), \xi, u, k\big)$$

$$A_\alpha\big(X(z), \theta(z), \xi, u, k\big) \;\equiv\; \text{SYM superfields, see next slide}$$

Also have an integrated vertex U containing further SYM superfields

$$U \;=\; \partial\theta^\alpha\, A_\alpha \;+\; A_m\, \Pi^m \;+\; d_\alpha\, W^\alpha \;+\; \frac{1}{4}\,(w\, \gamma^{mn}\, \lambda)\, \mathcal{F}_{mn}$$

such that $Q_{\mathrm{BRST}} U = \partial V$ and thus $Q_{\mathrm{BRST}} \int \mathrm{d}z\, U(z) = 0$.

More on the $h = 1$ conformal fields later ...

SYM equations of motion give rise to the following θ expansions

$$
\begin{aligned}
A_\alpha(x,\theta) &= \exp\bigl(ik\cdot x\bigr)\left\{\frac{\xi^m}{2}\,(\gamma^m\,\theta)_\alpha - \frac{1}{3}\,(u\,\gamma_m\,\theta)\,(\gamma^m\,\theta)_\alpha - \frac{i}{16}\,k_{[m}\,\xi_{n]}\,(\gamma_p\,\theta)_\alpha\,(\theta\,\gamma^{mnp}\,\theta)\right.\\
&\left. + \frac{i}{60}\,(\gamma_m\,\theta)_\alpha\,k_n\,(u\,\gamma_p\,\theta)\,(\theta\,\gamma^{mnp}\,\theta) - \frac{1}{576}\,(\gamma^m\,\theta)_\alpha\,k^r\,k_{[p}\,\xi_{q]}\,(\theta\,\gamma_{mrs}\,\theta)\,(\theta\,\gamma^{spq}\,\theta) \;+\; \mathcal{O}(\theta^6)\right\}\\
A_m(x,\theta) &= \exp\bigl(ik\cdot x\bigr)\left\{\xi_m - (u\,\gamma_m\,\theta) - \frac{i}{4}\,k_{[p}\,\xi_{q]}\,(\theta\,\gamma_m\,\gamma^{pq}\,\theta)\right.\\
&\qquad\left. + \frac{i}{12}\,k_p\,(\theta\,\gamma_m\,\gamma^{pq}\,\theta)\,(u\,\gamma_q\,\theta) \;+\; \mathcal{O}(\theta^4)\right\}\\
W^\alpha(x,\theta) &= \exp\bigl(ik\cdot x\bigr)\left\{u^\alpha - \frac{i}{2}\,k_{[m}\,\xi_{n]}\,(\gamma^{mn}\,\theta)^\alpha + \frac{i}{4}\,k_m\,(\gamma^{mn}\,\theta)^\alpha\,(u\,\gamma_n\,\theta)\right.\\
&\qquad\left. - \frac{1}{24}\,k_m\,k_{[p}\,\xi_{q]}\,(\gamma^{mn}\,\theta)^\alpha\,(\theta\,\gamma_n\,\gamma^{pq}\,\theta) \;+\; \mathcal{O}(\theta^4)\right\}\\
\mathcal{F}_{mn}(x,\theta) &= \exp\bigl(ik\cdot x\bigr)\left\{2i\,k_{[m}\,\xi_{n]} - 2i\,k_{[m}\,(u\,\gamma_{n]}\,\theta) + \frac{1}{2}\,k_{[p}\,\xi_{q]}\,k_{[m}\,(\theta\,\gamma_{n]}\,\gamma^{pq}\,\theta)\right.\\
&\qquad\left. + \frac{1}{6}\,(\theta\,\gamma_{[m}\,\gamma^{pq}\,\theta)\,k_{n]}\,k_p\,(u\,\gamma_q\,\theta) \;+\; \mathcal{O}(\theta^4)\right\}
\end{aligned}
$$

