Singly and doubly soft resummation of rapidity distributions

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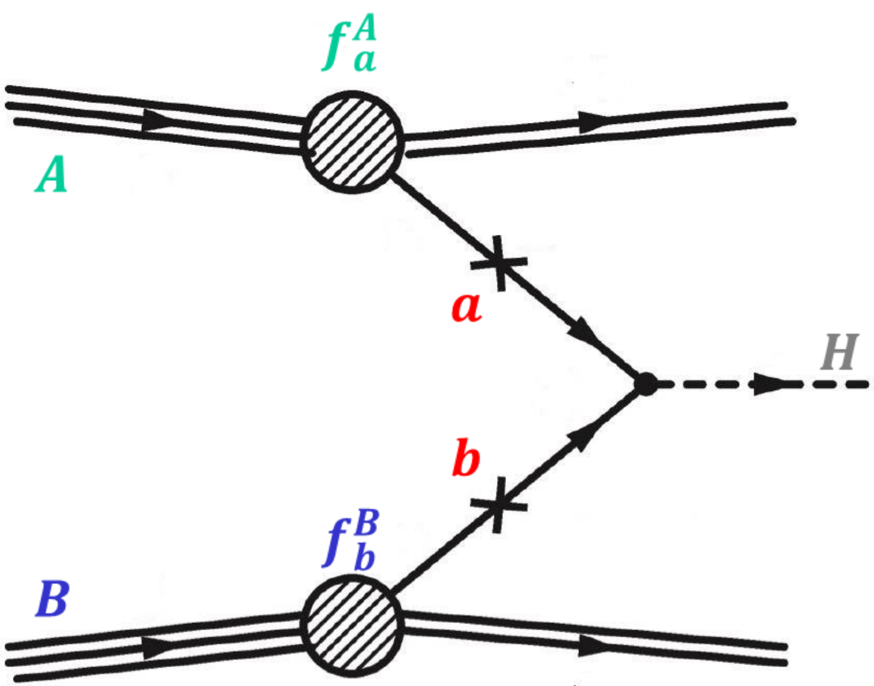
Outline

- Hadron collision and factorization
- Soft resummation (multi-scale)
- Rapidity distribution
- Kinematic thresholds and scales
- Resummed formulas

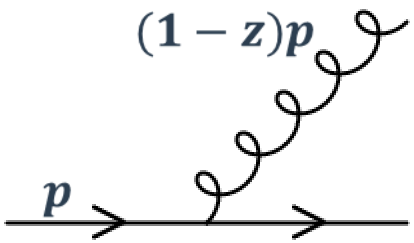
Hadron collision and factorization

$$\sigma_{A,B\to H}(\tau,\alpha_s) = \sum_{a,b} \iint_0^1 dx_1 dx_2 f_a^A(x_1,\mu_F) f_b^B(x_2,\mu_F) \hat{\sigma}_{a,b\to H}\left(\frac{\tau}{x_1 x_2}, \alpha_s, \mu_F\right)$$



- σ hadronic cross section
- f_a^A, f_b^B PDFs
- $\hat{\sigma}$ partonic cross section
- x_1, x_2 4-momentum fractions
- $\tau = {}^{M_H^2}/_s$ fraction of hadronic energy used to produce the Higgs
- $z = \frac{\tau}{x_1 x_2}$ fraction of partonic energy used to produce the Higgs

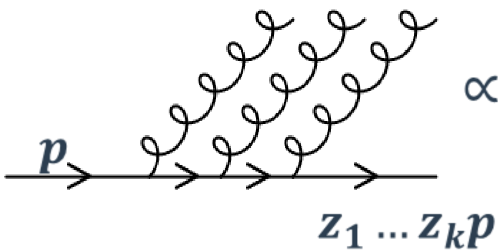
Soft gluons and big logs



$\propto \alpha_s \frac{1}{1-z}$, singular

Cancellation of soft divergences

$$\alpha_s \int_\tau^1 dz \, \frac{1}{(1-z)_+} \propto \alpha_s \ln(1-\tau)$$



$\propto \alpha_s^k \frac{1}{1-z_1} \dots \frac{1}{1-z_k}$, singular

$$\alpha_s^k \int_\tau^1 dz \left(\frac{\ln^{k-1}(1-z)}{1-z} \right)_+$$

$$\propto (\alpha_s \ln(1-\tau))^k$$

$$\alpha_s \ln(1-\tau) = O(1)$$

$$(\alpha_s \ln(1-\tau))^k = O(1)$$

$$\hat{\sigma} = \sum_{n=0}^{\infty} C_n \alpha_s^n (\ln(1-\tau))^n + \cdots$$

Perturbative series is spoiled

RESUMMATION is needed

RG and soft resummation

Re-organising the perturbative series

I Mellin-space factorization

$$\sigma(N, M_H^2) = \mathcal{L}(N, \mu^2)\hat{\sigma}\left(N, M_H^2/\mu^2\,, \alpha_s(\mu^2)\right)$$

