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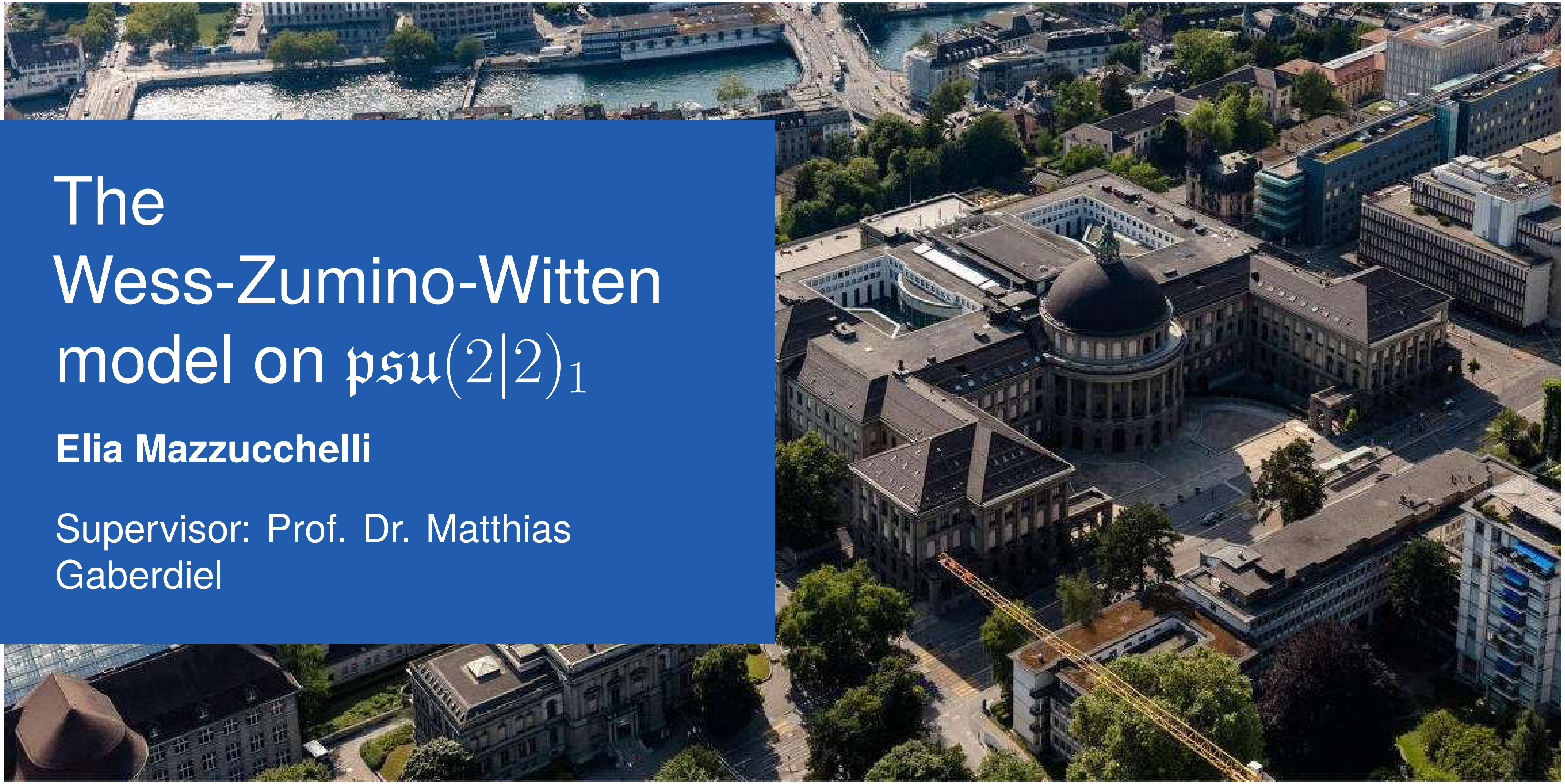

The Wess-Zumino-Witten model on $\mathfrak{psu}(2|2)_1$

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Outline

1. Collocation in the AdS/CFT correspondence
2. The affine Lie superalgebra $\mathfrak{su}(2|2)_k$
3. The free field realisation
4. The free field characters
5. Other results and overlook
6. Conformal embeddings

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Recent progress for $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$ and $\mathrm{AdS}_5 \times \mathrm{S}^5$

