Introduction to the AdS/CFT Correspondence

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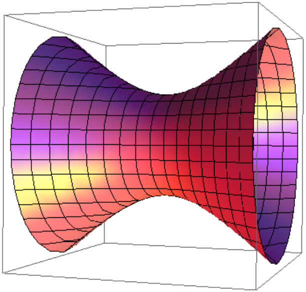
Anti-de Sitter Space

$$\left\{ -X_0^2 + \sum_{i=1}^{p+1} X_i^2 - X_{p+2}^2 = -L^2 \right\} \subset \mathbb{R}^{2,p+1}$$



A 1+1 dimensional AdS_2 space.

Poincaré Patch

$$ds^2 = \frac{L^2}{z^2}\left(\eta_{\mu\nu} dx^\mu dx^\nu + dz^2\right)$$

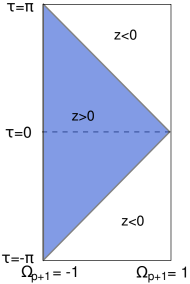


Figure: Penrose diagram for *AdS* space, viewed as a cylinder from the side. The shaded region is the Poincaré Patch, where $z > 0$.

Conformal Field Theory

- Conformal transformation:

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \Omega(x) g_{\alpha\beta}(x)$$

- Conformal field theory: invariant under conformal transformations.

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Conformal transformations

- Translations: $x^\mu \to x^\mu + a^\mu$
- Lorentz transformations: $x^\mu \to \Lambda^\mu{}_\nu x^\nu, \quad \Lambda \in SO(1,d)$
- Scalings: $x^\mu \to \lambda x^\mu, \quad \lambda \in \mathbb{R}$
- Special conformal transformations: $x^\mu \to \frac{x^\mu + b^\mu x^2}{1 + 2b\cdot x + b^2 x^2}$

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Quasi-primary fields

- Definition:

$$\phi(x) \to \phi'(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\frac{\Delta}{d}} \phi(x)$$

  Δ is the *scaling dimension* of the field $\phi(x)$.
- Can show that:

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle \sim \frac{1}{|x_1 - x_2|^{2\Delta}}$$

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The correspondence

- A field in the bulk, $\phi(x)$.
- $\phi(x) \to \phi_0(x)$ on the boundary.
- Generating functional:

$$Z_{Bulk} = \int \mathcal{D}\phi e^{iS[\phi]}$$

- Also one on the boundary, Z_{CFT}.

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The correspondence

Intuitively:

- Boundary action depends on ϕ_0.
- Perturb $\phi_0 \to \phi_0 + \delta\phi_0$
- $\Rightarrow \delta S = \int d^d x \sqrt{|g|} \delta\phi_0 \frac{\delta S}{\delta\phi_0(x)}$.
- $$Z_{CFT}[\phi_0 \to \phi_0 + \delta\phi_0] = \langle e^{i \int d^d x \sqrt{|g|} \delta\phi_0(x) \mathcal{O}(x)} \rangle$$

  where

  $$\mathcal{O}(x) = \frac{\delta S}{\delta\phi_0(x)}$$

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The master equation

$$Z_{CFT}[\phi_0] = Z_{QG}$$

What is Z_{QG}?

- Z_{QG} is the partition function for quantum gravity.
- Saddle point expansion

$$S[\phi] \approx S[\bar{\phi}] + \int d^d x \frac{\delta S}{\delta \phi} \delta\phi(x) + \dots$$

$$\Rightarrow Z_{QG} \approx e^{iS[\bar{\phi}]} \int \mathcal{D}[\phi]\, e^{iO(\delta\phi^2)}$$
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- Master equation becomes $Z_{CFT}[\phi_0] \approx e^{iS_{grav}}$.

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The Dictionary

- $\phi(x) \leftrightarrow \mathcal{O}(x)$:

$$Z_{CFT}[\phi_0] = \left\langle e^{i\int \phi_0 \mathcal{O}} \right\rangle \approx e^{iS_{grav}}$$

- $A^\mu \leftrightarrow J^\mu$:

$$Z_{CFT}[A_0^\mu] = \left\langle e^{i\int A_0^\mu J_\mu} \right\rangle \approx e^{iS_{grav}}$$

- $g_{\mu\nu} \leftrightarrow T^{\mu\nu}$:

$$Z_{CFT}\left[g_{(0)\mu\nu}\right] = \left\langle e^{i\int d^d x \sqrt{|g_{(0)}|} g_{(0)\mu\nu} T^{\mu\nu}} \right\rangle \approx e^{iS_{grav}}$$