II Physical anomalous dimension

$\gamma(N, M_H^2) = \dfrac{d\;\ln\sigma}{d\;\ln M_H^2}$ is RG-invariant

III Soft singularities factorization

$$\hat{\sigma}^0(N, M_H^2, \alpha_s^0, \epsilon) = \hat{\sigma}^{0c}(M_H^2, \alpha_s^0, \epsilon)\hat{\sigma}^{0l}(M_H^2/N^2, \alpha_s^0, \epsilon)$$

1 HARD scale: M_H^2

1 SOFT scale: M_H^2/N^2

Luminosity: $\mathcal{L} = f \otimes f$

Resummed formula

$$\hat{\sigma}\left(N, \frac{M_H^2}{\mu^2}, \alpha_s(\mu^2)\right) = \hat{\sigma}^c\left(N, \frac{M_H^2}{\mu^2}, \alpha_s(M_H^2)\right) \times \exp\left\{\int_1^{N^a} \frac{dn}{n} \int_{n\mu^2}^{M_H^2} \frac{dk^2}{k^2} \hat{g}\left(\alpha_s\left(\frac{k^2}{n}\right)\right)\right\}$$

Multi-scale resummation

2 **HARD** scales

$$Q_1^2,\ Q_2^2$$

2 **SOFT** scales

$$\Lambda_1^2(Q_1^2,N)=\frac{Q_1^2}{N^a},\ \ \Lambda_2^2(Q_2^2,N)=\frac{Q_2^2}{N^b}$$

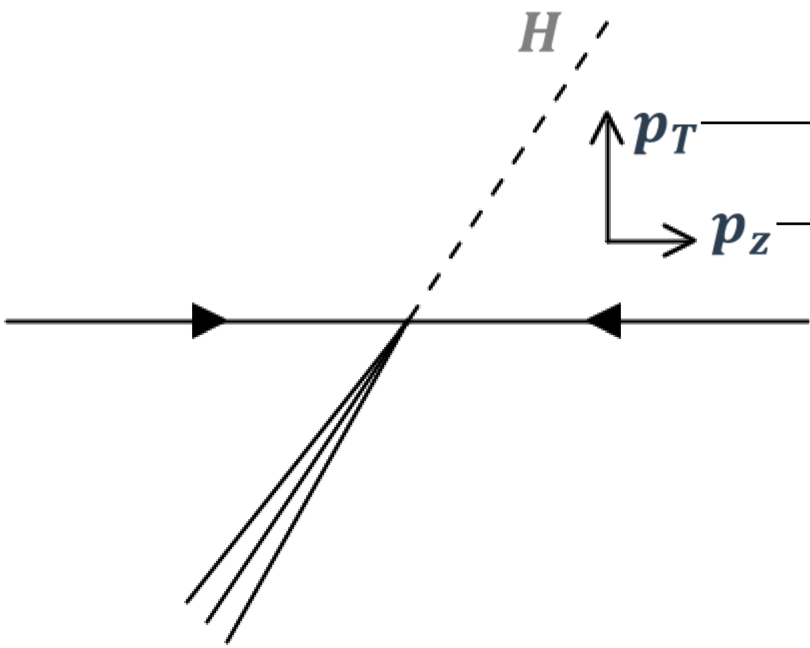
III Soft singularities factorization

$$\hat\sigma^0(N,Q_1^2,Q_2^2,\alpha_s^0,\epsilon)=\hat\sigma^{0c}(Q_1^2,Q_2^2,\alpha_s^0,\epsilon)\hat\sigma^{0l_1}(\Lambda_1^2(Q_1^2,N),\alpha_s^0,\epsilon)\hat\sigma^{0l_2}(\Lambda_2^2(Q_2^2,N),\alpha_s^0,\epsilon)$$

2-scale resummed formula

$$\hat{\boldsymbol\sigma}\left(N,\frac{Q_1^2}{\mu^2},\frac{Q_2^2}{\mu^2},\alpha_s(\mu^2)\right)=\hat{\boldsymbol\sigma}^c\left(N,\frac{Q_1^2}{\mu^2},\frac{Q_2^2}{\mu^2},\alpha_s(\mu^2)\right)\times$$
$$\times\exp\left\{\int_1^{N^a}\frac{dn}{n}\int_{n\mu^2}^{Q_1^2}\frac{dk^2}{k^2}\hat g_1\left(\alpha_s\left(k^2/n\right)\right)+\int_1^{N^b}\frac{dn}{n}\int_{n\mu^2}^{Q_2^2}\frac{dk^2}{k^2}\hat g_2\left(\alpha_s\left(k^2/n\right)\right)\right\}$$