- Hybrid formalism of $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$ = WZW model on $\mathfrak{psu}(1,1|2)_k$, well-defined also in the tensionless limit $k = 1$, in contrast with the RNS formalism. The physical spectrum reproduces exactly that of the large N limit 2d CFT of the free symmetric product orbifold $\mathrm{Sym}^N(\mathbb{T}^4)$; this has been shown at the level of correlators [Lorenz Eberhardt, Matthias R. Gaberdiel, and Rajesh Gopakumar, 2018]
- Tensionless limit worldsheet description of $\mathrm{AdS}_5 \times \mathrm{S}^5$ = sigma model on $\mathfrak{psu}(2,2|4)_1$. Imposing a set of residual gauge constraints determines the physical spectrum of the string theory. Remarkably, there is evidence that this prescription reproduces precisely the entire planar spectrum of single trace operators of the free $\mathcal{N} = 4$ SYM theory [Matthias R. Gaberdiel and Rajesh Gopakumar, 2021].

But...

For what concerns the proposal for $\mathrm{AdS}_5 \times \mathrm{S}^5$:

- The gauge constraints seem “unnatural” and a major part of the field content of the $\mathfrak{psu}(2,2|4)$ model needs to be gauged out.
- An actual WZW model on $\mathfrak{psu}(2,2|4)$ yields a spectrum generating algebra consisting of two copies of this superalgebra, while the symmetry algebra of $\mathcal{N} = 4$ SYM consists only of one copy.
- In the integrability approach, the $\mathfrak{psu}(2,2|4)$ symmetry of $\mathcal{N} = 4$ SYM is broken down to $\mathfrak{psu}(2|2) \oplus \mathfrak{psu}(2|2)$, and correlators are more naturally formulated in terms of bilinears of $\mathfrak{su}(2|2)$ bits.

So maybe the right world sheet description of $\mathrm{AdS}_5 \times \mathrm{S}^5$ is in terms of a WZW model on $\mathfrak{su}(2|2)$?

Outline

Matrix representation of the finite superalgebra

The elements of the superalgebra $\mathfrak{u}(2|2)$ can be represented as $2|2$-block matrices

$$\left(\begin{array}{c|c} \mathfrak{u}(2) & S^+ \\ \hline S^- & \mathfrak{u}(2) \end{array}\right) .$$

There are two distinct $\mathfrak{u}(1)$ generators

$$Z = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & I \end{array}\right) , \qquad Y = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & -I \end{array}\right) .$$

The superalgebra $\mathfrak{su}(2|2)$ is obtained by leaving out Y, and $\mathfrak{psu}(2|2)$ by additionally quotienting out Z.

The eight fermions $S^\pm = S^{\alpha\beta\pm}$ for $\alpha, \beta = 1, 2$ transform in the bosonic representation

$$(\mathbf{2}, \overline{\mathbf{2}}) \oplus (\overline{\mathbf{2}}, \mathbf{2}) .$$

Presence of non-unitary representations of $\mathfrak{su}(2)$

The affine Lie superalgebra $\mathfrak{su}(2|2)_k$ has bosonic subalgebra

$$\mathfrak{su}(2)_{-k} \oplus \mathfrak{su}(2)_k \oplus \mathfrak{u}(1)_0 \,.$$

The allowed representations for $k \in \mathbb{Z}_{>0}$ are:

- "compact" $\mathfrak{su}(2)_k$: integrable unitary affine representations of spin $0 \leq \ell \leq k/2$.
- "non-compact" $\mathfrak{su}(2)_{-k}$: only non-unitary affine modules; must consider also discrete highest/lowest weight $D_j^{\pm}$ and continuous C_λ^j representations. Actually

$$\mathfrak{su}(2)_{-k} \cong \mathfrak{sl}(2,\mathbb{R})_k \,.$$

Multiplets of $\mathfrak{su}(2|2)$

The eight fermionic generators form a 16-dimensional Clifford module. A general *long* $\mathfrak{su}(2|2)$-multiplet takes the form

