Toric Geometry in String Theory

(DIY Calabi-Yau Tool Kit)

Thorsten Rahn

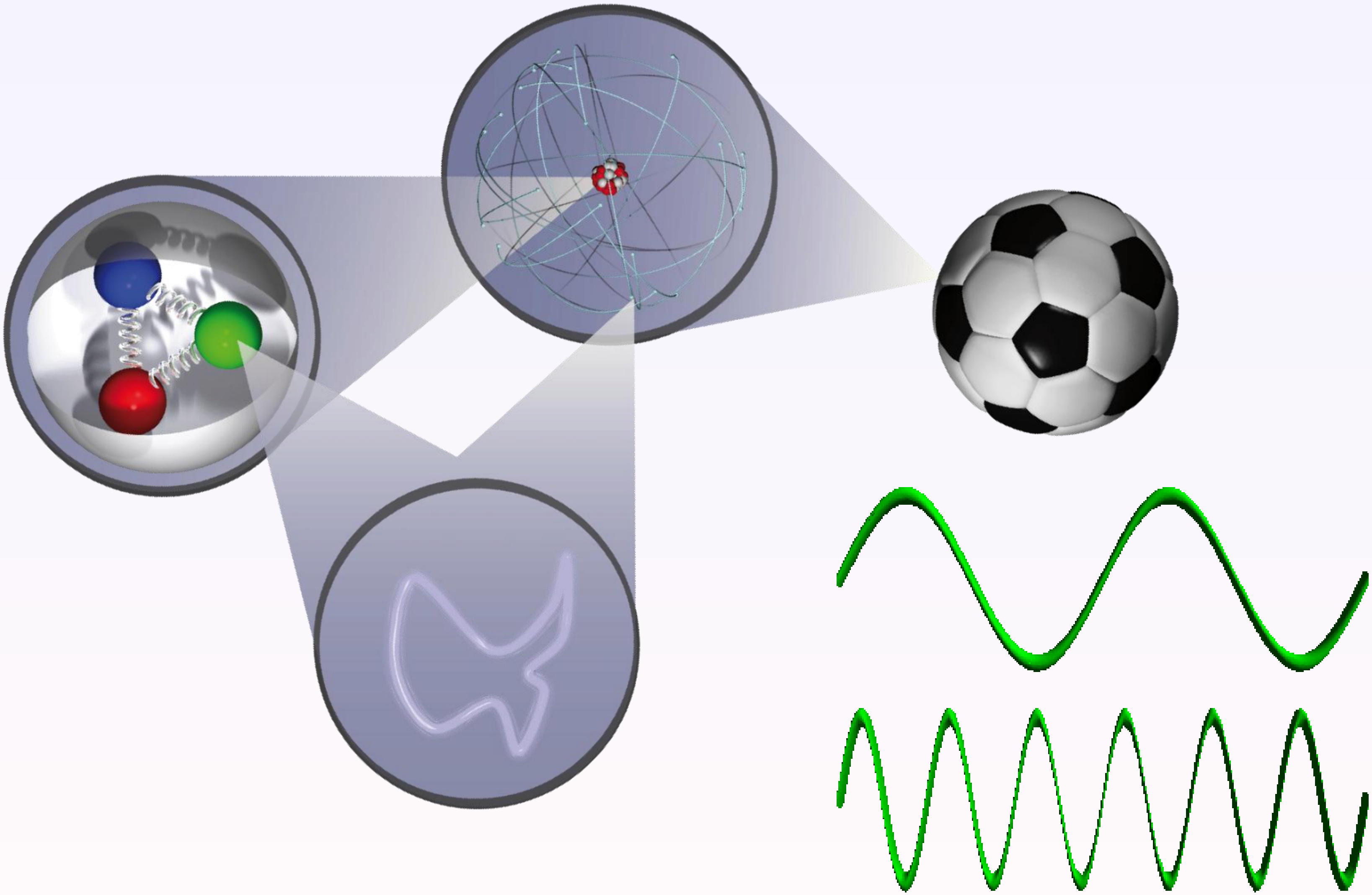
Max-Planck-Institut für Physik
(Werner-Heisenberg-Institut)

Munich — February 11th, 2011

Table of contents

1. Calabi-Yau Spaces in String Theory
2. Projective Spaces
3. Calabi-Yaus in Projective Spaces
4. Calabi-Yaus in Toric Varieties
5. Summary

