Magnetic Monopoles and the Callan-Rubakov Effect

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A dissertation for the Master of Scienece in Mathematical and Theoretical Physics under the supervision of Prateek Agrawal

- magnetic monopoles usually excluded from electromagnetism
- 1974, 't Hooft[1] and Polyakov[2]: automatically included in certain non-abelian gauge theories
- 1982, Callan[3] and Rubakov[4]: GUT monopoles induces baryon number non-conservation and proton decay without a GUT suppression

[1] G.'t Hooft. "Magnetic monopoles in unified gauge theories". In: *Nuclear Physics B* 79.2 (1974), pp. 276–284.

[2] Alexander M. Polyakov. "Particle Spectrum in Quantum Field Theory". In: *JETP Lett.* 20 (1974). Ed. by J. C. Taylor, pp. 194–195.

[3] Curtis G. Callan. "Monopole catalysis of baryon decay". In: *Nuclear Physics B* 212.3 (1983), pp. 391–400.

[4] V.A. Rubakov. "Adler-Bell-Jackiw anomaly and fermion-number breaking in the presence of a magnetic monopole". In: *Nuclear Physics B* 203.2 (1982), pp. 311–348.

The 't Hooft-Polyakov monopole

Solitons and Topology

field theory of n scalar fields ϕ
n-dimensional representation of a gauge symmetry with global group G

$$G \overset{V(\phi)}{\rightarrow} H$$

vacuum-manifold $\mathcal{M}$

$$\mathcal{M} \cong G/H$$

finite energy classical solution

$$\phi(\vec{x}) \in \mathcal{M} \qquad \text{for } |\vec{x}| \rightarrow \infty$$

ϕ induced map

$$\phi_\infty = \phi(\vec{x})\big|_{|\vec{x}|=\infty} : \mathcal{S}_\infty^{n-1} \rightarrow \mathcal{M}$$

Solitons and Topology

solitons described by ϕ_∞ modulo homotopies
$\implies$ classified by the (n-1)th homotopy class $\pi_{n-1}(\mathcal{M})$.
Theorem:

$$\pi_2(G/H) = \pi_1(H) \tag{1}$$

and

$$\pi_1(\mathrm{U}(1)) = \mathbb{Z}$$

define a winding number

$$N = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\theta \, \frac{\mathrm{d}\alpha}{\mathrm{d}\theta} \tag{2}$$

The 't Hooft Polyakov Monopole

SU(2) non-abelian gauge theory, with an adjoint Higgs field $\phi = \phi^a \tau^a$.

$$\mathcal{L} = \mathrm{Tr}\Big((D_\mu \phi)^2\Big) - \frac{1}{2}\mathrm{Tr}(F^{\mu\nu}F_{\mu\nu}) - V(\phi) \qquad (3)$$

$$V(\phi) = \frac{\lambda}{4}\left(\mathrm{Tr}(\phi^2) - \frac{v^2}{2}\right)^2 \qquad (4)$$

minimum of the potential at $\phi^a \phi^a = v^2$
soliton solutions are possible:

$$\pi_2(SU(2)/U(1)) = \pi_1(U(1)) = \mathbb{Z}$$

The 't Hooft Polyakov Monopole

solution:

$$\phi = \tau^a \hat{r}^a h(r)$$
$$A^i = \tau^a \epsilon^{aim} \hat{r}^m \frac{1 - u(r)}{er}$$
$$A_0 = 0$$

Monopoles in Higher Gauge Groups

SU(5) GUT Monopoles

embed 't Hooft-Polyakov

$$T^a = \frac{1}{2}\mathrm{diag}\left(0, 0, \sigma^a, 0\right)$$

SU(2) doublets

$$\begin{pmatrix} u_1 \\ u_2^C \end{pmatrix} \quad \begin{pmatrix} u_2 \\ u_1^C \end{pmatrix} \quad \begin{pmatrix} e^+ \\ d_3 \end{pmatrix} \quad \begin{pmatrix} d_3^C \\ e^- \end{pmatrix}$$