$$
\begin{array}{c}
\big(j\,,\mathbf{n}\big) \\
\big(j+\tfrac{1}{2}\,,\mathbf{n}+\mathbf{1}\big)\quad \big(j+\tfrac{1}{2}\,,\mathbf{n}-\mathbf{1}\big)\quad \big(j-\tfrac{1}{2}\,,\mathbf{n}+\mathbf{1}\big)\quad \big(j-\tfrac{1}{2}\,,\mathbf{n}-\mathbf{1}\big) \\
\big(j+1\,,\mathbf{n}\big)\qquad \big(j\,,\mathbf{n}+\mathbf{2}\big)\qquad 2\,\big(j\,,\mathbf{n}\big)\qquad \big(j\,,\mathbf{n}-\mathbf{2}\big)\qquad \big(j-1\,,\mathbf{n}\big) \\
\big(j+\tfrac{1}{2}\,,\mathbf{n}+\mathbf{1}\big)\quad \big(j+\tfrac{1}{2}\,,\mathbf{n}-\mathbf{1}\big)\quad \big(j-\tfrac{1}{2}\,,\mathbf{n}+\mathbf{1}\big)\quad \big(j-\tfrac{1}{2}\,,\mathbf{n}-\mathbf{1}\big) \\
\big(j\,,\mathbf{n}\big)\,,
\end{array}
$$

where j may label also non-unitary $\mathfrak{su}(2)$-representations seen above.

- As an $\mathfrak{su}(2|2)$-repr: additionally characterised by fixed $Z \in \mathbb{R}$.
- $\mathfrak{psu}(2|2)$: only those with $Z = 0$.

Shortening at $k = 1$

At $k = 1$ only $\mathbf{n} = 1$ and $\mathbf{n} = 2$ allowed $\implies$ this fixes Z in terms of j and gives a *short* multiplet:

$$\begin{array}{ccc} & \big(j\,,\mathbf{2}\big) & \\ \big(j+\tfrac{1}{2}\,,\mathbf{1}\big) & & \big(j-\tfrac{1}{2}\,,\mathbf{1}\big)\,. \end{array}$$

There are extra shortenings for $Z = 0, \pm\frac{1}{2}$. In particular, the only highest weight $\mathfrak{psu}(2|2)_1$ representations are:

$$\big(j=0\,,\mathbf{1}\big)\,, \qquad \begin{array}{ccc} & \big(D^{\pm}_{\mp 1/2}\,,\mathbf{2}\big) & \\ \big(D^{\pm}_{\mp 1}\,,\mathbf{1}\big) & & \big(D^{\pm}_{\mp 1}\,,\mathbf{1}\big)\,, \end{array} \qquad \begin{array}{ccc} & \big(C^{1/2}_{1/2}\,,\mathbf{2}\big) & \\ \big(C^{1}_{0}\,,\mathbf{1}\big) & & \big(C^{0}_{0}\,,\mathbf{1}\big)\,. \end{array}$$

Outline

Free field realisation of $\mathfrak{u}(2|2)_1$

Consider one pair of symplectic boson fields $(\lambda^\alpha, \mu^\dagger_\alpha)$ and two complex fermions $(\psi^\alpha, \psi^\dagger_\alpha)$ with $\alpha = 1, 2$, satisfying the commutation relations

$$[\lambda^\alpha_r, (\mu^\dagger_\beta)_s] = \delta^\alpha_\beta \, \delta_{r,-s} \, , \qquad \{\psi^\alpha_r, (\psi^\dagger_\beta)_s\} = \delta^a_b \, \delta_{r,-s} \, .$$

Writing $Y_J = (\mu^\dagger_\alpha, \psi^\dagger_\beta)$ and $X^I = (\lambda^\alpha, \psi^\beta)$, the normal ordered bilinears

$$J^I_J = Y_J X^I \, ,$$

generate $\mathfrak{u}(2|2)_1$.

Free field representations

- **NS sector**: modes are half-integered, we obtain the affine module to highest weight $(j = 0, \mathbf{1})$, the **vacuum**.
- **R sector**: integer-moded, states $|m_1, m_2\rangle$ are labelled by bosonic occupation numbers $m_i \in \frac{1}{2}\mathbb{Z}$. There is freedom in declaring the action of the symplectic bosons zero modes, which yields different representations:
  - For $m_i \in \frac{1}{2}\mathbb{N}$ and

$$\lambda_0^1 \,|m_1, m_2\rangle := 2m_1|m_1 - \tfrac{1}{2}, m_2\rangle \,, \qquad (\mu_1^\dagger)_0 \,|m_1, m_2\rangle := |m_1 + \tfrac{1}{2}, m_2\rangle \,,$$
$$\lambda_0^2 \,|m_1, m_2\rangle := 2m_2|m_1, m_2 - \tfrac{1}{2}\rangle \,, \qquad (\mu_2^\dagger)_0 \,|m_1, m_2\rangle := |m_1, m_2 + \tfrac{1}{2}\rangle \,,$$

    we obtain the above multiplets with finite-dimensional spin j repr.
  - For other choices we obtain the above multiplets with highest/lowest weight discrete repr. and continuous repr.