Rapidity distribution



p_T is invariant under z-boosts

p_z is not: additional difficulty

Instead of p_z, we choose to fix the **longitudinal rapidity**

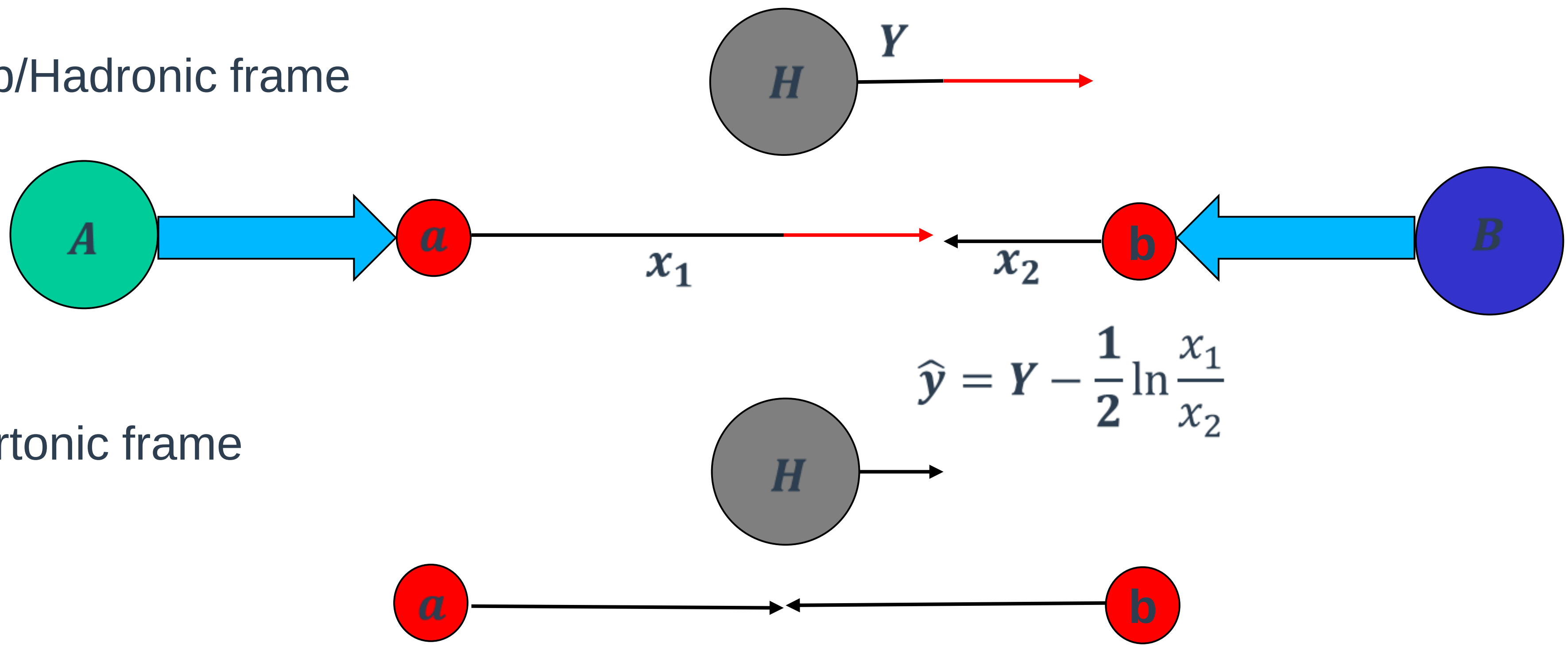
$$Y = \frac{1}{2} \ln \frac{E_H + p_z}{E_H - p_z}$$

Rate of Higgs production at **FIXED** longitudinal rapidity Y

$$\frac{d\sigma_{A,B\to H}}{dY}(\tau, Y) = \sum_{a,b} \iint_0^1 dx_1 dx_2 f_a^A(x_1) f_b^B(x_2) \frac{d\hat{\sigma}_{a,b\to H}}{d\hat{y}} \left(\frac{\tau}{x_1 x_2}, Y - \frac{1}{2} \ln \frac{x_1}{x_2} \right)$$

Rapidity distribution

Lab/Hadronic frame



$$\hat{y} = Y - \frac{1}{2}\ln\frac{x_1}{x_2}$$

Partonic frame

Rate of Higgs production at **FIXED** longitudinal rapidity Y

$$\frac{d\sigma_{A,B\to H}}{dY}(\tau, Y) = \sum_{a,b}\iint_0^1 dx_1 dx_2 f_a^A(x_1) f_b^B(x_2) \frac{d\hat{\sigma}_{a,b\to H}}{d\hat{y}}\left(\frac{\tau}{x_1 x_2}, Y - \frac{1}{2}\ln\frac{x_1}{x_2}\right)$$

Mellin-Fourier and Mellin-Mellin

$$\frac{d\sigma_{A,B\to H}}{dY}(\tau, Y) = \sum_{a,b} \iint_0^1 dx_1 dx_2 f_a^A(x_1) f_b^B(x_2) \frac{d\hat{\sigma}_{a,b\to H}}{d\hat{y}}\left(\frac{\tau}{x_1 x_2}, Y - \frac{1}{2}\ln\frac{x_1}{x_2}\right)$$

