Gauge/Gravity Duality and its Applications to Holographic Superconductors

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14.01.2011

Overview

Concept of Gauge/Gravity Duality

- Equivalence of some QFTs to (quantum) gravity theories in a particular limit: The gravity theory becomes classical and the QFT strongly coupled.
- It originates from string theory.
- Generalization of the AdS/CFT correspondence (AdS: Anti-de-Sitter spacetime, CFT: Conformal field theory)

Maldacena's Conjecture on AdS/CFT

4d CFT, i.e. supersymmetric SU(N) Yang-Mills theory, is equivalent to a superstring theory on a 10d spacetime involving AdS space!

CFT in d=4

- Conformal coordinate transformations leave angles unchanged.
- There are several possibilities for such transformations:
  Translations, rotations, dilations and so called special conformal transformations.
- In a conformal field theory fields transform covariantly under conformal coordinate transformations.

Anti-de-Sitter Space

- The (p+2)-dimensional AdS space is represented by the hyperboloid

$$X_0^2 + X_{p+2}^2 - \sum_{i=1}^{p+1} X_i^2 = L^2$$

  which is embedded in (p+3)-dimensional space.
  L denotes the AdS radius.

- The metric is given by

$$ds^2 = -dX_0^2 - dX_{p+2}^2 + \sum_{i=1}^{p+1} dX_i^2 \, .$$

- AdS space has constant negative curvature.

Open String Perspective

- D3-branes:
  (3+1)-dimensional hypersurfaces in flat (9+1)-dimensional space on which open strings can end (the "D"stands for Dirichlet boundary conditions).
- Low energy limit:
  Dynamics described by $\mathcal{N}$=4 SYM theory.
- Maldacena limit ($\alpha' \to 0$ but $\frac{r}{\alpha'}$ constant for an arbitrary r):
  Open strings and closed strings decouple.

RESULT:
$\mathcal{N}$=4 SYM and 10d SUGRA

Closed String Perspective

- D3-branes:
  Solutions of low energy limit of superstring theory (i.e. SUGRA). They are massive charged objects which curve the surrounding spacetime.
- Metric:

$$ds^2 = H(r)^{-\frac{1}{2}} \eta_{\mu\nu} dx^\mu dx^\nu + H(r)^{\frac{1}{2}} d\vec{y}^2$$

  with $H(r) = 1 + \frac{L^4}{r^4}$.
- Two asymptotic regions: $r \to \infty$ and $r \to 0$:
  $r \to \infty$: 10d flat space.
  $r \to 0$ (near horizon region): $H(r) \sim \frac{L^4}{r^4}$

- Maldacena limit:
  Closed strings in the two asymptotic regions decouple.
- Near horizon limit of the metric:

$$ds^2 = \frac{r^2}{L^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{L^2}{r^2}dr^2 + L^2 d\Omega^2$$

  This is the metric of $AdS_5 \times S^5$!

RESULT:
Type IIB SUGRA on $\mathbb{R}^{9+1}$ and SUGRA on $AdS_5 \times S^5$

Summary

The low energy limit gives two decoupled theories from both perspectives:

- Open strings:
  Type IIB SUGRA on $\mathbb{R}^{9+1}$ and $\mathcal{N}$=4 SYM in d=3+1
- Closed strings:
  Type IIB SUGRA on $\mathbb{R}^{9+1}$ and SUGRA on $AdS_5 \times S^5$

The symmetries agree:

- $\mathcal{N}$=4 SYM: $SO(4,2) \times SU(4)_R$
- $AdS_5 \times S^5$: $SO(4,2) \times SO(6)$

The Field-Operator-Map

- One-to-one map between gauge invariant operators in the CFT and SUGRA fields.
- Interpretation: SU(N) $\mathcal{N}$=4 SYM lives on the boundary of AdS_5 (“Holography“).
- In $AdS_5 \times S^5$ look at the Klein-Gordon equation for a scalar field:
  - Expand all fields in spherical harmonics, schematically:

$$\phi(\vec{z}, \vec{y}) = \sum_{\Delta \geq 0} \phi_\Delta(\vec{z}) Y_\Delta(\vec{y})$$

  - Solve the resulting wave equation on 5d AdS space.

- Two independent solutions with different asymptotic behavior for $z_0 \to 0$ ($\vec{z} = (z_0, z')$ with $z_0 \sim \frac{1}{r}$):

$$\phi_\Delta(z_0, z') \sim \begin{array}{ll} z_0^{\Delta} & \text{normalizable} \\ z_0^{4-\Delta} & \text{non}-\text{normalizable} \end{array}$$

- Define boundary fields: $\bar{\phi}_\Delta(z') = \lim_{z_0 \to 0} \phi_\Delta(z_0, z') z_0^{\Delta - 4}$
- General expansion for $z_0 \to 0$:

$$\phi_\Delta(z_0, z') \sim \langle \mathcal{O}_\Delta \rangle z_0^{\Delta} + \bar{\phi}_\Delta(z') z_0^{4-\Delta}$$

  Here $\langle \mathcal{O}_\Delta \rangle$ is the VEV of an Operator $\mathcal{O}_\Delta$ in the dual field theory and the boundary field $\bar{\phi}_\Delta$ is its source.
- The generating functional for all correlators of the operator $\mathcal{O}_\Delta$ is given in terms of the source fields!