Outline

Affine characters

Definition: The (normalised) **affine character** of a $\mathfrak{psu}(2|2)_1$-module $\mathcal{H}$ is

$$\chi_{\hat{\lambda}}(t, z; \tau) := \mathrm{Tr}_{\hat{\lambda}}\, e^{2\pi i \tau (L_0 - c/24)} e^{2\pi i (t J_0^3 + z K_0^3)} \quad \text{with } c = -2\,.$$

- $\mathfrak{su}(2)_1$: characters and modular invariants are well-known.
- $\mathfrak{su}(2)_{-1}$: less known.

In particular, I found all the singular vectors present in the $\mathfrak{su}(2)_{-1}$-model using the Kac-Kazhdan determinant and computed the characters and their modular behaviour.

Characters of the symplectic bosons

The zero modes of four symplectic bosons generate

$$\mathfrak{sp}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{u}(1) \oplus \mathbf{3}_1 \oplus \overline{\mathbf{3}}_{-1} \,,$$

so we expect the free field repr. to be reducible w.r. to $\mathfrak{su}(2)_{-1}$. Indeed,

$$\mathcal{H}^{(-1)}_{NS} \cong \bigoplus_{j \in \mathbb{N}} \mathcal{H}^{(-1)}_j \,,$$

$$\mathcal{H}^{(-1)}_{R,\,Z} \cong q^{-Z(Z\pm 1)} \bigoplus_{j \in \mathbb{N}+Z} \mathcal{H}^{(-1)}_j \quad \text{for every } Z \in \tfrac{1}{2}\mathbb{N} \,,$$

which has been proven at the level of characters, thanks to the denominator identity of a Lie superalgebra by Kac.

Affine vacuum character of $\mathfrak{psu}(2|2)_1$

I computed the vacuum character given by the NS sector in terms of affine $\mathfrak{su}(2)_{\pm 1}$-characters:

$$\mathrm{ch}[\mathcal{L}](t,z;\tau) = \chi_0^{(1)}(z;\tau)\sum_{j\in\mathbb{N}}(2j+1)\chi_j^{(-1)}(t;\tau) + \chi_{1/2}^{(1)}(z;\tau)\sum_{j\in\mathbb{N}+\frac{1}{2}}(2j+1)\chi_j^{(-1)}(t;\tau)$$
$$= \frac{\partial_t\left(\vartheta_2(\frac{t+z}{2};\tau)\vartheta_2(\frac{t-z}{2};\tau)\right)}{\pi\,\eta(\tau)\vartheta_1(t;\tau)}\,,$$

which explicitely shows the decomposition of the vacuum module in affine $\mathfrak{su}(2)_{\pm 1}$ highest weight representations:

Affine bosonic content of $\mathcal{L}$

$$
\begin{array}{cccccccccc}
L_0 - \frac{1}{12} & & & & & & & & & \\
0 & & & & & (0\,,\mathbf{1}) & & & & \\
1 & & & & (\frac{1}{2}\,,\mathbf{2}) & & (\frac{1}{2}\,,\mathbf{2}) & & & \\
2 & & & (1\,,\mathbf{1}) & & (1\,,\mathbf{1}) & & (1\,,\mathbf{1}) & & \\
4 & & (\frac{3}{2}\,,\mathbf{2}) & & (\frac{3}{2}\,,\mathbf{2}) & & (\frac{3}{2}\,,\mathbf{2}) & & (\frac{3}{2}\,,\mathbf{2}) & \\
6 & \iddots & & (2\,,\mathbf{1}) & & (2\,,\mathbf{1}) & & (2\,,\mathbf{1}) & & \ddots \\
\vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
$$

