Global Symmetries in Axion Electrodynamics

IMPRS Recruiting Workshop

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Axion Electrodynamics

Consider a periodic scalar field coupled to $U(1)$ gauge field

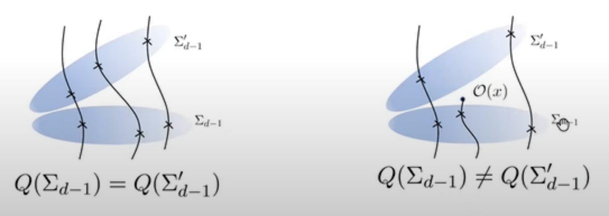
$$\mathcal{L}_{Ax} = \frac{1}{2} da \wedge \star da + \frac{1}{2g^2} F \wedge \star F - \frac{k}{8\pi^2 f} aF \wedge F \tag{1}$$

Axion field satisfies $a \sim a + 2\pi f$.
Axion famously has a $U(1)$ shift symmetry that is by the instantons of gauge theory broken down to $\mathbb{Z}_k$.

Our goal today is to understand symmetries of this model and to see that shift symmetry is broken down to $\mathbb{Q}/\mathbb{Z}$ noninvertible symmetry.

What is a Symmetry in Quantum Field Theory?

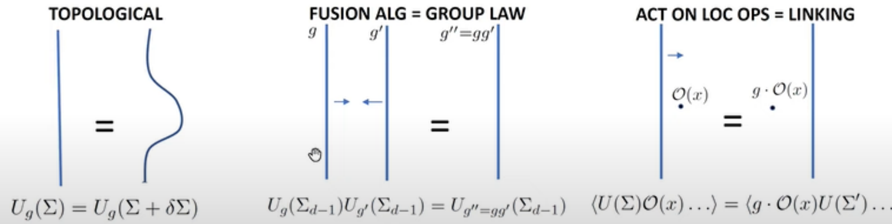
Prototipical example: $U(1)$ global symmetry
Noether theorem:
Conserved current $d \star j = 0 \rightarrow$ Conserved charge $\mathrm{Q}(\Sigma_{d-1}) = \int_{\Sigma_{d-1}} \star j$
Relativity: $[\mathrm{Q}, T^{\mu\nu}] = 0 \implies$ does not care about orientation.



What is a Symmetry in Quantum Field Theory

Modern viewpoint:
Symmetry is generated by *Topological Operator*

$$U_{g=e^{i\alpha}}(\Sigma_{d-1}) = e^{i\alpha \int_{\Sigma_{d-1}} \star j} \tag{2}$$

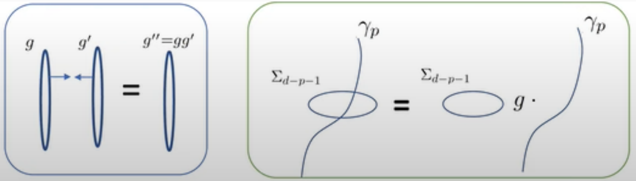


Easy to generalize: Relax some of these properties!

p-form Symmetries

Generator of symmetry lives on a codimension-(p+1) submanifold.

$U(1)$ p-form symmetry: $d \star J^{p+1} = 0$ $U_{g=e^{i\alpha}}(\Sigma_{d-p-1}) = e^{i\alpha \int_{\Sigma_{d-p-1}} \star j^{p+1}}$



General properties:

Acts by linking on operators supported on p-dimensional

Has to be Abelian

Can be gauged, leading to p-form gauge theory

Can be spontaneously broken

Can be anomalous

p-form gauge theory

Consider a pure $U(1)$ p-form gauge field in d dimensions

$$S = \frac{1}{2}\int_{\mathcal{M}_d} F^{(p+1)} \wedge \star F^{(p+1)} \tag{3}$$

Where $F^{(p+1)} = dA^{(p)}$ for $A^{(p)}$ p-form gauge field (For p=0 axion).

It has two conserved currents:
EOM: $d \star F^{(p+1)} = 0 \rightarrow J_e^{p+1} = F^{(p+1)}$ Electric p+1-form symmetry
Bianchi Identity: $dF^{(p+1)} = 0 \rightarrow J_m^{d-p-1} = \frac{1}{2\pi} \star F^{(d-p-1)}$ Magnetic d-p-1-form symmetry.