The Callan-Rubakov effect

Angular Momentum in Fermion Monopole Scattering and the J=0 Approximation

changed angular momentum

$$\mathsf{J} = \mathsf{L} + \mathsf{S} + \mathsf{T} \tag{5}$$

in monopole background:

$$\phi = \tau^a \hat{r}^a h(r)$$

$$A^i = \tau^a \epsilon^{aim} \hat{r}^m \frac{1 - u(r)}{er} \qquad A_0 = 0$$

Massive Fermions and Bosonization

(1+1) dimensional fermionic fields in terms of scalars ϕ:

$$(e^+, e^-) \to \phi_e^+ \qquad (d_3^C, d_3) \to \phi_{d_3^C} \qquad (u_{1/2}, u_{1/2}^C) \to \phi_{u_{1/2}}$$

boundary condition:

$$\Phi_{e^+}(0) = \Phi_{d_3^C}(0) \qquad \Phi'_{e^+}(0) = -\Phi'_{d_3^C}(0)$$

$$\Phi_{u_1}(0) = \Phi_{u_2}(0) \qquad \Phi'_{u_1}(0) = -\Phi'_{u_2}(0)$$

Lagrangian:

$$L = \int_0^\infty \mathrm{d}r \, (\mathcal{L}_K + \mathcal{L}_M + \mathcal{L}_C)$$

Massive Fermions and Bosonization

Lagrangian:

$$L = \int_0^\infty \mathrm{d}r \left(\mathcal{L}_K + \mathcal{L}_M + \mathcal{L}_C\right)$$

$$\mathcal{L}_K = \sum \frac{1}{2} \left(\partial_\mu \Phi_i\right)^2$$

$$\mathcal{L}_M = \sum \mu_i^2 \cos\left(2\sqrt{\pi}\Phi_i\right)$$

$$\mathcal{L}_C = \frac{e^2}{32\pi^2 r^2} \left(\Phi_{e^+} + \frac{1}{3}\Phi_{d_3^C} + \frac{2}{3}\Phi_{u_1} + \frac{2}{3}\Phi_{u_2}\right)^2 + \frac{g^2}{32\pi^2 r^2} \left(\sqrt{\frac{2}{3}}\Phi_{d_3^C} + \sqrt{\frac{1}{6}}\Phi_{u_1} + \sqrt{\frac{1}{6}}\Phi_{u_2}\right)^2 + \frac{g^2}{32\pi^2 r^2} \left(\frac{1}{2}\Phi_{u_1} - \frac{1}{2}\Phi_{u_2}\right)^2$$

Massive Fermions and Bosonization

fermions occur as solitons:

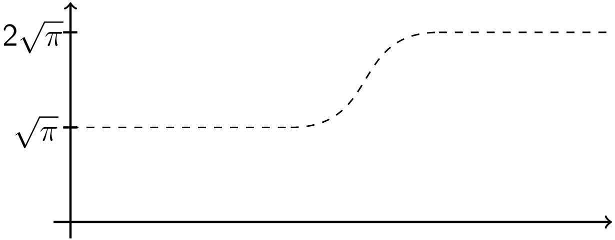


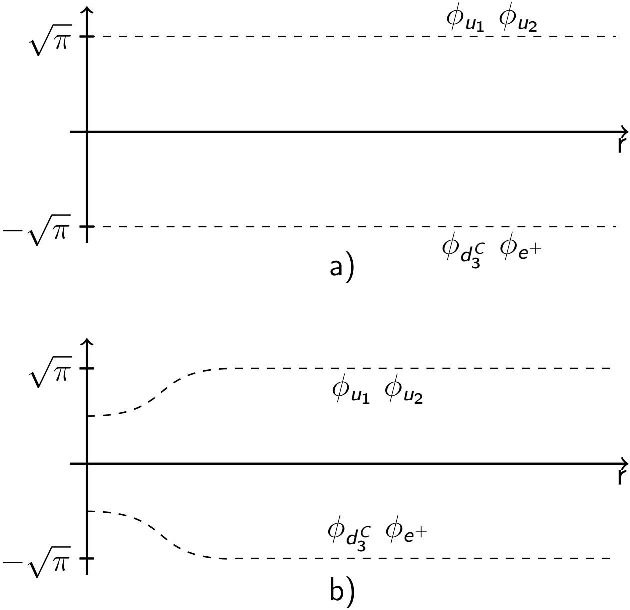
Figure: A kink like soliton in the bosonic theory equivalent to fermions in the original theory.