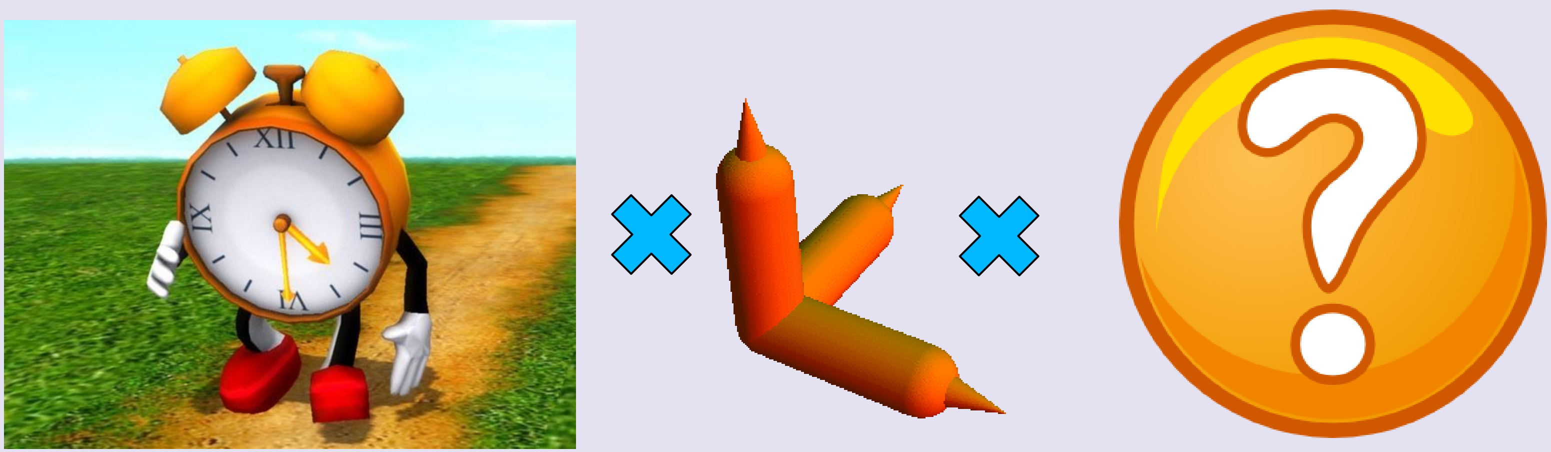
Unification of Particles and Forces with Strings



Consistency of Superstrings

Prediction of dimension

- Superstring theories are theories with world sheet (hence space time) super symmetry
- In Order to maintain consistency of the superstring, $D = 10$ is required



- Choose for example $X = \mathbb{R}^{1,3} \times M_6$

Shape of Extra Dimensions

= compactification space

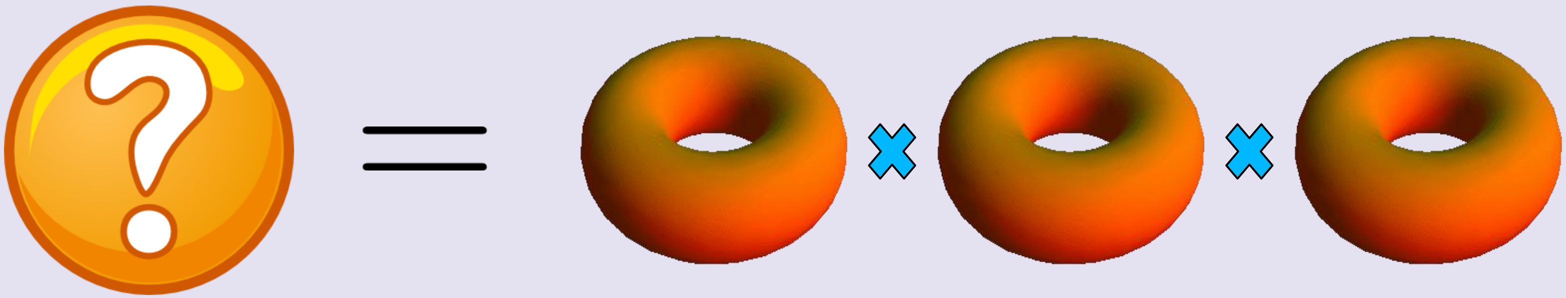
How does the compactification space look like?

- The heterotic super string has $\mathcal{N} = 1$ in 10 dimensions
- Depending on the compactification space some of them survive some don't
- For a realistic model we want $\mathcal{N} = 1$ in 4 dimensions
- Thus only $\frac{1}{4}$ of the original super symmetry survives

How to Determine the right Compactification Space?