Charged operators under these symmetries are Willson and 't Hooft surfaces.
$W(\Sigma_p) = e^{i\int_{\Sigma_p} A^{(p)}}$
$H(\Sigma_{d-p-2}) = e^{i\int_{\Sigma_{d-p-2}} \tilde{A}^{(d-p-2)}}$

Symmetries of Axion Electrodynamics

Consider axion electrodynamics without interaction:

$$\mathcal{L} = \frac{1}{2} da \wedge \star da + \frac{1}{2g^2} F \wedge \star F \tag{4}$$

It has four conserved currents

$$\star j^1 = f \star da \quad \star j_e^2 = \frac{1}{g^2} \star F \quad \star j_m^2 = \frac{1}{2\pi} F \quad \star j^3 = \frac{1}{2\pi f} da \tag{5}$$

So the complete symmetry of this system is

$$U(1)^{(0)} \times U(1)_e^{(1)} \times U(1)_m^{(1)} \times U(1)^{(2)} \tag{6}$$

Symmetries of Axion Electrodynamics

With the interaction term $L_{int} = -\frac{k}{8\pi^2 f} aF \wedge F$
equations of motion give

$$d \star j^1 = \frac{k}{8\pi^2} F \wedge F \quad d \star j_e^2 = -\frac{k}{4\pi^2 f} d(aF) \tag{7}$$

$\implies U^{(0)}_{g=e^{i\alpha}}(\Sigma_3) = e^{i\alpha \int_{\Sigma_3} \star j^1}$ and $U^{(1),e}_{g=e^{i\alpha}}(\Sigma_2) = e^{i\alpha \int_{\Sigma_2} \star j_e^2}$ are not topological operators.
Consider modifed operators:

$$\tilde{U}^{(0)}_{g=e^{i\alpha}}(\Sigma_3) = e^{i\alpha \int_{\Sigma_3} (\star j^1 - \frac{k}{8\pi^2} A \wedge F)} \quad \tilde{U}^{(1,e)}_{g=e^{i\alpha}}(\Sigma_3) = e^{i\alpha \int_{\Sigma_3} (\star j_e^2 + \frac{k}{4\pi^2 f} aF)} \tag{8}$$

These are now topological operators but, are not well defined, except for $\alpha \in \mathbb{Z}_k$. $\implies$ symmetry broken down to

$$\mathbb{Z}_k^{(0)} \times \mathbb{Z}_k^{(1,e)} \times U(1)_m^{(1)} \times U(1)^{(2)} \tag{9}$$

For $k = 1$ shift and electric 1-form symmetries completely broken.

More about Shift Symmetry

Can we modify $U(1)_g^{(0)}(\Sigma_3)$ to obtain a well defined topological operator?
Consider a new current defined as

$$\star\tilde{j}^1 = \star j^1 - \frac{1}{8\pi^2} A \wedge dA \tag{10}$$

This satisfies the conservation equation $d\star\tilde{j}^1 = 0$, but it is not a gauge invariant operator.
Interestingly, we can find a modification for $\alpha \in 2\pi\mathbb{Q}$. Consider, first $\alpha = 2\pi/N$. The gauge non-invariant term is $-\frac{i}{4\pi N}\oint_{\Sigma_3} A \wedge dA$. There is a trick to make this gauge invariant: A gauge invariant action for the fractional quantum Hall state is

$$i\oint_{\Sigma_3}\left(\frac{N}{4\pi} a \wedge da + \frac{1}{2\pi} a \wedge dA\right) \tag{11}$$

Where $a^{(1)}$ is a dynamical $U(1)$ gauge field on Σ_3. It is a $U(1)_N$ Chern-Simons theory coupled to dA.

Noninvertible shift symmetry

We define a new operator in the theory:

$$\mathcal{D}_{\frac{1}{N}}(\Sigma_3) = \int Da^{(1)}|_{\Sigma_3} exp(i \oint_{\Sigma_3} (\frac{2\pi}{N} \star j^1 + \frac{N}{4\pi} a^{(1)} \wedge da^{(1)} + \frac{1}{2\pi} a^{(1)} \wedge dA)) \tag{12}$$

This operator acts as a shift symmetry on axion field for $2\pi/N$. It is gauge invariant and topological.
As a well defined topological operator, it should be wieved as a generalized global symmetry. Interestingly, it is not a usual group-like symmetry. That is, this operator doesn't follow the group multiplication law under parallel fusions. In particular, $\mathcal{D}_{\frac{1}{N}}$ is not a unitary operator and it does not have an inverse operator $(\mathcal{D}_{\frac{1}{N}})^{-1}$.
For this reason, $\mathcal{D}_{\frac{1}{N}}$ is a non-invertible symmetry.