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&\quad + \frac{i}{60}\,(\gamma_m\,\theta)_\alpha\,k_n\,(u\,\gamma_p\,\theta)\,(\theta\,\gamma^{mnp}\,\theta) - \frac{1}{576}\,(\gamma^m\,\theta)_\alpha\,k^r\,k_{[p}\,\xi_{q]}\,(\theta\,\gamma_{mrs}\,\theta)\,(\theta\,\gamma^{spq}\,\theta) \;+\; \mathcal{O}(\theta^6)\Big\} \\
\longrightarrow\quad A_m(x,\theta) \;&=\; \exp\big(ik\cdot x\big)\,\Big\{\xi_m - (u\,\gamma_m\,\theta) - \frac{i}{4}\,k_{[p}\,\xi_{q]}\,(\theta\,\gamma_m\,\gamma^{pq}\,\theta) \\
&\quad + \frac{i}{12}\,k_p\,(\theta\,\gamma_m\,\gamma^{pq}\,\theta)\,(u\,\gamma_q\,\theta) \;+\; \mathcal{O}(\theta^4)\Big\} \\
W^\alpha(x,\theta) \;&=\; \exp\big(ik\cdot x\big)\,\Big\{u^\alpha - \frac{i}{2}\,k_{[m}\,\xi_{n]}\,(\gamma^{mn}\,\theta)^\alpha + \frac{i}{4}\,k_m\,(\gamma^{mn}\,\theta)^\alpha\,(u\,\gamma_n\,\theta) \\
&\quad - \frac{1}{24}\,k_m\,k_{[p}\,\xi_{q]}\,(\gamma^{mn}\,\theta)^\alpha\,(\theta\,\gamma_n\,\gamma^{pq}\,\theta) \;+\; \mathcal{O}(\theta^4)\Big\} \\
\mathcal{F}_{mn}(x,\theta) \;&=\; \exp\big(ik\cdot x\big)\,\Big\{2i\,k_{[m}\,\xi_{n]} - 2i\,k_{[m}\,(u\,\gamma_{n]}\,\theta) + \frac{1}{2}\,k_{[p}\,\xi_{q]}\,k_{[m}\,(\theta\,\gamma_{n]}\,\gamma^{pq}\,\theta) \\
&\quad + \frac{1}{6}\,(\theta\,\gamma_{[m}\,\gamma^{pq}\,\theta)\,k_{n]}\,k_p\,(u\,\gamma_q\,\theta) \;+\; \mathcal{O}(\theta^4)\Big\}
\end{aligned}
$$

Can even give closed formulae [Policastro, Tsimpis 0603165] such as

$$
\begin{aligned}
A_m \;&=\; e^{ik\cdot x}\,\Big\{\,[\cosh\sqrt{\mathcal{O}}]_m{}^q\,\xi_q \;+\; \Big[\frac{\sinh\sqrt{\mathcal{O}}}{\sqrt{\mathcal{O}}}\Big]_m{}^q\,(\theta\,\gamma_q\,u)\,\Big\} \\
\mathcal{O}_m{}^q \;&=\; \frac{k_p}{2i}\,(\theta\,\gamma_m\,\gamma^{qp}\,\theta)
\end{aligned}
$$

III. Tree level amplitudes for open string states

Recall the tree level prescription of RNS open string amplitudes

$$\mathcal{A}_N^{\rm RNS} \ = \ \left\langle c\, V_1^{(q_1)}(z_1)\, c\, V_2^{(q_2)}(z_2)\, c\, V_3^{(q_3)}(z_3) \prod_{j=4}^{N} \int_{\partial\Sigma} \mathrm{d}z_j \ V_j^{(q_j)}(z_j) \right\rangle$$

- three fixed positions $z_{1,2,3}$ at c ghost insertions
- $N-3$ integrations along worldsheet boundary $\partial\Sigma$
- overall superghost charge $\sum_{k=1}^N q_k = -2$ adjusted to genus $g=0$

In PS formalism, also fix 3 positions $V(z_{1,2,3})$ and integrate $N-3\; U_j(z_j)$'s:

$$\mathcal{A}_N^{\rm PS} \ = \ \left\langle V_1(z_1)\, V_2(z_2)\, V_3(z_3) \prod_{j=4}^{N} \int_{\partial\Sigma} \mathrm{d}z_j \ U_j(z_j) \right\rangle$$

To compute the correlator $\langle ... \rangle$, firstly integrate out the $h=1$ fields in

$$U \;=\; \partial\theta^\alpha\, A_\alpha \;+\; A_m\, \Pi^m \;+\; d_\alpha\, W^\alpha \;+\; \frac{1}{4}\,(w\,\gamma^{mn}\,\lambda)\,\mathcal{F}_{mn}\;.$$

