Soft and collinear scales in Higgs boson production

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Outline

State of the art

- Threshold resummation of production processes in QCD - *Sterman* (1987);
- Resummation in the renormalization group equation (RGE) approach – *Sterman* (1997), *Forte, Ridolfi* (2003);
- Threshold resummation of the rapidity distribution of Higgs boson production – *De Ros* (2022);

The problem

- Soft and collinear scales at NNLO

Hadronic and partonic cross sections

Consider the **Higgs boson production process.**

$$p + p \to H + X$$

We focus on the **gluon fusion** channel

$$g + g \to H + X$$

We define the partonic energy fraction

$$x := \frac{m_H^2}{\hat{s}}$$

At the **threshold limit** the center-of-mass energy to produce an Higgs boson with a given mass is the least possible.

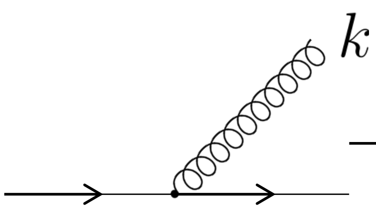
$$x \to 1$$

The Higgs boson is produced at rest and all the extra emissions are soft.

Threshold limit and resummation

In the threshold limit partonic cross sections feature a class of enhanced logarithms that spoil the perturbative series. **Resummation formulae** solve this problem, increasing the precision of the predictions.

Threshold logarithms arise from soft and collinear regions of the phase space.



$$\longrightarrow \int d^3k \quad \begin{cases} \displaystyle\int_0^{\Lambda_{soft}} \frac{dE}{E} \longrightarrow \ln \frac{\Lambda^2_{soft}}{\mu^2} \\[2ex] \displaystyle\int_0^{\Lambda^2_{coll}} \frac{dp_T^2}{p_T^2} \longrightarrow \ln \frac{\Lambda^2_{coll}}{\mu^2} \end{cases}$$

Scales are determined by studying the **phase space** of the process.
For the total cross section of the Higgs boson production process there's only one scale

$$\Lambda^2 = m_H^2(1-x)^2$$

Resummation and Renormalization Group Equations

The determination of the scales leads to the resummation formulae through the Renormalization Group Equations (RGE) approach.

1. Work in Mellin space $x \longleftrightarrow N$

2. RG invariance of the hadronic cross section.

3. Factorization of the bare partonic coefficient

Mellin transform of $m_H^2(1-x)^2$

$$C^{(0)}(N, m_H^2, \epsilon) = C^{(c)}(m_H^2, \epsilon) C^{(l)}(m_H^2/N^2, \epsilon)$$

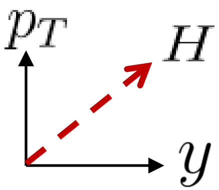
Resummation formula

$$C\left(N, m_H^2, \mu\right) = C^{(c)}\left(\frac{m_H^2}{\mu^2}, \alpha_s(Q^2)\right) \exp\left\{\int_1^{N^2} \frac{dn}{n} \int_{n\mu^2}^{m_H^2} \frac{dk^2}{k^2} \hat{g}(\alpha_s(k^2/n))\right\}$$

When there are more scales the formula can be generalized trivially.

RGE approach $\longleftrightarrow$ **Knowledge of scales**

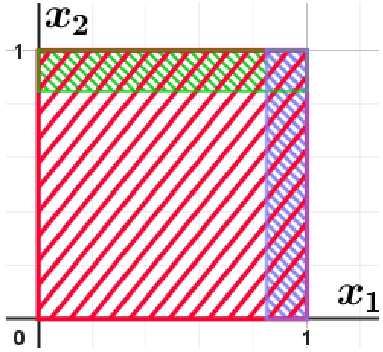
We consider the resummation of the **rapidity distribution** (rate of production at **fixed longitudinal momentum**).