Field Theories with Finite Temperature

- The AdS Schwarzschild black hole (in Euclidean signature) has the metric

$$ds^2 = \frac{r^2}{L^2}(f(r)d\tau^2 + dx^2) + \frac{L^2}{r^2}\frac{1}{f(r)}dr^2 + L^2 d\Omega_5^2$$

with $f(r) = 1 - \frac{r_H^4}{r^4}$ and r_H the horizon.

- The Hawking temperature of the black hole becomes the temperature of the field theory at the boundary:

$$r_H = T\pi L^2$$

- It is possible to add charges (e.g. adding electric charge gives the Reissner-Nordström solution).

Applications

- Quark-gluon-plasma (strongly coupled system)
- Condensed matter systems as e.g. superconductors

A Holographic Superconductor

Gravity dual of a superconductor: A hairy black hole!

Setup

- We consider a charged AdS black hole (Reissner-Nordström) in the probe limit (i.e. we neglect the back-reaction on the metric).
- This gives AdS Schwarzschild together with an electric field (i.e. we have a charged horizon but this does not affect the metric).
- Now: Add an additional component to the gauge field ("hair").
- More precisely: We solve the YM equations in the AdS_4 background for the following SU(2) gauge field ansatz:

$$A = \Phi(r)\tau^3 dt + w(r)\tau^1 dx$$

Conductivities and Green's Functions

- We have the following formula connecting the retarded Green's function and the conductivity:

$$\sigma_{ij}^{ab}(\omega) = \frac{i}{\omega} G_{ij}^{R,ab}(\omega, 0)$$

- For large r the components A_i^a of a non-Abelian gauge field can be expanded as

$$A_i^a = e^{-i\omega t}\left[A_i^{a,(0)} + \frac{A_i^{a,(1)}}{r} + \mathcal{O}(r^{-2})\right].$$

- This is related to the retarded Green's function by

$$G_{ij}^{R,ab} = -\frac{2}{\kappa^2}\frac{\delta A_i^{a,(1)}}{\delta A_j^{b,(0)}}.$$

Fluctuations

- We consider perturbations $A \to A + a$ with

$$a = e^{-i\omega t}(a_y^1 \tau^1 + a_y^2 \tau^2) dy \,.$$

and derive the e.o.m. at the linearized level.

- Ansatz for the near horizon expansion of the fluctuations:

$$\begin{aligned} a_y^1 &= (r-1)^{-i\omega/4\pi T}\left[1 + a_y^{1(1)}(r-1) + \dots\right], \\ a_y^2 &= (r-1)^{-i\omega/4\pi T}\left[1 + a_y^{2(1)}(r-1) + \dots\right]. \end{aligned}$$

This means we have ingoing boundary conditions at the horizon!

- Near boundary expansion for the fluctuations:

$$\begin{aligned} a_y^1 &= A_y^{1,(0)} + \frac{A_y^{1,(1)}}{r} + \ldots , \\ a_y^2 &= A_y^{2,(0)} + \frac{A_y^{2,(1)}}{r} + \ldots \end{aligned}$$

This can be used to calculate the Green's functions/ conductivities.

Analytical Calculations

- We can have a look at the normal state, i.e. a background with $w = 0$. This gives the Reissner-Nordström solution

$$\tilde{\Phi} = \mu \left(1 - \frac{1}{r} \right)$$

with μ the chemical potential.

- Expand the fluctuations in ω and μ (as long as these quantities are assumed to be small). This gives the Green's functions/conductivities.

- RESULT:

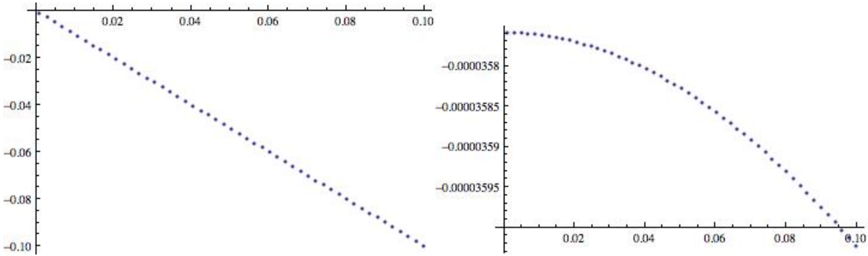
$$G_{yy}^{R,11}(k=0,\omega) = -i\omega - 0.358\mu^2 + \dots ,$$
$$\sigma_{yy}^{11} = 1 - i\frac{0.358\mu^2}{\omega} + \dots$$

For $\omega \to 0$ $\mathrm{Im}(\sigma_{\mathrm{yy}}^{11})$ has a pole with negative residue!

- CONCLUSION:
From the Kramers-Kronig relations (which connect real and imaginary part of a holomorphic function) follows that the real part of σ_{yy}^{11} must have a negative delta function contribution at $\omega = 0$.
- This "wrong" sign seems to be special in $d = 4$!

Numerical Calculations

The numerical results for the normal state agree with the analytical calculations:



Imaginary and real part of $G_{yy}^{R,11}$ as a function of ω for $\mu = 0.01$

Numerics

- The conductivities in the superconducting state can only be calculated numerically.
- The same problem concerning the sign of the residue appears!
- We can calculate the dependence of the conductivity on the frequency ω in both states.
- We can examine the μ dependence of the residue in the normal state.
- Calculation of the residue for different temperatures in the superconducting state is possible.

Conclusion and Outlook

- At the moment we need to understand what the negative residue actually means.
- We trust in the numerics since the results agree with the analytical calculations, so what is the correct interpretation of the negative sign in the conductivity? Does it indicate an instability?
- A solution to this problem might provide a better understanding of non-Abelian gauge fields in gauge/gravity duality.