Mellin $\int_0^1 d\tau\, \tau^{N-1}[\quad]$

Fourier $\int_{-\infty}^{+\infty} dY\, e^{iMY}[\quad]$

Factorization in **Mellin-Fourier** space

$$\frac{d\sigma_{A,B\to H}}{dY}(N, M) = \sum_{a,b} f_a^A\left(N + i\frac{M}{2}\right) f_b^B\left(N - i\frac{M}{2}\right) \frac{d\hat{\sigma}_{A,B\to H}}{d\hat{y}}(N, M)$$

Convenient variables

$\left(z_1 = \sqrt{z}e^{\hat{y}}, z_2 = \sqrt{z}e^{-\hat{y}}\right)$

Mellin $\int_0^1 dz_i\, z_i^{N_i-1}[\quad]$

Factorization in **Mellin-Mellin** space

$$\frac{d\sigma_{A,B\to H}}{dY}(N_1, N_2) = \sum_{a,b} f_a^A(N_1) f_b^B(N_2) \frac{d\hat{\sigma}_{A,B\to H}}{d\hat{y}}(N_1, N_2)$$

Kinematic thresholds

DOUBLY SOFT region *(Catani, Trentadue)*

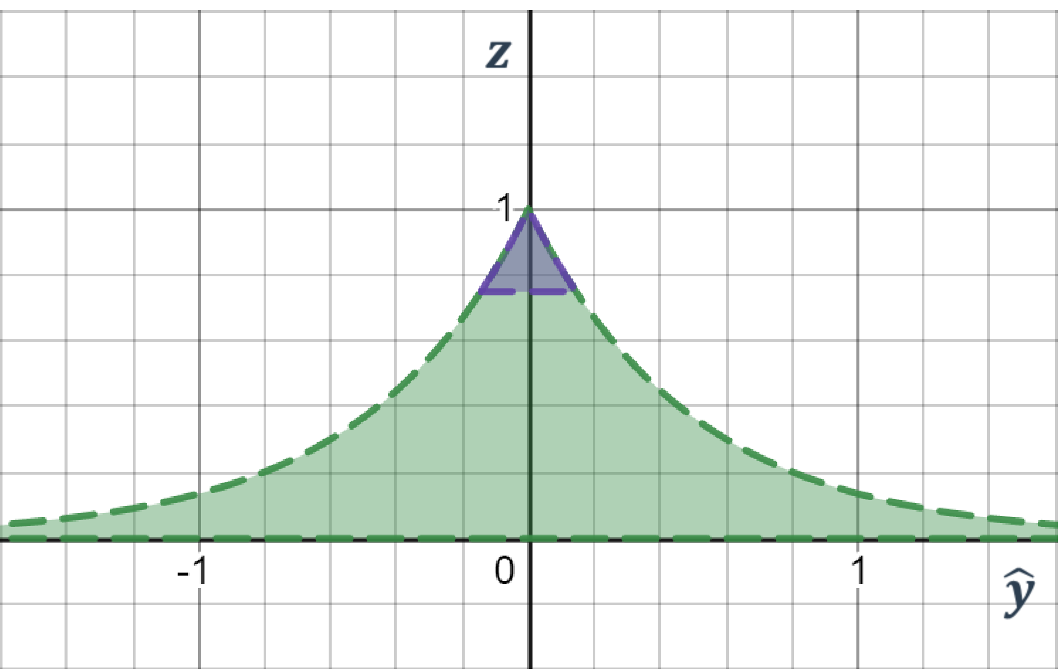
The partonic energy is approaching its **minimum** $\boldsymbol{M_H}$

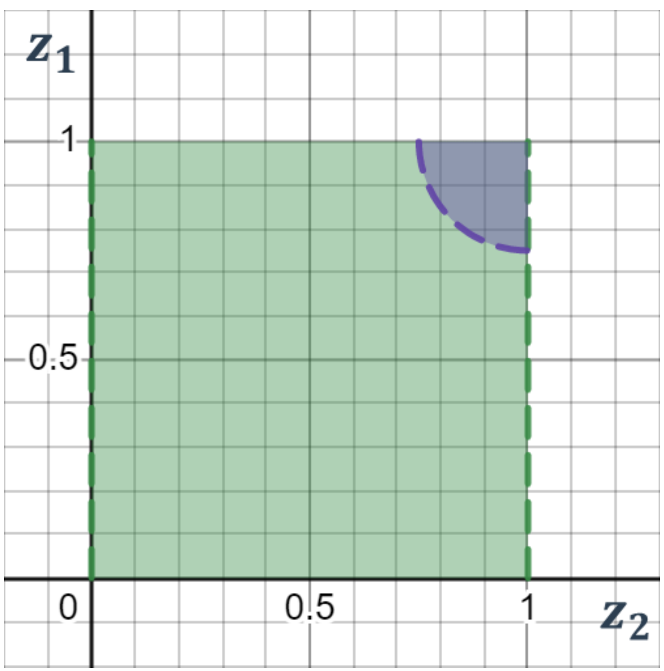
No more energy for the Higgs rapidity $\hat{y} \to 0$