They do not have zero modes at tree level, so correlations are completely

determined by OPE singularities $\Rightarrow$ can use Wick's theorem, e.g.

$$\left\langle \Pi^m(z)\,\prod_{j=1}^n \Phi_j(w_j)\right\rangle \;=\; \sum_{j=1}^n \frac{-\,k_j}{z-w_j}\,\left\langle \prod_{j=1}^n \Phi_j(w_j)\right\rangle$$

On the RNS side, however, $\exists$ interaction $\psi^m(z)\,\Xi_\alpha(w) \sim \frac{\gamma^m_{\alpha\beta}\,\Xi^\beta(w)}{(z-w)^{1/2}}$

$\Rightarrow \quad \langle \psi^{m_1}(z_1)\ldots\psi^{m_n}(z_n)\,\Xi_{\alpha_1}(w_1)\ldots\Xi_{\alpha_r}(w_r)\rangle$ quite tedious to get

- $D=4$: Problem solved [DH, OS, StSt 0911.5168] see Daniel's talk

- $D>4$: Group theory more involved [DH, OS work in progress]

After integrating out the $h = 1$ fields, deal with the zero modes in λ, θ.

In RNS formalism, have the c ghost zero mode prescription

$$\langle c(z_1)\,\partial c(z_2)\,\partial^2 c(z_3)\rangle \;=\; 1$$

PS analogoue reads

$$\langle(\lambda\,\gamma^m\,\theta)\,(\lambda\,\gamma^n\,\theta)\,(\lambda\,\gamma^p\,\theta)\,(\theta\,\gamma_{mnp}\,\theta)\rangle \;=\; 1$$

Have to pick out θ^5 pieces from the superfields!

IV. Higher point couplings in closed string sector

Closed strings states obtained by tensoring two open string sectors

$$(G_{mn}, B_{mn}, \varphi) \;\leftrightarrow\; \xi_m \otimes \bar{\xi}_n \,, \qquad \psi_m^\alpha \;\leftrightarrow\; u^\alpha \otimes \bar{\xi}_m$$

$$\not{F}^{\alpha\hat{\beta}} \;\sim\; \sum_p (\gamma_{m_1 m_2 \dots m_p})^{\alpha\hat{\beta}} \, F^{m_1 m_2 \dots m_p} \;\leftrightarrow\; u^\alpha \otimes \bar{u}^{\hat{\beta}}$$

... where chirality of right movers distinguishes type II A/B:

type IIA	$\hat{\beta} = \dot{\beta}$	p even
type IIB	$\hat{\beta} = \beta$	p odd

On the level of amplitudes, use KLT relations

$$\mathcal{A}_4^{\rm cl} \;\sim\; \sin\big(\pi\alpha'\,u\big)\,\mathcal{A}_4^{\rm op}(s,u)\,\otimes\,\bar{\mathcal{A}}_4^{\rm op}(u,t)$$

$\exists$ higher point generalization with more terms ...

$$\mathcal{A}_N^{\rm cl} \;\sim\; \sum_{P,P'\in S_N} \mathcal{A}_N^{\rm op}(P)\,\otimes\,\bar{\mathcal{A}}_N^{\rm op}(P')\,e^{i\pi f(P,P')}$$

For mixed open–closed string disk amplitudes see [Stieberger 0907.2211]

Let us now focus on open string amplitudes $\mathcal{A}_N^{\rm op}$ ...

... 4 point amplitude

$$
\begin{aligned}
\mathcal{A}_4^{\mathrm{op}} &= \langle V_1(z_1)\, V_2(z_2)\, V_3(z_3) \int \mathrm{d}z_4\, U_4(z_4)\rangle \\
&= \int \frac{\mathrm{d}z_4}{z_{41}}\, |z_{41}|^{\alpha' u}\, |z_{42}|^{\alpha' t}\, |z_{12}|^{\alpha' s} \Big\langle\, i k_1^m\, (\lambda A_1)\, (\lambda A_2)\, (\lambda A_3)\, A_m^4 \\
&\qquad\qquad + A_m^1\, (\lambda A_2)\, (\lambda A_3)\, (\lambda\, \gamma^m\, W_4)\, \Big\rangle \\
&\quad - \int \frac{\mathrm{d}z_4}{z_{42}}\, |z_{41}|^{\alpha' u}\, |z_{42}|^{\alpha' t}\, |z_{12}|^{\alpha' s} \Big\langle\, i k_2^m\, (\lambda A_1)\, (\lambda A_2)\, (\lambda A_3)\, A_m^4 \\
&\qquad\qquad + A_m^2\, (\lambda A_1)\, (\lambda A_3)\, (\lambda\, \gamma^m\, W_4)\, \Big\rangle
\end{aligned}
$$