We now consider the threshold limit at fixed rapidity. New threshold variables are necessary:

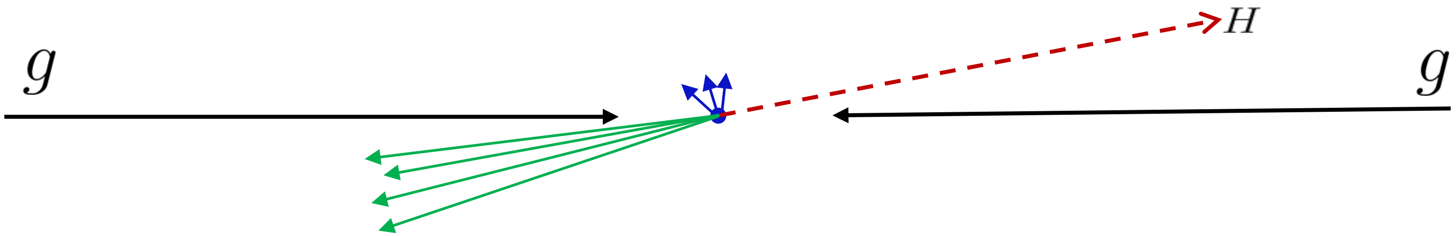
$$\begin{cases} x_1 := \sqrt{x}e^{y} \\ x_2 := \sqrt{x}e^{-y} \end{cases}$$

whose limits are



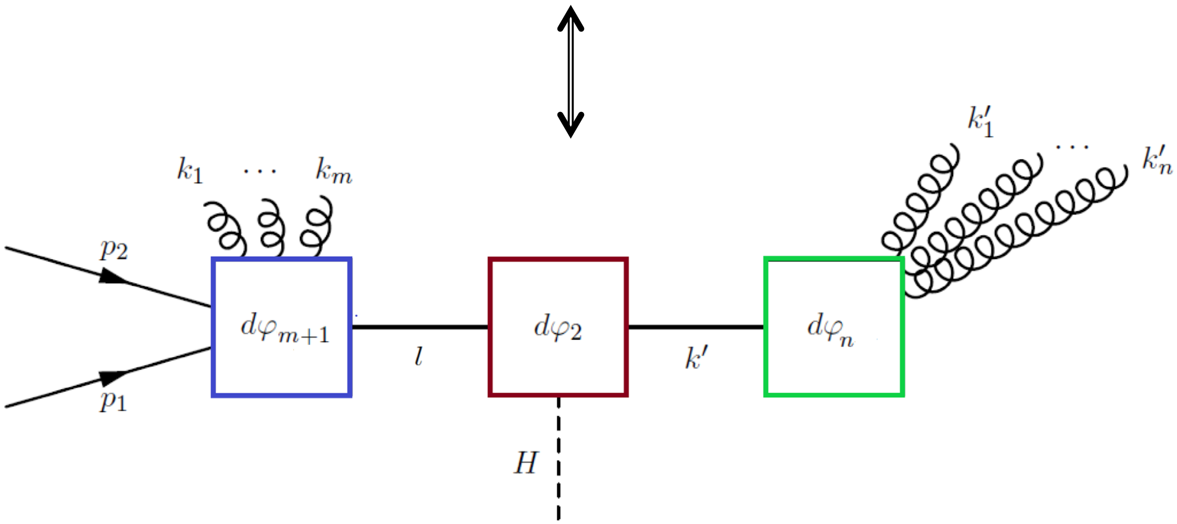
- For fixed x_2 the threshold limit is $x_1 \to 1$;
- For fixed x_1 the threshold limit is $x_2 \to 1$.
- When both variables go to 1 we recover the previous threshold limit or, **doubly soft limit.**

To identify soft and collinear scales we separate soft and collinear emissions at threshold...