No more energy for extra radiation → **soft gluons**

$$z = \frac{M_H^2}{\hat{s}} \to 1 \qquad \Rightarrow \qquad \hat{y} \to 0$$

$z_1 \to 1$ and $z_2 \to 1$, big $\ln N_1, \ln N_2$





Kinematic thresholds

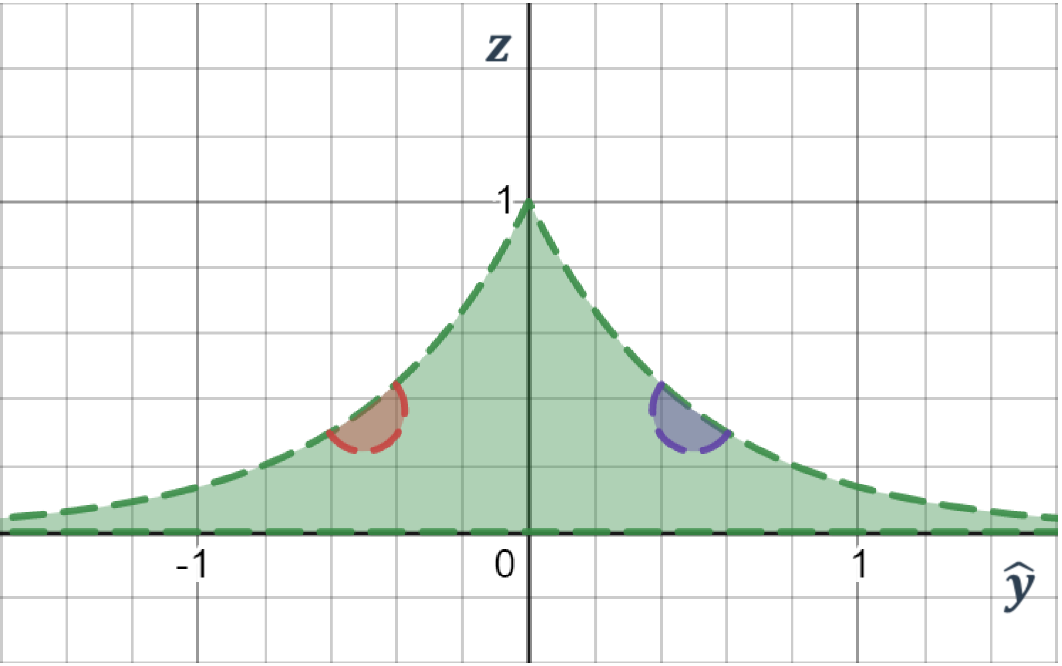
SINGLY SOFT region *(our contribution)*

The partonic energy is **fixed**, the rapidity is approaching its **maximum** $\hat{y}_0$

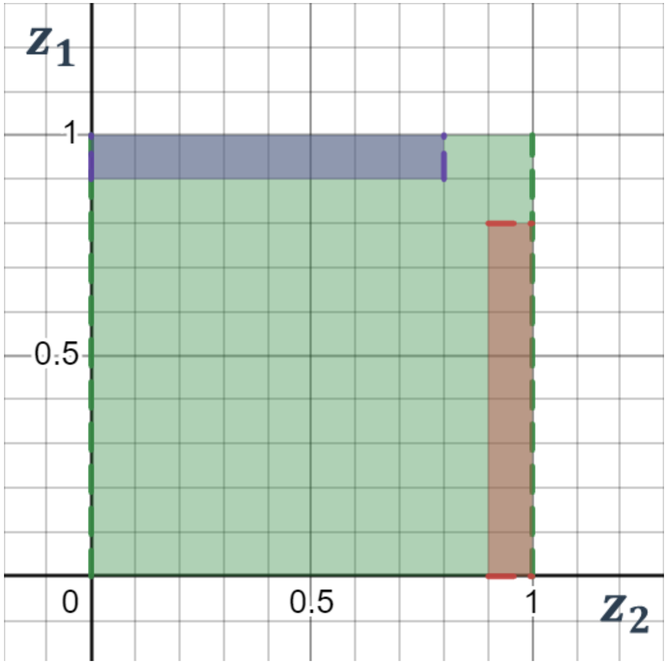
All the energy is used for the Higgs rapidity