Noninvertible shift symmetry

We can generilize this construction to any rational angle $\alpha = \frac{2\pi p}{N}$.
The new topological operator $\mathcal{D}_{\frac{p}{N}}$ associated with the shift for $2\pi p/N$ is defined as

$$\mathcal{D}_{\frac{p}{N}}(\Sigma_3) = exp(i \oint_{\Sigma_3} (\frac{2\pi i p}{N} \star j^A + \mathcal{A}^{N,p}[dA/N]) \tag{13}$$

where $\mathcal{A}^{N,p}[B^{(2)}]$ is a TQFT with a $\mathbb{Z}_N^{(1)}$ one-form symmetry with its t'Hooft anomaly labeled by p and coupled to a $\mathbb{Z}_N^{(1)}$ background two-form gauge field $B^{(2)}$.
An example of a fusion rule is:

$$\mathcal{D}_{\frac{1}{N}} \times \mathcal{D}_{\frac{1}{N}} = \mathcal{A}^{N,2} \mathcal{D}_{\frac{2}{N}} \tag{14}$$

So, we see that fusion coefficients are actually not ordinary numbers but TQFTs.

More about electric 1-form symmetry

Now we use the similar idea and try to modify $U(1)_g^{(1),e}(\Sigma_2)$ to obtain a well defined topological operator. Start with a new current

$$\star\tilde{j}_e^2 = \star j_e^2 + \frac{1}{4\pi^2 f} aF \tag{15}$$

The second term is again not well defined operator when multiplied with arbitrary phase.
To cure this for $\alpha = \frac{2\pi}{N}$ consider a well defined action:

$$i \oint_{\Sigma_2} \left(\frac{N}{2\pi} c^{(1)} \wedge d\phi^{(0)} - \frac{1}{2\pi}\phi^{(0)} F^{(2)} - \frac{1}{2\pi f} c^{(1)} \wedge da^{(0)}\right) \tag{16}$$

where $c^{(1)}$ and $\phi^{(0)}$ are dynamical $U(1)$ gauge field and periodic scalar on Σ_2. It is a $2d - \mathbb{Z}_N$ gauge theory coupled to $da^{(0)}$ and $dA^{(1)}$.

Noninvertible 1-form Symmetry

Now we define a new topological operator in the theory:

$$\mathcal{C}_{\frac{1}{N}}(\Sigma_2) = \int D\phi^{(0)} Dc^{(1)}|_{\Sigma_2} expi \oint_{\Sigma_2} (\frac{2\pi}{N} \star j_e^2 + \frac{N}{2\pi} c^{(1)} \wedge d\phi^{(0)} - \frac{1}{2\pi}\phi^{(0)} F^{(2)} - \frac{1}{2\pi f} c^{(1)} \wedge da^{(0)} \tag{17}$$

This could be generalized for any rational angle $\alpha = \frac{2\pi p}{N}$. The new topological operator $\mathcal{C}_{\frac{p}{N}}$ implementing 1-form electrical symmetry is defined as:

$$\mathcal{C}_{\frac{p}{N}}(\Sigma_2) = expi \oint_{\Sigma_2} (\frac{2\pi}{N} \star j_e^2 + \mathcal{B}^{N,p}[da/N, dA/N]) \tag{18}$$

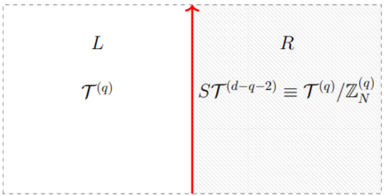
where $\mathcal{B}^{N,p}[B^{(1)}, C^{(2)}]$ is a 2d TQFT with a $\mathbb{Z}_N^{(0)} \times \mathbb{Z}_N^{(1)}$ symmetry with mixed t'Hooft anomaly labeled by p and coupled to a $\mathbb{Z}_N^{(0)}$ background 1-form gauge field $B^{(1)}$ and a $\mathbb{Z}_N^{(1)}$ background 2-form gauge field $C^{(2)}$.

Alternative construction of a noninvertible shift symmetry

Our theory has a magnetic one-form global symmetry $U(1)_m^{(1)}$ whose Noether current is a two-form $j_m^2 = \frac{1}{2\pi} \star F$. The background gauge field is a 2-form field $B^{(2)}$ coupled by $iB^{(2)} \wedge \star j_m^2 = \frac{i}{2\pi} B^{(2)} \wedge F$.
If we want to gauge a discrete subgroup $\mathbb{Z}_N^{(1)}$ of $U(1)_m^{(1)}$ we can write a Lagrangian

$$\mathcal{L}_{Ax}[a, A] + \frac{i}{2\pi} b^{(2)} \wedge F + \frac{iN}{2\pi} b^{(2)} \wedge dc^{(1)} + \frac{iNk}{4\pi} b^{(2)} \wedge b^{(2)} \qquad (19)$$

Idea is now to gauge only in the half of space-time and modify a bit to end with the same theory with the noninvertibe defect in bewteen.