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An example: Massive scalar field

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In Fourier space

- Metric: $ds^2 = \frac{L^2}{z^2}\left(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2\right)$
- Can Fourier transform the μ components: $\phi(z, x^\mu) \to \tilde{\phi}(z, k^\mu)$
- Equation of motion becomes:

$$0 = -z^2\partial_z^2\tilde{\phi} - (1-d)z\partial_z\tilde{\phi} + z^2k^2\tilde{\phi} + L^2m^2\tilde{\phi}$$

- Use ansatz $\tilde{\phi}(z, k) = z^\Delta$, limit $z \to 0$,

$$-\Delta(\Delta-1)z^\Delta - (1-d)\Delta z^\Delta + k^2z^{\Delta+2} + L^2m^2z^\Delta = 0$$

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ϕ is...

- As $z \to 0$,
- $\phi \sim z^{\Delta_\pm}$,
- and $\Delta_\pm = \frac{d}{2} \pm \sqrt{\left(\frac{d}{2}\right)^2 + L^2 m^2}$
- We thus need

$$\phi(z,x) \to z^{\Delta_-}\phi_0(x) + z^{\Delta_+}\phi_1(x)$$

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Finding the action

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\begin{aligned}
S_{\text{bulk}}[\phi] &= \frac{\kappa}{2}\int d^d x \sqrt{|\gamma|}\, n^A \phi \nabla_A \phi \\
&= \ldots \\
&\sim \int d^d k\, \tilde{\phi}_0(k)\tilde{\phi}_0(-k)\mathcal{F}(k)
\end{aligned}
$$

Finding the correlator

$$
\begin{aligned}
\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle &= \frac{\delta^2 W}{\delta\phi_0(x_1)\delta\phi_0(x_2)} \\
&= -\frac{\delta^2 S_{\text{bulk}}}{\delta\phi_0(x_1)\delta\phi_0(x_2)} \\
&\sim \int d^d k e^{-ik_\mu(x_1^\mu - x_2^\mu)}\mathcal{F}(k) \\
&\sim \frac{1}{(x_1 - x_2)^{2\Delta_+}}
\end{aligned}
$$

Extensions

- Three-point functions.
- Fields with spin.
- Statistical mechanics: create a finite temperature in the CFT by introducing a black hole in the bulk.
  $Z = \mathrm{tr} e^{H/T} = e^{F/T}$
- Transport coefficients, shear viscosity: $\frac{\eta}{S} = \frac{1}{4\pi}$.

Partition function

$$\langle \phi_2, t_2 | \phi_1, t_1 \rangle = \int_{\phi=\phi_1(t_1)}^{\phi=\phi_2(t_2)} d[\phi] e^{i \int_{t_1}^{t_2} dtL}$$

$$\langle \phi_2, t_2 | \phi_1, t_1 \rangle = \langle \phi_2 | e^{-iH(t_2 - t_1)} | \phi_1 \rangle$$

Set $t_2 - t_1 = -i\beta$, with $\beta = \frac{1}{T}$.

$$\begin{aligned} \mathrm{tr}\, e^{-\beta H} &= \sum_n \langle n | e^{-\beta H} | n \rangle \\ &= \sum_n \int_{\phi=n(0)}^{\phi=n(\beta)} d[\phi] e^{-\int_0^\beta dtL} \\ &= \int_{PBC} d[\phi] e^{-\int_0^\beta dtL} \end{aligned}$$

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Black hole

$$ds^2 = \frac{L^2}{z^2}\left(-f(z)dt^2 + \frac{dz^2}{f(z)} + d\vec{x}\cdot d\vec{x}\right)$$

where $f(z) = 1 - \left(\frac{z}{z_s}\right)^d$, and z_s is the position of the event horizon.

Near-horizon limit

Define $\rho^2 \equiv \alpha(z - z_s) \ll 1$, $\varphi \equiv i\beta t$. Rescaling coordinates,

$$ds^2 \approx \rho^2 d\varphi^2 + d\rho^2 + \frac{1}{z_s^2} dx_\mu dx^\mu$$

where we had to choose $\beta = \frac{d}{2z_s}$. To avoid a singularity, $\varphi \sim \varphi + 2\pi$. Therefore Euclidean time $\tau = -it$ has restriction $\tau \sim \tau + \frac{4\pi z_s}{d}$.

For Further Reading I

J. McGreevy
Holographic duality with a view toward many-body physics.
arXiv:hep-th/09090518v2

S. Hartnoll
Lectures on holographic methods for condensed matter physics.
arXiv:hep-th/09033246v2