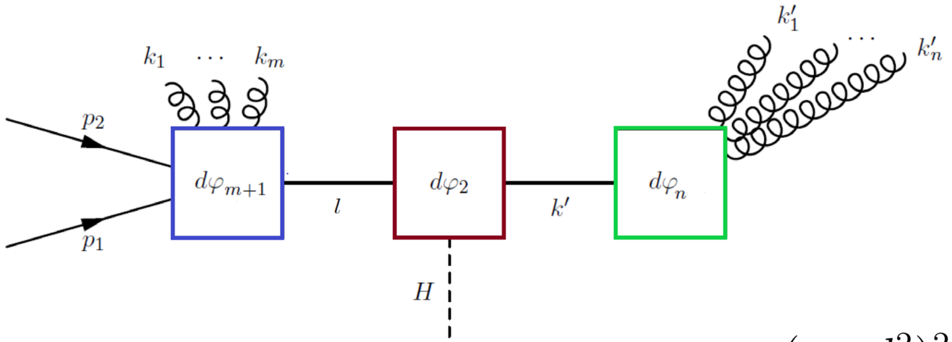


...and decompose the phase space accordingly

$$d\phi_{n+m+1} = \int \frac{dl^2}{2\pi} d\phi_{m+1}(p_1, p_2; l, k_1, \ldots, k_m) \int \frac{d(k')^2}{2\pi} d\phi_2(l; p_H, k') d\phi_n(k'; k'_1, \ldots, k'_n)$$



This decomposition directly leads to the desired scales.



- The **soft emissions** phase space is DY-like $\Lambda^2_{DY} = \frac{(s-l^2)^2}{s}$
- The **collinear emissions** phase space is DIS-like $\Lambda^2_{DIS} = (k')^2$

$$\Lambda^2_{soft} = m^2_H (1-x_1)^2$$

$$\Lambda^2_{coll} = m^2_H (1-x_1)(1-x_2)$$

$$C\left(N_1, \frac{m_H^2}{\mu^2}, \frac{m_H^2/N_2}{\mu^2}, \alpha_s(\mu^2)\right) = C^c\left(\frac{m_H^2}{\mu^2}, \alpha_s(\mu^2)\right)$$
$$\exp\left\{ \int_1^{N_1^2} \frac{dn}{n} \int_{n\mu^2}^{m_H^2} \frac{dk^2}{k^2} g_1(\alpha_s(k^2/n), N_2) + \int_1^{N_1 N_2} \frac{dn}{n} \int_{n\mu^2}^{m_H^2/N_2} \frac{dk^2}{k^2} g_2(\alpha_s(k^2/n)) \right\}$$

L. De Ros (2022)

The missing soft scale

In the doubly soft limit, the soft scale produces subleading logs.

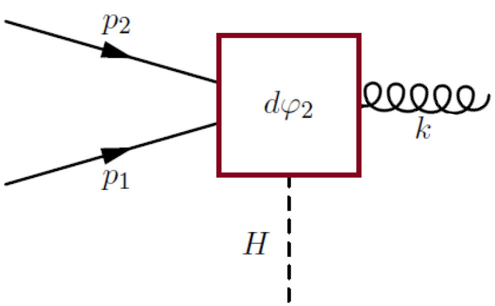
$$C\left(N_1, N_2, \frac{m_H^2}{\mu^2}, \alpha_s(\mu^2)\right) = C^c\left(\frac{m_H^2}{\mu^2}, \alpha_s(\mu^2)\right) \exp\left\{ \int_1^{N_1 N_2} \frac{dn}{n} \int_{n\mu^2}^{m_H^2} \frac{dk^2}{k^2} g_2(\alpha_s(k^2/n)) \right\}$$

Old result – *Catani, Trentadue* (1989)

A first result for the resummation in the threshold limit is due to *F. Tackmann, G. Lustermanns, J. Michel* (*2019*) in the SCET approach.

→ Fixed-order approximations give no signs of the soft scale!

Can we actually see the soft scale at fixed order?
The phase space at NLO has only one emission recoiling against the Higgs.



We need the NNLO!