Alternative construction of a noninvertible shift symmetry

We gauge theory in half of space-time (i.e. $x \geq 0$). Next, we add to the total Lagrangian the following:

$$\frac{2\pi ip}{N}\oint_{x=0}\star j^1+\frac{ip}{4\pi N}\int_{x\geq 0}F\wedge F \tag{20}$$

which is trivial.
Composing the discrete gauging and the axion shift in the x ≥ 0 region, the total Lagrangian becomes

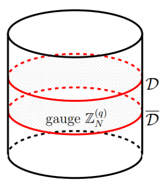
$$\int_{x<0}\mathcal{L}_{Ax}[a,A]+\frac{2\pi pi}{N}\oint_{x=0}\star j^1+$$

$$\int_{x\geq 0}(\mathcal{L}_{Ax}[a,A]+\tfrac{ip}{4\pi N}F\wedge F+\tfrac{i}{2\pi}b^{(2)}\wedge F+\tfrac{iN}{2\pi}b^{(2)}\wedge dc^{(1)}+\tfrac{iNk}{4\pi}b^{(2)}\wedge b^{(2)}) \tag{21}$$

Alternative construction of a noninvertible shift symmetry

The bulk Lagrangians in the x < 0 and x > 0 regions are both the original theory Lagrangian, but the discrete gauging leaves behind exactly the defect $\mathcal{D}_{\frac{p}{N}}$ at x = 0.

We can use this to compute fusion of defects to see that it is realy noninvertible:

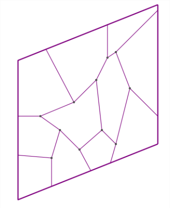


We can find that

$$\mathcal{D}_{\frac{1}{N}} \times \mathcal{D}^{\dagger}_{\frac{1}{N}} = \mathcal{C} \neq 1 \tag{22}$$

Higher gauging

The notion of gauging a global symmetry within lower dimensional submanifolds of spacetime is called higher gauging. Higher gauging of a discrete symmetry produces a noninvertible topological defect. Gauging a q-form symmetry on codimension-p closely resembles the gauging of a $(q-p)$-form symmetry.



Alternative construction of a noninvertible 1-form symmetry

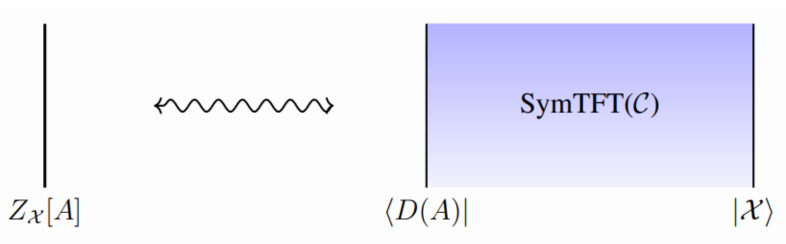
To construct topological defect on a codimension 2-surface Σ_2 idea is to extend it to a codimension one hypersurface Σ_3 s.t. $\Sigma_2 = \partial\Sigma_3$ and use higher gauging of discrete subgroups of $U(1)^{(2)}$ and $U(1)^{(1)}_m$.

We need to sum over 1-form $\mathbb{Z}_N^{(1)}$ background gauge field $b^{(1)}$, coupled to the magnetic symmetry, and a 2-form $\mathbb{Z}_N^{(2)}$ background gauge field $c^{(2)}$, coupled to the magnetic symmetry along Σ_3.
In total we have to add to the theory:

$$\int_{\Sigma_3} \frac{N}{2\pi} b^{(1)} \wedge de^{(1)} + \frac{N}{2\pi} c^{(2)} \wedge df^{(1)} + \frac{iNk}{4\pi} b^{(1)} \wedge c^{(2)} + \frac{i}{2\pi} b^{(1)} \wedge F^{(2)} + \frac{i}{2\pi} c^{(2)} \wedge da^{(0)} \tag{23}$$

What I woud like to understand

Symmetry TFT in d+1 dimensions describing symmetries of quantum field theory in d dimensions.



Can we capture these noninvertible symmetries of Axion Electrodynamics in this framework?

Some thoughts about Noninvertible symmetries

- Noninvertible topological operators are ubiquitous in 2d CFTs
- Can maybe give new selection rules, or naturalness
- Can be gauged sometimes
- Existence can give dynamical applications
- Implications for Swampland