Integral yields Euler beta function with well-known α' expansion.

... 5 point amplitude [Mafra 0909.5206]

$$\mathcal{A}_5^{\rm op} \;=\; \int \mathrm{d}z_2\,\mathrm{d}z_3 \prod_{i<j} |z_{ij}|^{2\alpha' k_i\cdot k_j} \left\{ \frac{L_{2131}}{z_{21}\,z_{31}} + \frac{L_{2134}}{z_{21}\,z_{34}} \right.$$
$$\left. - \frac{L_{2434}}{z_{24}\,z_{34}} + \frac{L_{2431}}{z_{24}\,z_{31}} + \frac{L_{2331}}{z_{23}\,z_{31}} - \frac{L_{2334}}{z_{23}\,z_{34}} + \frac{L_{2314}}{z_{23}^2} \right\}$$

Each of the seven kinematic structures L_{ijkl} goes along with a different

hypergeometric integral of type

$$F(a,b,c,d,e) \;=\; \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y\; x^a\, y^b\, (1-x)^c\, (1-y)^d\, (1-xy)^e$$

Good news [Stieberger, Taylor]:

α' expansion under control & $\exists$ a 2 dim basis $\{F_1, F_2\}$

The seven shorthands L_{ijkl} represent superspace expressions like

$$
\begin{aligned}
L_{2131} \;=\; & \left[A^1_m \,(\lambda\gamma^m W^2) \;+\; (\lambda A_1)\,(k^1\cdot A^2)\right] (\lambda A_4)\,(\lambda A_5)\left[(k^1+k^2)\cdot A^3\right] \\
& - (W^1\gamma^m W^2)\,(\lambda\gamma_m W^3)\,(\lambda A_4)\,(\lambda A_5) \;+\; (A^1\cdot A^2)\,k^2_m\,(\lambda\gamma^m W^3)\,(\lambda A_4)\,(\lambda A_5) \\
& + A^1_m\,(\lambda\gamma^m W^3)\,(k^1\cdot A^2)\,(\lambda A_4)\,(\lambda A_5) \;-\; A^2_m\,(\lambda\gamma^m W^3)\,(k^2\cdot A^1)\,(\lambda A_4)\,(\lambda A_5) \\
& + (k^1\cdot k^2)\,(A_1 W^3)\,(\lambda A_2)\,(\lambda A_4)\,(\lambda A_5) \;-\; (k^1\cdot k^2)\,(A_2 W^3)\,(\lambda A_1)\,(\lambda A_4)\,(\lambda A_5) \\
& + (k^1\cdot k^3)\,(A_1 W^2)\,(\lambda A_3)\,(\lambda A_4)\,(\lambda A_5) \;+\; (k^2\cdot k^3)\,(A_1 W^2)\,(\lambda A_3)\,(\lambda A_4)\,(\lambda A_5)
\end{aligned}
$$

... 6 point amplitude

More complex hypergeometric integrals with 6 dim basis [Oprisa, Stieberger]

Superspace expressions L_{ijklmn} at 6pt level:

Work in progress [Mafra, Policastro, OS, Stieberger, Tsimpis]

V. Conclusion

- PS formalism is an appealing approach to superstrings overcoming some shortcomings of RNS and GS
- SUSY tree level amplitudes can be computed in PS framework by integrating out free fields in the worldsheet CFT
- Aim to get tree level 5pt and 6pt couplings of closed string states by means of KLT

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- PS formalism is an appealing approach to superstrings overcoming some shortcomings of RNS and GS

- SUSY tree level amplitudes can be computed in PS framework by integrating out free fields in the worldsheet CFT

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Thank you for your attention !