The fully differential distribution at NNLO

We have explicitly only the fully differential distribution – *Glosser, Schmidt* (2003)

$$\frac{d^2\sigma_{gg}}{dp_T^2 dy} = \frac{\sigma_0}{s}\left[\frac{\alpha_s(\mu_R)}{2\pi}C_{gg}^{(1)} + \left(\frac{\alpha_s(\mu_R)}{2\pi}\right)^2 C_{gg}^{(2)}\right]$$

$$\begin{aligned}
C_{sing}^{(2)} =& \delta(Q^2)\left\{(11+\delta+N_cU)g_{gg} + (N_c-n_f)\frac{N_c}{3}\left[\frac{m_H^4}{s}+\frac{m_H^4}{t}+\frac{m_H^4}{u}+m_H^2\right]\right\} \\
&+\left\{\left(\frac{1}{-t}\right)\left[-P_{gg}(z_t)\log\frac{\mu_F^2 z_t}{-t} + p_{gg}(z_t)\left(\frac{\log(1-z_t)}{1-z_t}\right)_+\right]g_{gg,t}(z_t)\right. \\
&+\left(\frac{1}{-t}\right)\left[-2n_f P_{qg}(z_t)\log\frac{\mu_F^2}{Q_{max}^2 q} + 2n_f z_t(1-z_t)\right]g_{qg,t}(z_t) \\
&+\left(\frac{z_t}{-t}\right)\left[\left(\frac{\log(1-z_t)}{1-z_t}\right)_+ - \log\frac{Q_T^2 z_t}{-t}\left(\frac{1}{1-z_t}\right)_+\right]\frac{N_c^2}{2}\left[\frac{m_H^8+s^4+t^4+u^4+Q^8+z_t z_u(m_H^8+s^4+Q^8+(u/z_u)^4+(t/z_t)^4)}{sut}\right] \\
&\left.-\left(\frac{z_t}{-t}\right)\left(\frac{1}{1-z_1}\right)\frac{\beta_0}{2}N_c\left(\frac{m_H^8+s^4+z_t z_u((u/z_u)^4+(t/z_t)^4)}{sut}\right) + (t\leftrightarrow u)\right\} \\
&+N_c^2\left[\frac{(m_H^8+s^4+Q^8+(u/z_u)^4+(t/z_t)^4)(Q^2+Q_T^2)}{s^2Q^2Q_T^2} + \frac{2m_H^4((m_H^2-u)^4+(m_H^2-t)^4+t^4+u^4)}{sut(m_H^2-t)(m_H^2-u)}\right]\frac{1}{p_T^2}\log\frac{p_T^2}{Q_T^2}
\end{aligned}$$

Many different variables with logarithms that may be combined in different ways.

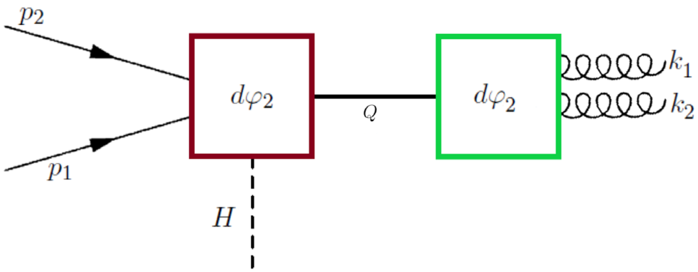
$$\ln(1-x_1)+\ln(1-x_2)$$

Is this the collinear scale $\ln(1-x_1)(1-x_2)$ or a sum of soft scales?

In the case of two emissions the possible decompositions of the phase space are two:

$$g + g \to H + g + g$$

Extra emission is collinear

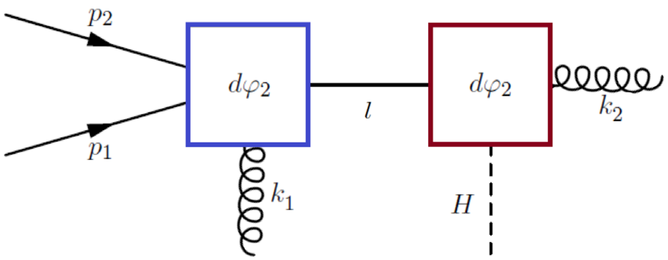


We can now write the collinear scale as

$$\Lambda^2_{coll} = Q^2_{max}$$

Where $Q = (k_1 + k_2)^2$

Extra emission is soft



The soft emission phase space gives again

$$\Lambda^2_{DY} = \frac{(s - l^2)^2}{s}$$

But now we write it as

$$\Lambda^2_{soft} = t^2$$

Thus we have now a dictionary:

$$\Lambda^2_{coll} = m^2_H(1 - x_1)(1 - x_2) \longrightarrow Q^2_{max}$$

$$\Lambda^2_{soft} = m^2_H(1 - x_1)^2 \longrightarrow t^2$$

Setting the integration on transverse momentum and changing variables to plus distributions

$$C^{(2)}_{sing,rap} = \int_0^1 dq J(x_1, x_2, q) \Bigg\{ Q^2_{max}\left[\mathcal{E} + \frac{\mathcal{B}_{2,t}}{-t} + \frac{\mathcal{B}_{1,t}}{-t}\ln\frac{1}{qQ^2_{max}}\right] + \delta(q)\Bigg[\mathcal{A}_t\beta_0 \ln t + N_c\mathcal{A}_t \ln^2 Q^2_{max} - N_c\mathcal{A}_t \ln^2 t - \mathcal{C}_t \ln Q^2_{max} \ln p^2_{T,max} + \frac{1}{2}\mathcal{C}_t \ln^2 Q^2_{max} - \frac{1}{2}\mathcal{C}_t \ln^2 t + \mathcal{C}_t \ln p^2_{T,max} - \mathcal{D}_t \ln Q^2_{max} + \mathcal{D}_t \ln t + \mathcal{C}_t \ln Q^2_T \ln t + \mathcal{V}\Bigg] + \left(\frac{1}{q}\right)\left[\frac{\mathcal{A}_t p_{gg}}{z_t} \ln Q^2_{max} + \mathcal{C}_t \ln Q^2_{max} - \mathcal{C}_t \ln Q^2_T - \mathcal{D}_t\right] + \left(\frac{\ln q}{q}\right)_+ \left[\frac{\mathcal{A}_t p_{gg}}{z_t} + \mathcal{C}_t\right] + (t \leftrightarrow u)\Bigg\}$$

With the considerations made above the soft and the collinear scales are immediately identified.

Retracing the origin of the logarithms

The logarithms can be retraced from the intermediate passages of the calculation.

1. Logarithms of the collinear scale come explicitly from the phase space decomposition chose by *GS*.

$$C_{ij}^{(2R)}(\epsilon) = \frac{1}{2\pi\Gamma(1-2\epsilon)}\left(\frac{4\pi\mu^2}{p_T^2}\right)^{\epsilon}\left(\frac{4\pi\mu^2}{Q^2}\right)^{\epsilon}\int d\Omega_\epsilon |\mathcal{M}_{ij}|^2$$

2. Logarithms of the soft scale always appear multiplied by $\delta(q)$

3. Logarithms of the soft scale originate from the Altarelli-Parisi collinear term, coming from the same region of the phase space of the soft scale.

$$C_{gg}^{(2AP)}(\epsilon) = \frac{1}{\epsilon\Gamma(1-\epsilon)}(4\pi)^{\epsilon}\sum_k\left[\frac{1}{-t}P_{kg}(z_t)g_{kg,t}(z_t;\epsilon) + \frac{1}{-u}P_{kg}(z_u)g_{gk,u}(z_u;\epsilon)\right]$$

Conclusions

- At fixed-order there are both the collinear and the soft scale;
- Their origin agrees with the phase space argument.

Future developments

- RGE vs. SCET approach;
- Fully differential distribution?

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- RGE vs. SCET approach;
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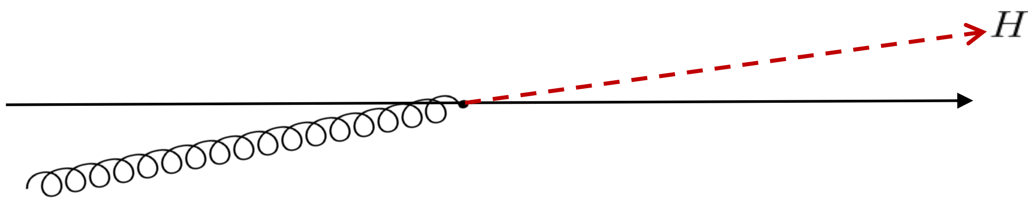
Thank you