Massive Fermions and Bosonization

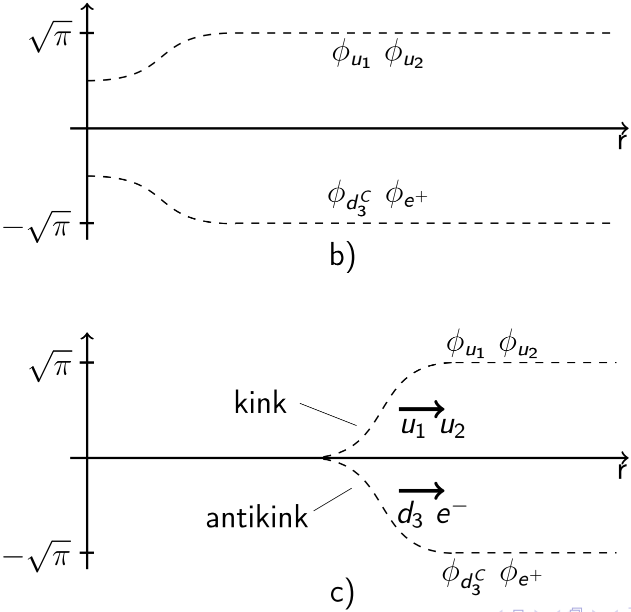
ground state $|N\rangle$:

$$\Phi_{u_1} = \Phi_{u_2} = N\sqrt{\pi} \quad \Phi_{e^+} = \Phi_{d_3^C} = -N\sqrt{\pi}$$

Massive Fermions and Bosonization



Massive Fermions and Bosonization



Massive Fermions and Bosonization

ground state $|N\rangle$:

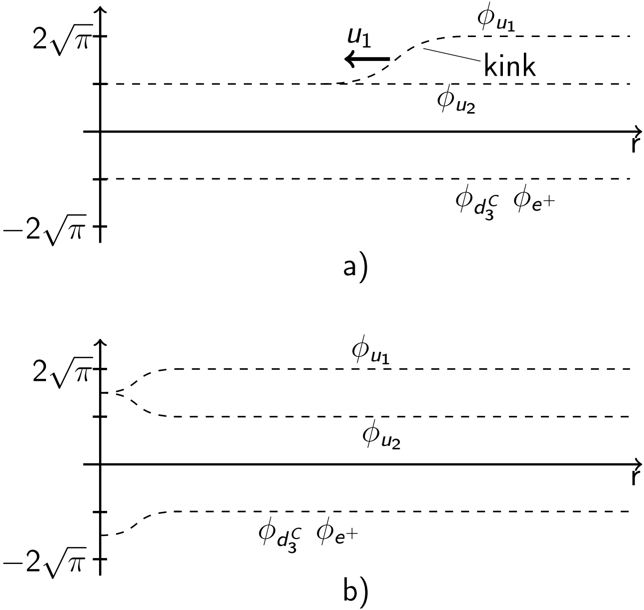
$$\Phi_{u_1} = \Phi_{u_2} = N\sqrt{\pi} \quad \Phi_{e^+} = \Phi_{d_3^C} = -N\sqrt{\pi}$$