Modular transformation of the vacuum character

For better modular properties, we look at the **supercharacter**

$$\mathsf{sch}[\mathcal{L}](t,z;\tau) = \frac{\partial_t\left(\vartheta_1(\frac{t+z}{2};\tau)\vartheta_1(\frac{t-z}{2};\tau)\right)}{\pi\,\eta(\tau)\vartheta_1(t;\tau)}\,,$$

obtaining the S-transformation

$$\mathsf{sch}[\mathcal{L}](\tfrac{t}{\tau},\tfrac{z}{\tau};-\tfrac{1}{\tau}) = \mathrm{e}^{\frac{\pi i}{2\tau}(z^2-t^2)}\,\frac{\pi x\,\vartheta_1(\frac{t+z}{2};\tau)\vartheta_1(\frac{t-z}{2};\tau) - i\tau\,\partial_t(\vartheta_1(\frac{t+z}{2};\tau)\vartheta_1(\frac{t-z}{2};\tau))}{\pi\,\eta(\tau)\vartheta_1(t;\tau)}\,,$$

containing an explicit linear τ-dependence:

$$\mathcal{S}_{\mathcal{L},\mathcal{L}} = -i\tau\,,$$

which suggests that the $\mathfrak{psu}(2|2)$-WZW model is a logarithmic CFT.

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Outlook

Follow-up:

- **Finding modular invariants of the $\mathfrak{su}(2)_{-1}$-model**: it is tricky, because of the non-holomorphicity in the chemical potential of the affine characters. In particular, character formulas in terms of theta functions hold only on specific annuli of convergence which are generally not preserved under modular transformations. Also, there are several recursive relations between character functions, which cause the ring of characters to be “smaller” then the ring of affine modules.
- **Understanding the logarithmic CFT properties of $\mathfrak{psu}(2|2)$**: maybe the vacuum $\mathcal{L}$ and the discrete multiplets form together reducible but indecomposable modules, such as those corresponding to continuous multiplets. Then, it is possible that a better basis for the ring of characters is given by the continuous ones.
- **Understanding the modular behaviour of the $\mathfrak{su}(2|2)_1$ characters**: the $\mathfrak{su}(2|2)_1$-representations for different values of Z seem to be related by some sort of spectral flow, maybe this can help in computing the modular transformation, or to relate the problem to the case of the vacuum $\mathcal{L}$.

Other results in the thesis

- **Spectral flow**: these are automorphisms of the affine Lie superalgebra and map in general highest weight representations to non-highest weight ones. They have been defined at the free field level, and the spectrally flowed vacuum characters of $\mathfrak{psu}(2|2)_1$ have been computed. It is likely that in order to obtain a modular invariant partition function, the spectrally flowed modules must be included.
- **Free field characters of $\mathfrak{su}(2|2)_1$**: they have been also explicitely computed and affine branching rules for the bosonic subalgebra have been given. They seem to have a more complicated form than the characters of $\mathcal{L}$. In particular, I could not write these characters in terms of the usual theta and eta functions in order to capture their modular behaviour.

Outline

A criterion for superalgebras

Conformal embedding: affine embedding $\widetilde{\mathfrak{g}}_{\tilde{k}} \hookrightarrow \mathfrak{g}_k$ such that

$$T^{\mathfrak{g}_k} = T^{\tilde{\mathfrak{g}}_{\tilde{k}}} \iff c(\mathfrak{g}_k) = c(\widetilde{\mathfrak{g}}_{\tilde{k}})\,.$$

Proposition: The embedding $\bigoplus_i \mathfrak{g}^i_{k_i} \hookrightarrow \mathfrak{g}_k$ of a semisimple bosonic subalgebra into a Lie superalgebra is conformal iff

$$\sum_i \frac{C_i}{k_i + h_i^\vee} = 1\,,$$

where

- $C_i =$ Casimir of the adjoint action of $\mathfrak{g}^i$ on the fermions,
- $h_i^\vee =$ dual Coxeter number of $\mathfrak{g}^i$.

Examples of bosonic conformal embeddings

Some examples:

1. $\mathfrak{sl}(2,\mathbb{R})_1 \oplus \mathfrak{su}(2)_1 \hookrightarrow \mathfrak{psu}(1,1|2)_1 \implies$ constrains allowed $\mathfrak{sl}(2,\mathbb{R})$ spins to $j = 1/2$.
2. $\mathfrak{su}(N)_{-k} \oplus \mathfrak{su}(N)_k \hookrightarrow \mathfrak{psu}(N|N)_k$ only possible for $k = 1$, but then true for every $N > 1$. In particular, for $N = 2$ it relates the Casimirs:

$$C^{\mathfrak{psu}(2|2)_1} = j(j+1) + \frac{\ell(\ell+1)}{3}\,, \tag{1}$$

   which shows that $C^{\mathfrak{psu}(2|2)_1} = 0$ on every $\mathfrak{psu}(2|2)$-multiplet given above.
3. For $\mathfrak{u}(N|N)_1$ also true, but for $\mathfrak{su}(N|N)$ not.

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