Supplementary material

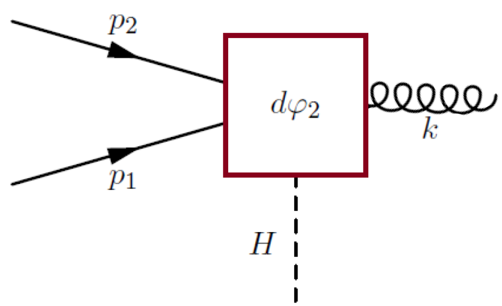
Rapidity distribution at NLO

Consider the NLO order result (*Anastasiou, Dixon, Melnikov 2002*

$$g + g \to H + g\,.$$



The momentum of the gluon is fixed by the momentum of the Higgs boson, hence the only integration that takes place is on the transverse momentum of the Higgs boson and the PS is trivial.



$$\frac{d\sigma^{real}_{NLO}}{dy} = \frac{(4\pi)^{\epsilon}|p_{T,max}|^{-2\epsilon}}{16\pi\Gamma(1-\epsilon)}\frac{s+m^2}{2s\cosh^2 y}|\mathcal{M}_{gg\to Hg}|^2$$

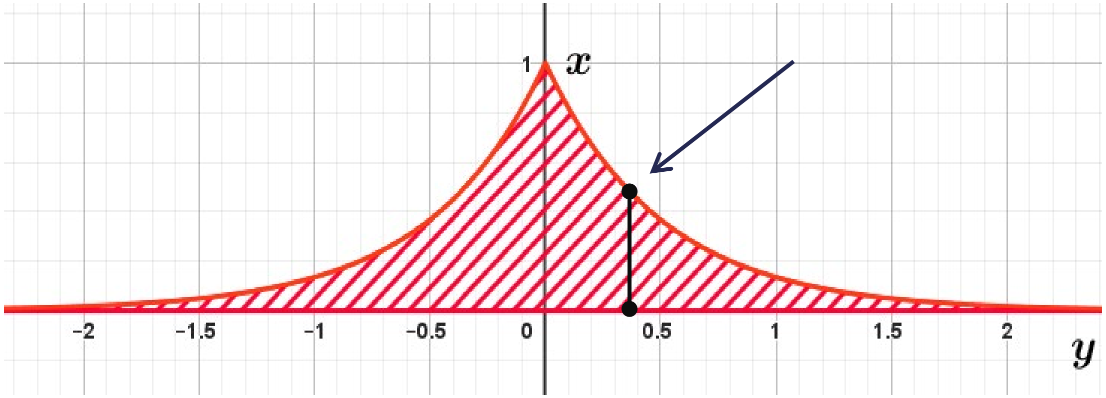
$$p^2_{T,max} \propto m^2_H(1-x_1)(1-x_2)$$ → We recover the collinear scale

Kinematics of the rapidity distribution

For fixed rapidity the extrema of x are

$$\begin{cases} 0 \leq x \leq e^{-2y} & \text{if } y > 0 \\ 0 \leq x \leq e^{2y} & \text{if } y < 0 \end{cases}$$

Graphically the domain is



The new threshold variables are simply defined by interpolating on the two extrema in the two cases.

The scales of the total cross sections are found by decomposing the phase space in two-body phase spaces and writing all the integrals in terms of dimensionless variables.
Consider for example a DY phase space

$$d\phi_{n+1}(p+p';Q,k_1,\dots,k_n)=\frac{d^{d-1}Q}{(2\pi)^{d-1}2Q^0}\frac{d^{d-1}k_1}{(2\pi)^{d-1}2k_1^0}\cdots\frac{d^{d-1}k_n}{(2\pi)^{d-1}2k_n^0}(2\pi)^d\delta^{(d)}(p_1+p_2-Q-k_1-\cdots-k_n)$$