transitions

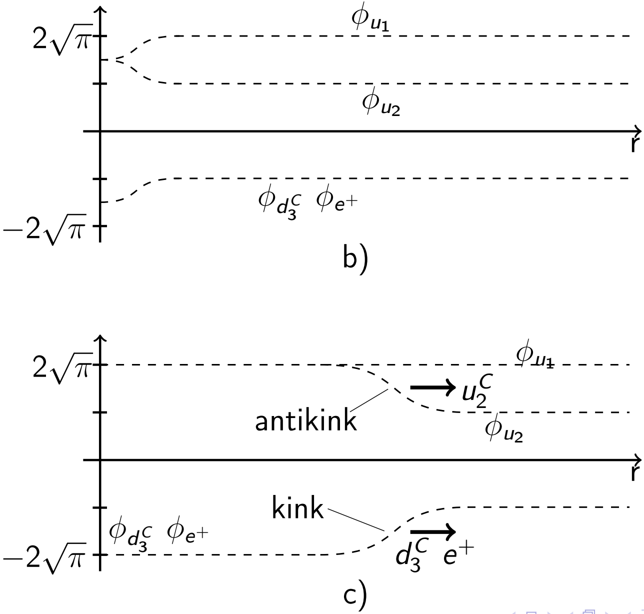
$$|N\rangle + \gamma \to |N-1\rangle + u_1 + u_2 + d_3 + e^- = |N-1\rangle + p + e^- \quad (6)$$

true vacuum: $\sum |N\rangle$.

Massive Fermions and Bosonization



Massive Fermions and Bosonization



Massive Fermions and Bosonization

decays are possible:

$$u_1 + |N\rangle \to |N+1\rangle + u_2^C + d_3^C + e^+$$

vacuum leads to catalyzation:

$$p + M \to M + e^+ + \pi + \pi$$

cross section at typical QCD scales:

$$\sigma = CE^{-2}$$

with an order one constant C

Thank you!

Questions?

Erick J. Weinberg. *Classical solutions in quantum field theory : solitons and instantons in high energy physics*. Cambridge monographs on mathematical physics. Cambridge, 2012

M. Shifman. *Advanced Topics in Quantum Field Theory: A Lecture Course*. Cambridge University Press, 2012. DOI: 10.1017/CBO9781139013352

The 't Hooft Polyakov Monopole

winding number:

$$\nu = \frac{1}{8\pi} \int_{S^2_\infty} \mathrm{d}^2 S_i \, \epsilon^{ijk} \epsilon_{abc} \hat{\phi}^a \partial_j \hat{\phi}^b \partial_k \hat{\phi}^c$$

asymptotic behaviour: $D_i \phi \to 0$ faster than $r^{-\frac{3}{2}}$, so that $\int \mathrm{d}r \, r^2 (D_i^a \phi)^2$

$$0 = \frac{1}{8\pi} \epsilon^{ijk} \int_{S^2_\infty} \mathrm{d}^2 S_i \left[\epsilon_{abc} \hat{\phi}^a \partial_j \hat{\phi}^b \partial_k \hat{\phi}^c + e \hat{\phi}^a F^a_{jk} \right]$$

$$Q_\mathrm{M} = \int_{S^2_\infty} \mathrm{d}^2 S_i \, \mathcal{B}^i = \int_{S^2_\infty} \mathrm{d}^2 S_i \, \hat{\phi}^a B^{i,a} = \frac{4\pi}{e} \nu$$

Zero Modes and Dyons

zero mode: zero energy excitation around the monopole solution
define gauge fixing:

$$0 = \langle \delta\Lambda | \delta \rangle \qquad \text{with} \qquad \delta\Lambda = \{ D_i\Lambda, ie\,[\Lambda, \phi] \}$$

zero modes are related to broken symmetries, e.g. translational zero modes

$$\delta_i A_j = F_{ij} \qquad \delta_i \phi = D_i \phi \tag{7}$$

global $U(1)_{\mathrm{EM}}$

$$U = \exp(i\alpha \cdot \phi(\mathrm{r})/v) \tag{8}$$

action of U on fields:

$$A_\mu \to U A_\mu U^{-1} + \frac{i}{e} U(\partial_\mu U^{-1})$$
$$\phi \to U \phi U^{-1} = \phi$$

symmetry is broken in the monopole case:

$$\delta A_\mu = i\frac{\alpha}{v}[\phi, A_\mu] + \frac{\alpha}{ve}\partial_\mu \phi = \frac{\alpha}{ve} D_\mu \phi = \begin{cases} \frac{\alpha}{ve} D_i \phi & \text{if } \mu = i \\ 0 & \text{if } \mu = 0 \end{cases}$$

Zero Modes and Dyons

U(1) Zero Mode and Dyons

movement along this coordinate has measurable effect
In analytic limit:

$$E_i^a = F_{0i}^a = \partial_0 A_i^a + \partial_i A_0^a - ie[A_0, A_i]^a = \partial_0 \delta A_i^a = \frac{\dot{\alpha}}{ve} D_i \phi^a = \frac{\dot{\alpha}}{ve} B_i^a$$

$\implies$ electric field with charge

$$Q_{\mathrm{E}} = \frac{\dot{\alpha}}{ev} Q_{\mathrm{M}} \tag{9}$$

Lagrangian description:

$$L_\alpha = \int \mathrm{d}^3 x \left[\frac{1}{2} F_{0i}^a F_{0i}^a + \frac{1}{2} \left(D_0 \phi^a \right)^2 \right] = \frac{1}{2} \frac{M_{\mathrm{M}}}{e^2 v^2} \dot{\alpha}^2$$

Zero Modes and Dyons

U(1) Zero Mode and Dyons

quantization:

$$H_\alpha = \frac{1}{2}\frac{m_W^2}{M_\mathrm{M}}\pi_\alpha^2$$

$$\dot{\alpha} = \frac{e^2 v^2}{M_\mathrm{M}} k$$

$$Q_\mathrm{E} = \frac{\dot{\alpha}}{ev} Q_\mathrm{M} = k\frac{e^2 v^2}{v Q_\mathrm{M} \cdot ev} Q_\mathrm{M} = ke \qquad k \in \mathbb{Z}$$

$\implies$ dyon solution with magnetic charge $Q_\mathrm{M} = \frac{4\pi}{e}$ and electric charge $Q_\mathrm{E} = k \cdot e$.

Zero Modes and Dyons

Witten Effect and the θ-Term

$$\mathcal{L}_\theta = -\frac{\theta e^2}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{a\mu\nu} = \frac{\theta e^2}{8\pi^2} \mathrm{E}^a \cdot \mathrm{B}^a$$

with dual field strength tensor $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$
using the 't Hooft Polyakov solution

$$L_\theta = \frac{\theta}{2\pi}\dot{\alpha} \qquad L_\alpha = \frac{1}{2}\frac{M_\mathrm{M}}{e^2 v^2}\dot{\alpha}^2 + \frac{\theta}{2\pi}\dot{\alpha}$$

and so the Hamiltonian is

$$H_\alpha = p_\alpha \dot{\alpha} - L = H_\alpha = \frac{1}{2}\frac{e^2 v^2}{M_\mathrm{M}}\left(p_\alpha - \frac{\theta}{2\pi}\right)^2$$

p_α is quantized

$$Q_\mathrm{E} = ke - \frac{\theta e}{2\pi} \qquad \text{with } k \in \mathbb{Z}$$

Fermionic Zero Modes

Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^{a,\mu\nu}F^a_{\mu\nu} + \frac{1}{2}(D_\mu\phi)^a(D^\mu\phi)^a - V(\phi) + i\bar{\psi}_n\gamma^\mu(D_\mu\psi)_n - G\bar{\psi}T^a_{nm}\psi_m\phi_a$$