Two examples

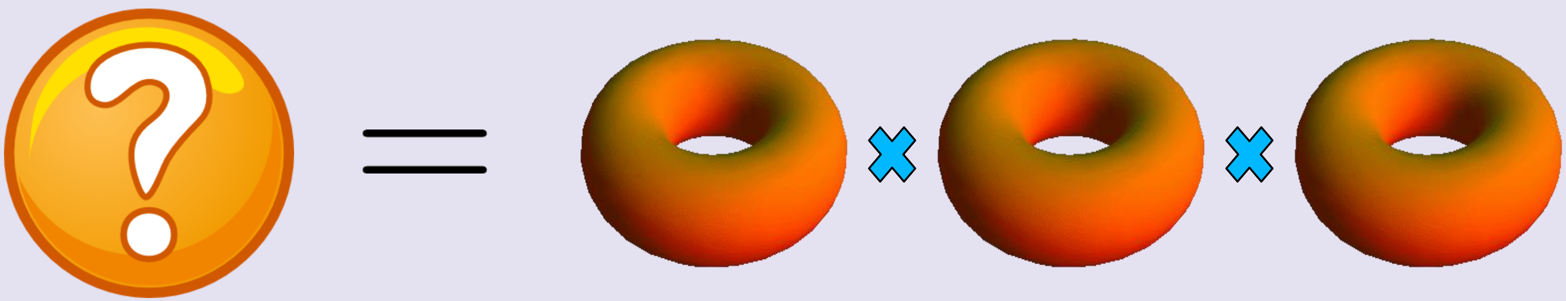
- Toroidal compactification



How to Determine the right Compactification Space?

Two examples

- Toroidal compactification



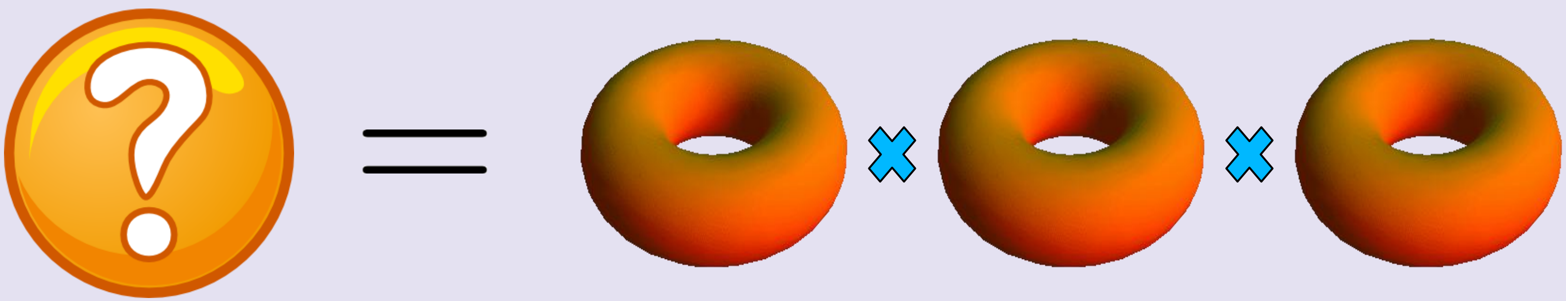
$\Rightarrow \mathcal{N} = 4$ in 4 dimensions 

- Calabi-Yau compactification

How to Determine the right Compactification Space?