This is equivalent to

$$=\int_{Q^2}^{s}\frac{M_n^2}{2\pi}d\phi_2(p+p';k_n,P_n)\int_{Q^2}^{M_n^2}\frac{M_{n-1}^2}{2\pi}d\phi_2(p+p';k_{n-1},P_{n-1})\dots\int_{Q^2}^{M_3^2}\frac{M_2^2}{2\pi}d\phi_2(k_1;k_2,P_2)d\phi_2(P_2;k_1,Q).$$

Computing each phase space in its C.O.M. frame and moving onto dimensionless variables we find

$$=2\pi\left[\frac{N(\epsilon)}{2\pi}\right]^n s^{-n(1-\epsilon)}(s-Q^2)^{2n-1-2n\epsilon}d\Omega_n\dots d\Omega_1\int_0^1 dz_n z_n^{n-2+(n-1)(1-2\epsilon)}(1-z_n)^{1-2\epsilon}\dots\int_0^1 dz_2 z_2^{1-2\epsilon}(1-z_2)^{1-2\epsilon}$$

The pre-factor finally gives rise to the soft scale

$$\left[\frac{(s-Q^2)^2}{s}\right]^{n-n\epsilon}\longrightarrow[Q^2(1-x)^2]^{n-n\epsilon}$$

Diagrammatic arguments on loop and tree contributions lead us to

$$(Q^2)^{-1-k\epsilon}[(1-x)]^{-1-2k\epsilon}$$

The scales of total cross sections (2)

This is the combination we expect since through the following distributional identity we get the plus distributions of logarithms, that are $\ln^p N$ in Mellin space.

$$(1-z)^{-1+\epsilon} = \frac{\delta(1-z)}{\epsilon} + \left(\frac{1}{1-z}\right)_+ + \epsilon\left(\frac{\ln(1-z)}{1-z}\right)_+ + \dots$$

Combining the phase space and the diagrammatic arguments we get an expression for the bare soft coefficient

$$C_n^{(0)}(x, Q^2, \epsilon) = (Q^2)^{-n\epsilon}\left[C_0(\epsilon)\delta(1-x) + \sum_{k=1}^{n} C_k(\epsilon)(1-x)^{-1-2k\epsilon}\right] + \mathcal{O}[(1-x)^0]$$

Poles of order $2n-1$

Or, in Mellin space

$$C_n^{(0)}(N, Q^2, \epsilon) = \sum_{k=0}^{n} C_k(\epsilon)(Q^2)^{-(n-k)\epsilon}\left(\frac{Q^2}{N}\right)^{-k\epsilon} + \mathcal{O}\left(\frac{1}{N}\right)$$

Factorization of the rapidity distribution

The factorization of the rapidity distribution in terms of the energy fraction and the rapidity reads

$$\frac{d\sigma}{dY}(\tau, Y, m_H^2) = \sigma_0 \sum_{i,j=g,q,\bar{q}} \int_{\tau_1}^{1} d\xi_1 \, f_i(\xi_1, \mu_F^2) \int_{\tau_2}^{1} d\xi_2 \, f_j(\xi_2, \mu_F^2) \frac{dC_{ij}}{dy} \left(x = \frac{\tau}{\xi_1 \xi_2}, y = Y - w, m_H^2, \mu_F^2 \right)$$

In terms of the newly defined threshold variables factorization assumes a more symmetrical form

$$\frac{d\sigma}{dY}(\tau_1, \tau_2, m_H^2) = \sigma_0 \int_{\tau_1}^{1} d\xi_1 \, f_i(\xi_1) \int_{\tau_2}^{1} d\xi_2 \, f_j(\xi_2) \frac{dC_{ij}}{dy} \left(x_1 = \frac{\tau_1}{\xi_1}, x_2 = \frac{\tau_2}{\xi_2}, m_H^2, \mu_F^2 \right)$$

The convolution can now be factorized in Mellin-Mellin space

$$\frac{d\sigma}{dY}(N_1, N_2, m_H^2) = \sigma_0 f_i(N_1) f_j(N_2) \frac{dC_{ij}}{dy}(N_1, N_2, m_H^2)$$