No more energy for extra radiation → **soft** and **collinear gluons**

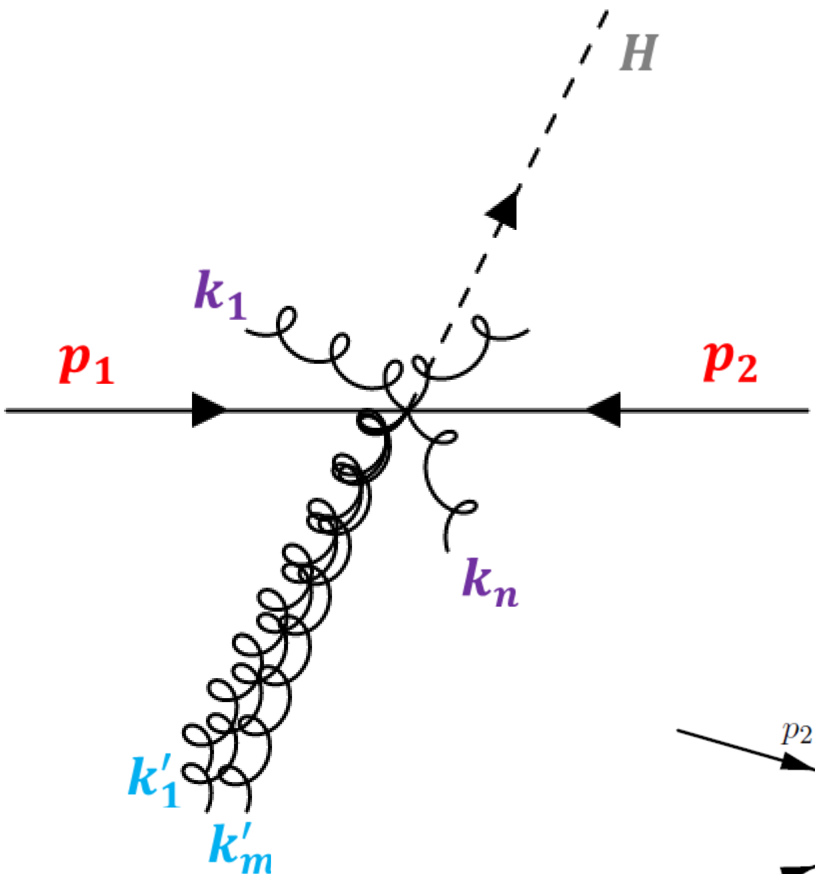
z **fixed** and $\hat{y} \to \pm\hat{y}_0$



$z_1 \to 1$ and z_2 **fixed**, big $\ln N_1$



Phase space and scales



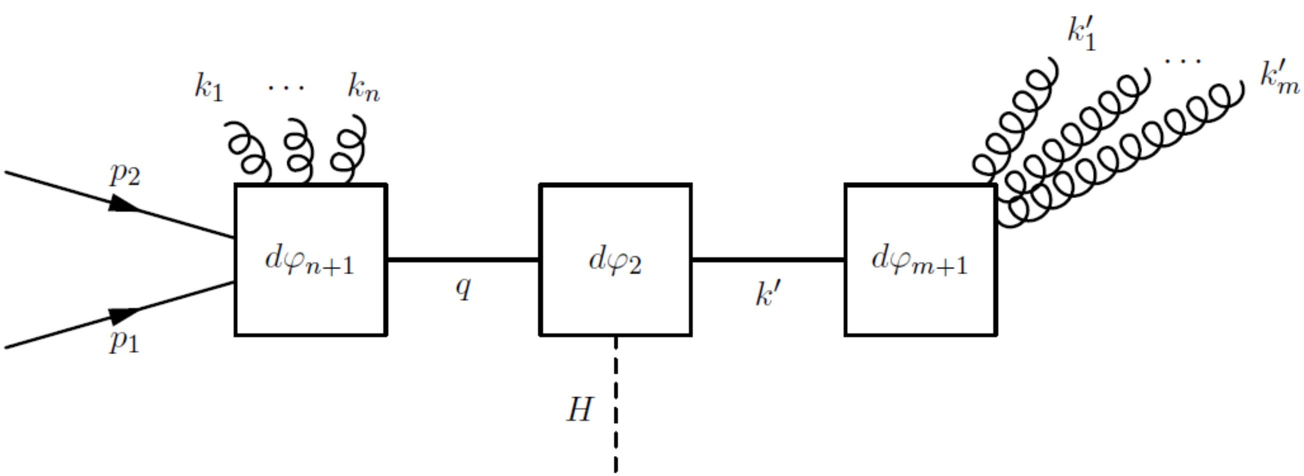
Factorizing **SOFT** and **COLLINEAR** emissions

$$d\varphi_{n+m+1} = \int dq^2\, d\varphi_{n+1}(p_1, p_2; q, k_1, \dots, k_n) \times$$

$$\times \int d(k')^2 d\varphi_2(q; p_H, k')\, d\varphi_m(k'; k'_1, \dots, k'_m)$$