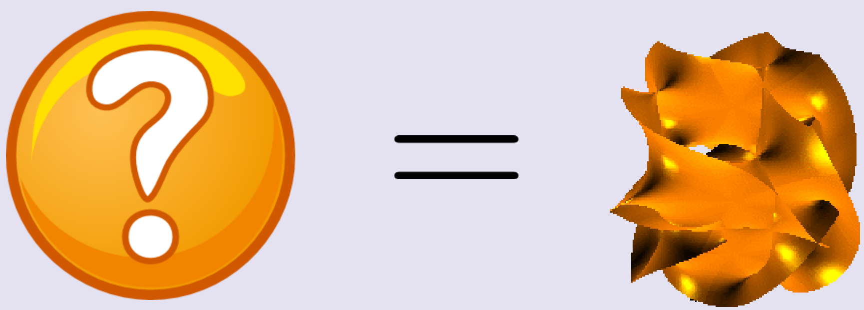
Two examples

- Toroidal compactification



$\Rightarrow \mathcal{N} = 4$ in 4 dimensions ✗

- Calabi-Yau compactification

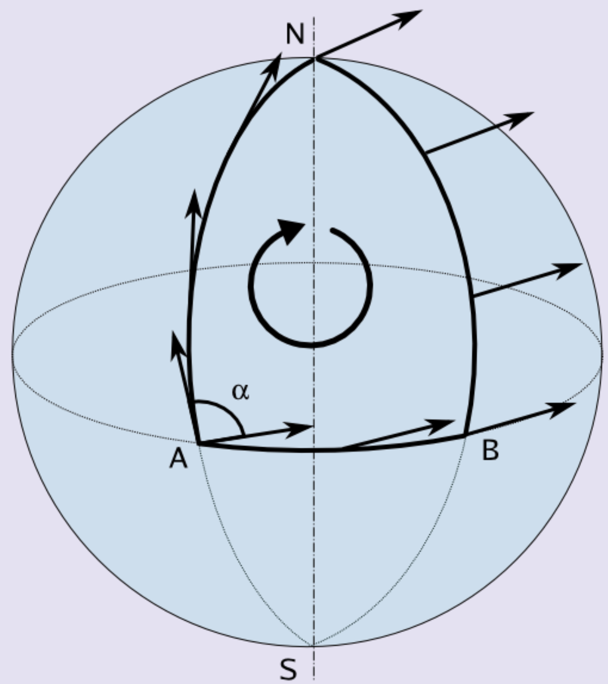


$\Rightarrow \mathcal{N} = 1$ in 4 dimensions ✓

What is a Calabi-Yau?

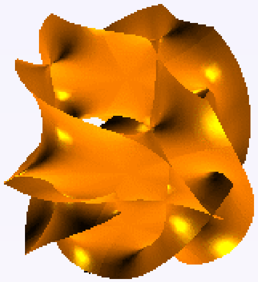
Holonomy

- The holonomy group of a space the group transformations that can be performed on a tangent vector via parallel transport



- For the 2-sphere the holonomy group is simpy $SO(2, \mathbb{R})$
- A space M **is called Calabi-Yau** if its holonomy group is $SU(n)$, $n = dimM$.

Calabi-Yau it is



- In fact the $SU(3)$ holonomy is necessary to break the correct amount of SUSY in this context
- Hence we need a way to construct Calabi-Yau Manifolds in string theory
- How?

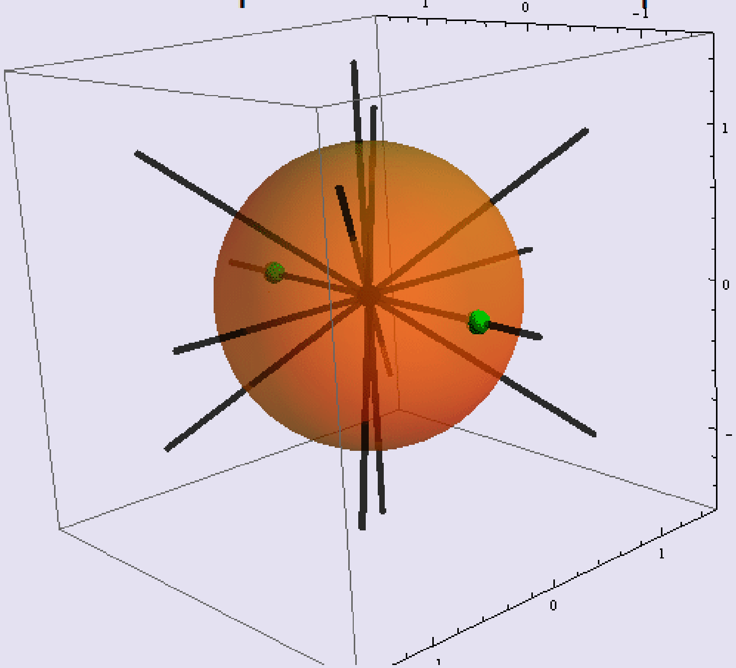
Projective Spaces

1. Calabi-Yau Spaces in String Theory
2. Projective Spaces
3. Calabi-Yaus in Projective Spaces
4. Calabi-Yaus in Toric Varieties
5. Summary

What is a projective space?

The real projective space $\mathbb{RP}^n$

- $\mathbb{RP}^n$ is the space consisting of all straight lines in $\mathbb{R}^{n+1}$ through the origin
- Consider for instance $\mathbb{RP}^2$:
  It is given by the unit sphere where antipodal points are identified