ψ_n, also transforming under the SU(2) gauge group
The Euler-Lagrange equations

$$-i\gamma^0\partial_0\psi = i\gamma^i\partial_i\psi + \left(\gamma_i\epsilon^{aim}\hat{r}^m e\frac{1-u(r)}{er} - G\hat{r}^a h(r)\right)T^a\psi$$

fundamental fermion where $T^a = \frac{1}{2}\sigma^a$

$$(\psi_0)_n = \begin{pmatrix} N\exp\left(\int_0^r \mathrm{d}r'\left[\frac{1-u(r)}{r} - \frac{1}{2}Gh(r')\right]\right)\sigma^2_{\alpha n} \\ 0 \end{pmatrix}$$

Monopoles in Higher Gauge Groups
SU(5) GUT Monopoles

$$\mathrm{SU}(5) \xrightarrow{\Lambda_{\mathrm{GUT}}} \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1) \xrightarrow{\Lambda_{\mathrm{EW}}} \mathrm{SU}(3) \times \mathrm{U}(1)$$

monopole from first breaking:

$$\pi_2(\mathrm{SU}(5)/(\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1))) = \pi_1(\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)) = \mathbb{Z}$$

Massive Fermions and Bosonization

details of the scattering process: temporary go back to the massless limit.

$$e_L^+ + M \to M + \frac{1}{2}e_R^+ + \frac{1}{2}u_{1,R} + \frac{1}{2}u_{2,R} + \frac{1}{2}d_{3,L} \qquad (10)$$

asymptomatically need well defined states
Problem for positron scattering Csáki, Shirman et al.[5].
limitation of the $J = 0$ approximation
extra U(1) factor under Lorenz transformation[6] with charge

$$q_{ij} = e_i g_j - e_j g_i \qquad (11)$$

[5] Csaba Csáki et al. “Monopoles Entangle Fermions”. In: (2021). arXiv: 2109.01145 [hep-th].

[6] Csaba Csáki et al. “Scattering amplitudes for monopoles: pairwise little group and pairwise helicity”. In: *Journal of High Energy Physics* 2021.8 (Aug. 2021).

Fermion states in the J=0 Approximation

Table: $J = 0$ fermion states in a monopole background

Fermion number	Direction	Charge	Helicity
+1	in	-1/2	left
+1	out	+1/2	left
+1	out	-1/2	right
+1	in	+1/2	right
-1	out	-1/2	left
-1	in	+1/2	left
-1	in	-1/2	right
-1	out	+1/2	right

Dirac Charge Quantization Condition

magnetic monopole with charge Q_{M}: has magnetic field B

$$\mathsf{B} = \frac{Q_{\mathrm{M}}}{4\pi r^2}\hat{\mathsf{r}} = \frac{g}{r^2}\hat{\mathsf{r}} \tag{12}$$

vector potential:
Poincaré-Lemma: total magnetic charge $\int \mathrm{d}\mathsf{S}\cdot\mathsf{B}$ zero.
consider only the space outside of the monopole:

$$A^{Ii} = -\epsilon^{ij3}\hat{r}^j \frac{g}{z+r}$$
$$A^{IIi} = -\epsilon^{ij3}\hat{r}^j \frac{g}{z-r}$$

Dirac Charge Quantization Condition

Difference on the overlap region:

$$A^{IIi} - A^{Ii} = -\epsilon^{ij3}\hat{r}^j g \left(\frac{1}{z-r} - \frac{1}{z+r} \right) = -g\epsilon^{ij3}\hat{r}^j \left(\frac{2r}{z^2 - r^2} \right)$$
$$= -2g\epsilon^{ij3} \frac{r^j}{x^2 + y^2} = -\partial_i (2g\phi) = \partial_i \Lambda$$

Transformation of matter fields:

$$\psi \xrightarrow{\text{U(1) EM}} e^{-iq\Lambda}\psi = e^{2iqg\phi}\psi$$

Dirac quantization condition:

$$g \cdot q = \frac{n}{2} \qquad n \in \mathbb{N}$$