$d\varphi_{n+1} \sim M_H^2 \left(1 - \max_{i=1,2} z_i\right)^2$ **SOFT** scale

$d\varphi_m \sim M_H^2 (1 - z_1)(1 - z_2)$ **COLLINEAR** scale



Resummation formulas

DOUBLY SOFT $(z_1 \to 1, z_2 \to 1)$

HARD scale

M_H^2

SOFT scale

$M_H^2\left(1 - \max_{i=1,2} z_i\right)^2$

$M_H^2 / \left(\max_{i=1,2} N_i\right)^2$

COLLINEAR scale

$M_H^2(1-z_1)(1-z_2)$

$M_H^2/(N_1 N_2)$

N.B. $\hat{y} \to 0$

$$\hat{\sigma}\left(N_1, N_2, \frac{M_H^2}{\mu^2}, \alpha_s(\mu^2)\right) = \hat{\sigma}^c\left(\frac{M_H^2}{\mu^2}, \alpha_s(M_H^2)\right)$$

$$\exp\left\{\cancel{\int_1^{N_1^2} \frac{dn}{n} \int_{n\mu^2}^{M_H^2} \frac{dk^2}{k^2} \hat{g}_1\left(\alpha_s\left(k^2/n\right), N_2\right)} + \int_1^{N_1 N_2} \frac{dn}{n} \int_{n\mu^2}^{M_H^2} \frac{dk^2}{k^2} \hat{g}_2\left(\alpha_s\left(k^2/n\right)\right)\right\}$$

SOFT becomes subleading

COLLINEAR becomes SOFT

Resummation formulas

SINGLY SOFT $(z_1 \to 1, z_2 \text{ fixed})$

HARD scale	**SOFT** scale	**COLLINEAR** scale
M_H^2 and $M_H^2(1-z_2)$	$M_H^2(1-z_1)^2$	$M_H^2(1-z_1)(1-z_2)$
M_H^2 and M_H^2/N_2	M_H^2/N_1^2	$M_H^2/(N_1N_2)$

$$\hat{\sigma}\left(N_1, \frac{M_H^2}{\mu^2}, \frac{M_H^2/N_2}{\mu^2}, \alpha_s(\mu^2)\right) = \hat{\sigma}^c\left(\frac{M_H^2}{\mu^2}, \alpha_s(M_H^2)\right)$$

$$\exp\left\{\int_1^{N_1^2}\frac{dn}{n}\int_{n\mu^2}^{M_H^2}\frac{dk^2}{k^2}\hat{g}_1\left(\alpha_s\left({}^{k^2}\!/_{n}\right), N_2\right) + \int_1^{N_1N_2}\frac{dn}{n}\int_{n\mu^2}^{M_H^2/N_2}\frac{dk^2}{k^2}\hat{g}_2\left(\alpha_s\left({}^{k^2}\!/_{n}\right)\right)\right\}$$

Conclusions: past, present and future

- **Phase space + RG => Resummation**
- p_T**-distribution case known** ✓
- **Rapidity doubly soft checked** ✓
- **Rapidity singly soft derived** ✓
- **Apply to fully differential distributions**

Thank you for your attention

Factorization in Mellin-Fourier space

$$\frac{d\sigma}{dY}(\tau,Y)=\int_0^1 dx_1 f_a^A(x_1)\int_{\frac{\tau}{x_1}}^1 dx_2 f_b^B(x_2)\frac{d\hat{\sigma}_{ab}}{d\hat{y}}\left(\frac{\tau}{x_1x_2},Y-\frac{1}{2}\ln\frac{x_1}{x_2}\right)$$

$$=\iiint_0^1 dx_1\,dx_2\,dz\,f_a^A(x_1)f_b^B(x_2)\int_{-\hat{y}_0}^{\hat{y}_0}d\hat{y}\;\delta(\tau-x_1x_2z)\delta\left(Y-\frac{1}{2}\ln\frac{x_1}{x_2}-\hat{y}\right)\frac{d\hat{\sigma}_{ab}}{d\hat{y}}(z,\hat{y})$$

$$\frac{d\sigma}{dY}(N,M)=\int_0^1 d\tau\;\tau^{N-1}\int_{-Y_0}^{Y_0}d\hat{y}\;e^{iMY}\frac{d\sigma}{dY}(\tau,Y)$$

$$=\left[\int_0^1 dx_1\,x_1^{N+i\frac{M}{2}-1}\,f_a^A(x_1)\right]\times\left[\int_0^1 dx_2\,x_2^{N-i\frac{M}{2}-1}\,f_b^B(x_2)\right]\times$$

$$\times\left[\int_0^1 dz\,z^{N-1}\int_{-\hat{y}_0}^{\hat{y}_0}d\hat{y}\;e^{iM\hat{y}}\frac{d\hat{\sigma}_{ab}}{d\hat{y}}(z,\hat{y})\right]$$

$$=f_a^A\left(N+i\frac{M}{2}\right)f_b^B\left(N-i\frac{M}{2}\right)\frac{d\hat{\sigma}_{ab}}{d\hat{y}}(N,M)$$