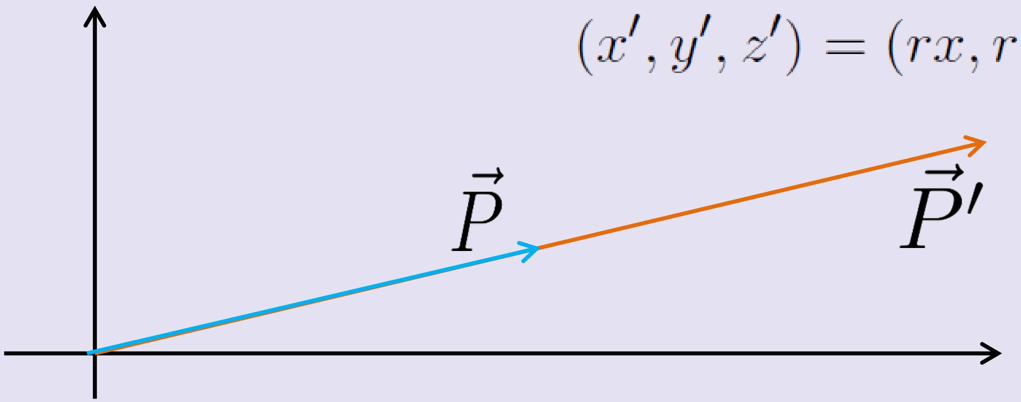


Definition of $\mathbb{RP}^n$

Points on a line are identified

- As mentioned, elements of the projective space $\mathbb{RP}^2$ are straight lines through the origin of $\mathbb{R}^3$
- Therefore a point $P = (x, y, z) \in \mathbb{R}^3$ is identified with a point $P' = (x', y', z') \in \mathbb{R}^3$ iff $\exists$ a constant $r \in \mathbb{R}^* = \mathbb{R} - \{0\}$, such that

$$(x', y', z') = (rx, ry, rz)$$



- We say then $(x', y', z') \sim_{\mathbb{R}^*} (x, y, z)$. Notice: The point $(0, 0, 0)$ is NOT an element of $\mathbb{RP}^2$
- Hence $\mathbb{RP}^2 = \frac{\mathbb{R}^3}{\sim_{\mathbb{R}^*}}$

From $\mathbb{C}^2$ to $\mathbb{CP}^2$

The same story applies to complex spaces

- Define $\mathbb{CP}^2$ as the space of all straight lines through the origin
- This time a point $u = (u_1, u_2, u_3) \in \mathbb{C}^3$ is identified with a point $u' = (u'_1, u'_2, u'_3)$ iff there is a constant $c \in \mathbb{C}^* = \mathbb{C} - \{0\}$, such that

$$(u'_1, u'_2, u'_3) = (c\, u_1, c\, u_2, c\, u_3)$$

- Notice: Since we deal with complex spaces, the exponent, denoting the complex dimension is twice the real dimension, e.g. dim$(\mathbb{C}^3) = 6$ and dim$(\mathbb{CP}^2) = 4$. Again as before, e.g. $\mathbb{CP}^2 = \frac{\mathbb{C}^3}{\sim_{\mathbb{C}^*}}$

Calabi-Yaus in Projective Spaces

1. Calabi-Yau Spaces in String Theory
2. Projective Spaces
3. Calabi-Yaus in Projective Spaces
4. Calabi-Yaus in Toric Varieties
5. Summary

Algebraic varieties in $\mathbb{R}^3$

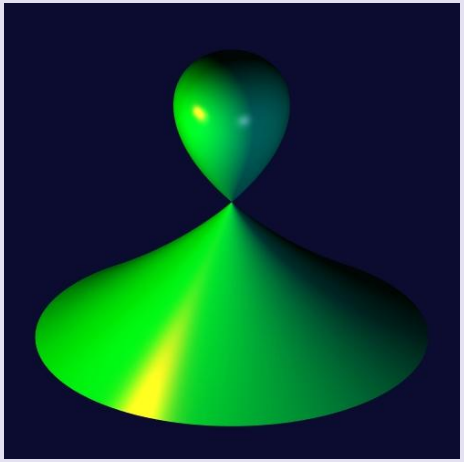
What is an algebraic variety V of $\mathbb{R}^3$?

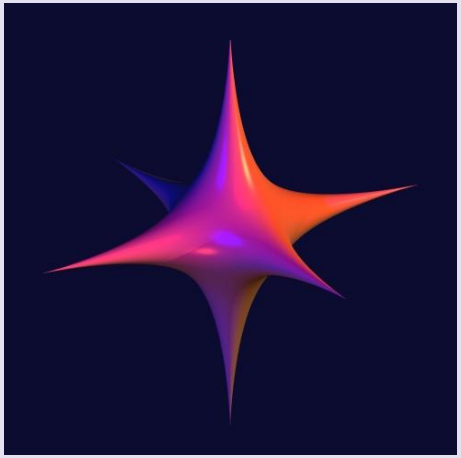
- V in $\mathbb{R}^3$ is the zero set of a function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$
- Formally $V := \{(x,y,z) \in \mathbb{R}^3$ such that $f(x,y,z) = 0\}$
- Unless $f \equiv 0$, $\dim(V) = \dim(\mathbb{R}^3) - 1 = 2$.
- Choose for instance $f(x,y,z) =$ $x^2y^2 + y^2z^2 + x^2z^2 + 100$

