Characterisations of $Pic^0(X)$

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The problem

Let X be a compact complex Kähler manifold

The Picard group $Pic(X)$ is the group of isomorphism classes of holomorphic line bundles on X and is much used in algebraic geometry, and the theory of complex manifolds. For practical applications one needs as many different ways to represent it (called *characterisations*) as possible. We use that the Picard group can be defined as the sheaf cohomology group $H^1(X, \mathcal{O}_X^*)$.

$Pic^0(X)$ is the subgroup of elements L in the Picard group $Pic(X)$ with $c_1(L) = 0$ (there exists a connection on L with zero curvature).

$$
\begin{aligned}
Pic^0(X) \;&:=\; \left\{ L \text{ holomorphic line bundel on } X \,\middle|\, c_1(L) \;=\; 0 \right\} \\
&=\; \underbrace{H^1(X, \mathcal{O}_X)}_{\text{n}-\text{dim. } \mathbb{C}-\text{vector space}} \Big/ \underbrace{H^1(X, \mathbb{Z})}_{\text{n}-\text{dim. lattice}}
\end{aligned}
$$

is a complex torus.

Task:
Find alternative characterisations for $Pic^0(X)$ in the mathematical literature and express them in a modern, coherent notation.
Lit: e.g. Pierre Colmez: “Intégration sur les variétés p-adiques” (Société Mathématique de France, 1998)

Background: G - fibre bundles and extensions

G is a complex commutative Lie group.

Definition:

A G-fibre bundle over X is a complex manifold Y with a morphism $Y \xrightarrow{\pi} X$ such that

- G operates on Y
- π has holomorphic local sections $\{s_U\}$ (choice of element on each fibre), which induce local trivializations

$$U \times G \cong \pi^{-1}(U)$$

on $U \subseteq X$.

Holomorphic line bundles are $\mathbb{C}^* = \mathbb{C} \backslash \{0\}$-fibre bundles.

Let X be a complex torus and G a complex commutative Lie group. An extension of X by G is an exact sequence of commutative complex Lie groups

$$0 \to G \to Y \to X \to 0$$

This is a vector extension if G is a vector space.

If Y is an extension of X by G, it is also a G-fibre bundle over X.

First result

$$
\begin{array}{ccccc}
0 \to \Omega^1(X) & \to & Pic^{0,\#}(X) & \to & Pic^0(X) \to 0 \\
 & & \| & & \\
 & & \{(L,\nabla) | L \text{ holom. line bundle on } X, & & \\
 & & \nabla \text{ integrable connection on } L\}/\text{Isom.} & & \\
 & & \| & & \\
 & & H^1(X,\mathbb{C})/H^1(X,\mathbb{Z}) & &
\end{array}
$$

is a vector extension of $Pic^0(X)$ by the vector space of holomorphic differential 1-forms on X, called $\Omega^1(X)$.

If X is an algebraic variety the above is an exact sequence of commutative algebraic groups.

A theorem and resulting corollaries – more results

From now on X is assumed to be a proper smooth algebraic variety. If X is even an Abelian variety the following theorem holds: ($Ext(X, G)$ is the group of isomorphism classes of extensions of X by G).

Theorem:

$$\begin{aligned} Ext(X, \mathbb{C}) &\cong H^1(X, \mathcal{O}_X) \\ Ext(X, \mathbb{C}^*) &\cong Pic^0(X) \subseteq H^1(X, \mathcal{O}_X^*) \end{aligned}$$

Consequences:

- $$\begin{aligned} H^1(X, \mathcal{O}_X) &\cong \{\ \mathbb{C}\text{−fiber bundles over } X\ \} \Big/ \text{Isomorphism} \\ &\cong Ext(X, \mathbb{C}) \end{aligned}$$

  i.e. up to isomorphism each $\mathbb{C}$-fibre bundle is an extension of X by $\mathbb{C}$, implying that it has a unique commutative group structure.

- For each other vector extension

$$0 \to E \to Y \to Pic^0(X) \to 0 \quad (*)$$

there exists a unique homomorphism f such that the second line of

$$\begin{array}{ccccccccc} 0 & \to & \Omega^1(X) & \to & Pic^{0,\#}(X) & \to & Pic^0(X) & \to & 0 \\ & & \downarrow f & & \downarrow f^* & & \| & & \\ 0 & \to & \mathrm{E} & \to & f^* \, Pic^{0,\#}(X) & \to & Pic^0(X) & \to & 0 \end{array}$$

is isomorphic to $(*)$.

Conclusions

- I have found five alternative characterisations (of which I only discussed two in this presentation).
- I have found and corrected errors and devised intermediate steps previously omitted in the literature, making the subject more accessible.
- The characterisations found can now be used for future work of our group
- Ongoing work: Generalisation to non-algebraic X.