From $(z,\hat{y})$ to (z_1,z_2)

$$\frac{d\sigma}{dY}(N,M) = f_a^A\left(N+i\frac{M}{2}\right) f_b^B\left(N-i\frac{M}{2}\right)\frac{d\hat{\sigma}_{ab}}{d\hat{y}}(N,M)$$

$$= f_a^A\left(N+i\frac{M}{2}\right) f_b^B\left(N-i\frac{M}{2}\right)\int_0^1 dz\, z^{N-1}\int_{-\hat{y}_0}^{\hat{y}_0} d\hat{y}\, e^{iM\hat{y}}\frac{d\hat{\sigma}_{ab}}{d\hat{y}}(z,\hat{y})$$

Variable change

$\left(z_1=\sqrt{z}e^{\hat{y}}, z_2=\sqrt{z}e^{-\hat{y}}\right)$ and $\left(N_1=N+i\frac{M}{2}, N_2=N-i\frac{M}{2}\right)$

$$= f_a^A(N_1)f_b^B(N_2)\int_0^1 dz_1\, z_1^{N_1-1}\int_0^1 dz_2\, z_2^{N_2-1}\frac{d\hat{\sigma}_{ab}}{d\hat{y}}(z_1,z_2)$$

$$= f_a^A(N_1)f_b^B(N_2)\frac{d\hat{\sigma}_{ab}}{d\hat{y}}(N_1,N_2)$$

Plus distributions (total cross section)

Definition of plus distribution

$$\int_0^1 dx\,[f(x)]_+\,g(x) = \int_0^1 dx\,f(x)\,\big(g(x) - g(1)\big)$$

Mellin transform of plus distributions

$$\int_0^1 dx\,x^{N-1}[f(x)]_+ = \int_0^1 dx\,\left(x^{N-1} - 1\right) f(x)$$

Mellin transform of typical contributions

$$\int_0^1 dx\,x^{N-1}\left[\frac{\ln^k(1-x)}{1-x}\right]_+ = \int_0^1 dx\,\frac{x^{N-1}-1}{1-x}\,\ln^k(1-x) =$$

$$= \frac{d^k}{d\eta^k}\bigg|_{\eta=0}\int_0^1 dx\,\left(x^{N-1} - 1\right)(1-x)^{\eta-1} = \frac{d^k}{d\eta^k}\bigg|_{\eta=0}\left[B(N,\eta) - \frac{1}{\eta}\right]$$

Plus distributions (rapidity distribution)

Inclusive cross section

$$\left[\frac{\ln^k(1-z)}{1-z}\right]_+$$

big logarithms

$$\ln^k N$$

Rapidity distributions

$\left[\frac{\ln^k(1-z)}{1-z}\right]_+$ and $\left[\frac{\ln^p(1-u)}{1-u}\right]_+$

where $u = \frac{1+\cos\theta}{2}$

difficult to obtain big logs

$\ln^m N$ and $\ln^n M$

M is not the conjugate of u

$\hat{y}$ does not range in $[0, 1]$

$\left[\frac{\ln^k(1-z_1)}{1-z_1}\right]_+$ and $\left[\frac{\ln^p(1-z_2)}{1-z_2}\right]_+$

easier to obtain big logs

$\ln^k N_1$ and $\ln^p N_2$

Drell-Yan example

Drell-Yan NLO

$$\frac{\eta_{q\bar{q}}}{Q_q^2} = \frac{8}{3}\,\frac{z^2}{1+z}\left[\delta(u) + \delta(1-u)\right]$$

$$\left[\delta(1-z)(2\zeta_2 - 4) + 4\left[\frac{\ln(1-z)}{1-z}\right]_+ - 2(1+z)\ln(1-z) - \frac{1+z^2}{1-z}\ln z + 1 - z\right]$$

$$+\frac{8}{3}\,\frac{z^2}{1+z}\left(1 + \frac{(1-z)^2}{z}u(1-u)\right)\left[\frac{1+z^2}{[1-z]_+}\left(\frac{1}{u_+} + \frac{1}{[1-u]_+}\right) - 2(1-z)\right]$$

Higgs example

Higgs production NLO

$$\eta_{gg} = \frac{1}{2}\delta\big(u(1-u)\big)\left[\delta(1-z)\left(6\zeta_2+\frac{11}{2}\right)+12\left[\frac{\ln(1-z)}{1-z}\right]_+ - 12z\big(z^2-z+2\big)\ln(1-z) - 6(z^2-z+1)^2\frac{\ln z}{1-z}\right]$$

$$+3\frac{1}{[u(1-u)]_+}\left(\frac{1}{[1-z]_+}-z\big(z^2-z+1\big)\right)-3(1-z)^3[2-u(1-u)]$$