$(x^2 + \frac{9}{4}y^2 + z^2 - 1)^3 - x^2z^3 - \frac{9}{80}y^2z^3$

$x^2 + y^2 - z^2 + z^3$

$(x^2 + y^2 + z^2 - 1)^3$





Algebraic varieties in $\mathbb{CP}^4$

One can of course also define an algebraic variety in $\mathbb{CP}^4$ (lack of imagination)

- $\mathbb{CP}^4 = \frac{\mathbb{C}^5}{\sim_{\mathbb{C}^*}}$, hence an algebraic variety Q inside $\mathbb{CP}^4$ can be defined by function $G : \mathbb{CP}^4 \longrightarrow \mathbb{C}$:

$$Q := \{(u_1, u_2, u_3, u_4, u_5) \in \mathbb{CP}^4 : \; G(u_1, u_2, u_3, u_4, u_5) = 0\}$$

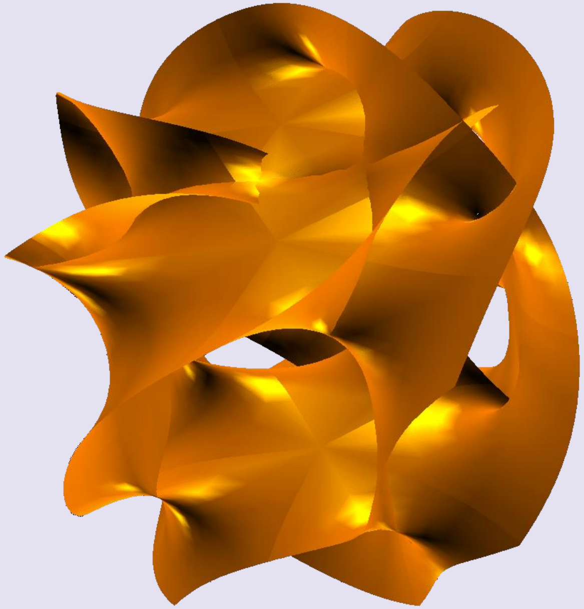
- One can prove that a variety in $\mathbb{CP}^4$ is Calabi-Yau, iff it is a polynomial, homogeneous of degree 5:

$$Q(u_1, u_2, u_3, u_4, u_5) := c_1 \; u_1^5 + c_2 \; u_1^4 u_2 + c_3 \; u_1^3 u_2^2 + ... \\ + u_1 u_2 u_3 u_4 u_5 + ... + c_{124} \; u_4^2 u_5^3 + c_{125} \; u_4^1 u_5^4 + c_{126} \; u_5^5$$

For arbitrary coefficients $c_1, ..., c_{126} \in \mathbb{C}$.

The Quintic and the $(n+1)$-tic

- The variety Q, since it is a degree 5 polynomial is also referred to as "'the Quintic"' in $\mathbb{CP}^4$ and denoted as $Q =: \mathbb{CP}^4\,[5]$ in order to indicate the degree of the homogeneous polynomial.
- Taking a certain two dimensional section of the Quintic, we obtain the following plot:



- In general it holds: A homogeneous polynomial of degree $n+1$ in $\mathbb{CP}^n$, namely $\mathbb{CP}^n[n+1]$ is always Calabi-Yau

Calabi-Yaus in Toric Varieties

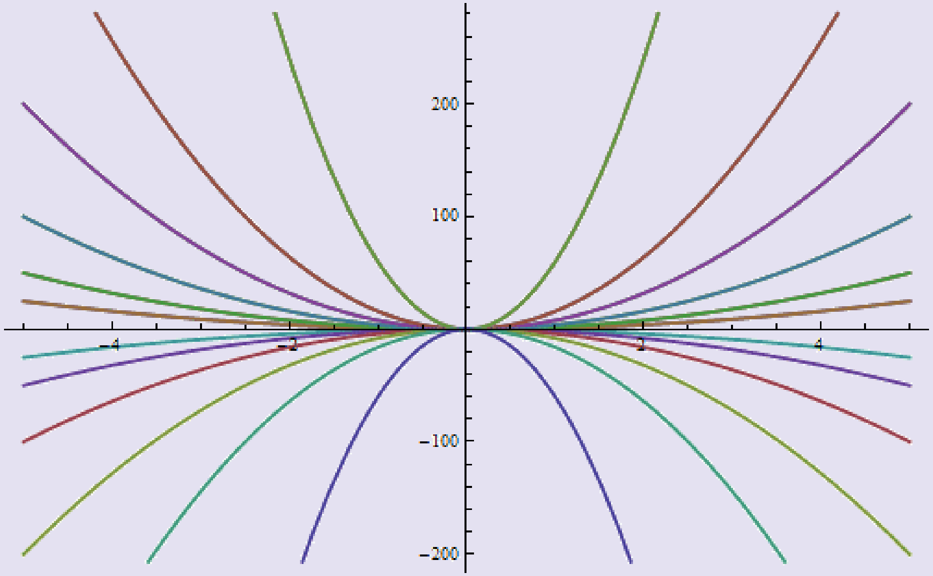
1. Calabi-Yau Spaces in String Theory
2. Projective Spaces
3. Calabi-Yaus in Projective Spaces
4. Calabi-Yaus in Toric Varieties
5. Summary

Weighted Projective Spaces

Curves instead of straight lines in $\mathbb{RP}^1$

- We introduced $\mathbb{RP}^1$ to be the set of straight lines through the origin, e.g. $(1,1) \sim_{\mathbb{R}^*} (r,r)$.
- This can be **generalized** in a straight forward way. For instance **identify parabola** of $\mathbb{R}^2$, namely

$$(x,y) \sim_{\mathbb{R}^*} (rx, r^2 y) \text{ for example: } (1,1) \sim_{\mathbb{R}^*} (r, r^2)$$



Weighted Projective Spaces

Curves instead of straight lines in $\mathbb{RP}^1$

$$(x, y) \sim_{\mathbb{R}^*} (rx, r^2 y) \text{ for example: } (1, 1) \sim_{\mathbb{R}^*} (r, r^2)$$

- Such a space we call a **weighted projective space**, where the **weights** are defined by the powers of r
- Our example here has weights 1 and 2 and is denoted by $\mathbb{RP}_{1,2}$
- Hence a projective space can then be written as

$$\mathbb{RP}^n = \mathbb{RP}_{\underbrace{1,1,...,1}_{(n+1)-\text{times}}}$$

Weighted Projective Spaces

The complex case:

- The same thing can be done to define a **complex weighted projective space**.
- For instance choose $\mathbb{CP}_{1,1,1,1,2}$. It is again $\mathbb{C}^5 - \{0\}$ where points are identified via:

$$\{(u_1, u_2, u_3, u_4, u_5) \sim_{\mathbb{C}^*} (c^1 u_1, c^1 u_2, c^1 u_3, c^1 u_4, c^2 u_5)\} \ , \ c \neq 0$$

- We now say that $u_1, ..., u_4$ have degree 1 and u_5 has degree 2
- Using this definition of the degree of a polynomial in $(u_1, ..., u_5)$ one can show that **every polynomial** $G(u_1, ..., u_5)$ **homogeneous of degree 6** in this space is Calabi-Yau
- For instance, terms in G may be u_1^6, $u_1^5 u_3$, u_5^3, $u_2^2 u_5^2$

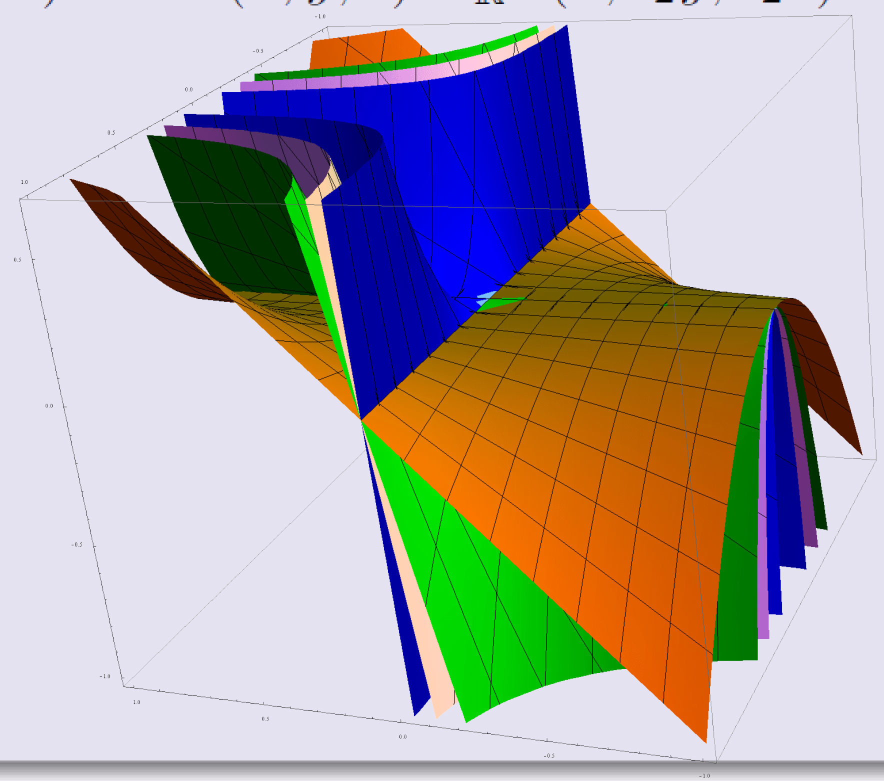
Toric Varieties

One further generalization

- Now **introduce more identifications** in $\mathbb{R}^3$. For instance let $(x, y, z) \in \mathbb{R}^3 - \{"0"\}$. Then define two relations:

$$(x, y, z) \sim_{\mathbb{R}^*} (r_1 x, r_1^2 y, z) \quad \text{and} \quad (x, y, z) \sim_{\mathbb{R}^*} (x, r_2 y, r_2 z)$$

$$(1, 1, 1) = (r_1, r_1^2 r_2, r_2)$$



Toric Varieties

Same story for the complex case

- **Introduce more identifications** in $\mathbb{C}^n$. For instance let $(u_1, ..., u_6) \in \mathbb{C}^6 - \{0\}$. Then define two relations:

$$(u_1, ..., u_6) \sim_{\mathbb{C}^*} (c^1 u_1, c^1 u_2, c^2 u_3, c^2 u_4, c^2 u_5, c^0 u_6) \quad \text{and}$$
$$(u_1, ..., u_6) \sim_{\mathbb{C}^*} (c^0 u_1, c^0 u_2, c^0 u_3, c^1 u_4, c^1 u_5, c^1 u_6)$$

- Here now every coordinate u_1 has not one degree but two. u_1 for instance has **multidegree** $\binom{1}{0}$ while u_5 has multidegree $\binom{2}{1}$
- Such a space is called a **toric variety**. Using the notation above, you may write it as:

$$\mathbb{CP}_{\begin{pmatrix} 1 & 1 & 2 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}}$$

Toric Varieties

Calabi-Yau hypersurface in toric varieties

- Due to the size of the matrix of degrees one often does not write the $\mathbb{CP}$ in front of it an simply specifies the space X by:

$$X = \begin{pmatrix} 1 & 1 & 2 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

- Now the **Calabi-Yau condition** on the degrees of a homogeneous polynomial G **applies to every single line** of the matrix above. This means that the variety

$$V := \{(u_1, ..., u_6) \in X \text{ such that } G(u_1, u_2, u_3, u_4, u_5, u_6) = 0\}$$

is Calabi-Yau iff G is a polynomial of multidegree $\binom{8}{3}$

Summary

What have we learnt?

Summary

What have we learnt?

- Many scenarios need Calabi-Yaus for compactification 10 dim $\rightarrow$ 4 dim

Summary

What have we learnt?

- Many scenarios need Calabi-Yaus for compactification 10 dim $\rightarrow$ 4 dim
- We can construct Calabi-Yaus in complex projective spaces $\mathbb{CP}^n$

Summary

What have we learnt?

- Many scenarios need Calabi-Yaus for compactification 10 dim $\rightarrow$ 4 dim
- We can construct Calabi-Yaus in complex projective spaces $\mathbb{CP}^n$
- Complex projective spaces $\rightarrow$ weighted projective spaces by changing the equivalence relations of points in $\mathbb{C}^n$ $\rightarrow$ new degrees to for coordinates.

Summary

What have we learnt?

- Many scenarios need Calabi-Yaus for compactification 10 dim $\rightarrow$ 4 dim
- We can construct Calabi-Yaus in complex projective spaces $\mathbb{CP}^n$
- Complex projective spaces $\rightarrow$ weighted projective spaces by changing the equivalence relations of points in $\mathbb{C}^n$ $\rightarrow$ new degrees to for coordinates.
- Toric varieties arise by introducing more equivalence relation a$\rightarrow$ multidegrees for coordinates

Summary

What have we learnt?

- Many scenarios need Calabi-Yaus for compactification 10 dim $\rightarrow$ 4 dim
- We can construct Calabi-Yaus in complex projective spaces $\mathbb{CP}^n$
- Complex projective spaces $\rightarrow$ weighted projective spaces by changing the equivalence relations of points in $\mathbb{C}^n$ $\rightarrow$ new degrees to for coordinates.
- Toric varieties arise by introducing more equivalence relation a$\rightarrow$ multidegrees for coordinates
- Calabi-Yau spaces can be obtained as the zero set of a homogeneous polynomial in a (wheighted) projective space/toric variety that has the same (multi)degree as the sum of (multi)degrees of all coordinates $\Rightarrow$ **Calabi-Yau condition**

Thank you!