The singular part of the real contribution is

$$C^{(2R)}_{sing} = \sigma_\epsilon \Bigg\{ - \frac{1}{N_c \epsilon} \frac{p_{gg}(z_t)}{-t} g_{gg,t}(z_t)$$

$$- \frac{1}{\epsilon} \left[\left(\frac{Q^2}{Q_T^2} \right)^{-\epsilon} \left(1 + \frac{\pi^2 \epsilon^2}{6} \right) - 1 \right] \frac{g_{gg,t}(z_t)}{Q^2}$$

$$- \frac{1}{2} \left(\frac{11}{6} + \frac{67\epsilon}{18} \right) \frac{g_{gg,t}(z_t)}{Q^2} + (t \leftrightarrow u)$$

The pre-factor σ_ϵ contains $(Q^2)^\epsilon$ and $(p_T^2)^\epsilon$, that combined with the corresponding denominators in the singular part give the logarithms and the plus distributions.

Intermediate results from *Glosser (2001)* – the virtual contribution

The virtual part is

$$
\begin{aligned}
U_\epsilon \;=\; & -\frac{1}{\epsilon^2}\left[\left(\frac{\mu^2}{-t}\right)^\epsilon + \left(\frac{\mu^2}{-u}\right)^\epsilon + \left(\frac{\mu^2}{s}\right)^\epsilon\right] + \frac{\pi^2}{6} + \frac{11}{N_c} + \frac{3}{\epsilon}\frac{\beta_0}{N_c} \\
& + \log\left(\frac{m_H^2}{s}\right)^2 + \log\left(\frac{m_H^2}{m_H^2 - t}\right)^2 + \log\left(\frac{m_H^2}{m_H^2 - u}\right)^2 \\
& - \log\left(\frac{s}{m_H^2}\right)\log\left(\frac{-t}{m_H^2}\right) - \log\left(\frac{s}{m_H^2}\right)\log\left(\frac{-u}{m_H^2}\right) - \log\left(\frac{-u}{m_H^2}\right)\log\left(\frac{-t}{m_H^2}\right) \\
& + 2\mathrm{Li}_2\left(1 - \frac{m_H^2}{s}\right) + 2\mathrm{Li}_2\left(\frac{m_H^2}{m_H^2 - t}\right) + 2\mathrm{Li}_2\left(\frac{m_H^2}{m_H^2 - u}\right)
\end{aligned}
$$

We see a trace of the soft scale from the integration on virtual partons inside loop diagrams.

Variable change of plus distributions

In order to make explicit the soft and singular scales we have used the identities

$$\frac{z_t}{-t}\left(\frac{1}{1-z_t}\right)_+ = \frac{1}{Q^2_{max}}\left\{\left(\frac{1}{q}\right)_+ + \delta(q)\ln\left(\frac{Q^2_{max}}{-t}\right)\right\}$$

$$\frac{z_t}{-t}\left(\frac{\ln(1-z_t)}{1-z_t}\right)_+ = \frac{1}{Q^2_{max}}\left\{\left(\frac{\ln q}{q}\right)_+ + \ln\frac{Q^2_{max}z_t}{-t}\left(\frac{1}{q}\right) + \frac{\delta(q)}{2}\ln^2\frac{Q^2_{max}}{-t}\right\}$$

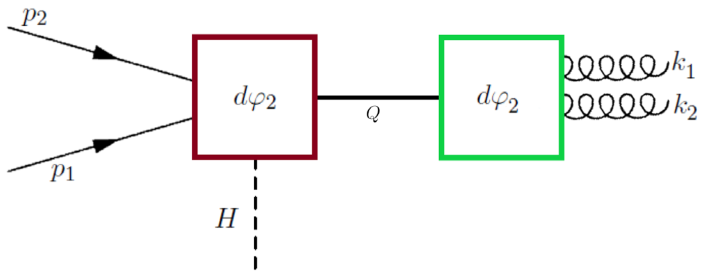
They arise by taking the relation between the two variables involved

$$q = \frac{-t}{Q^2_{max}}\frac{1-z_t}{z_t}$$

And expanding and comparing order-by-order the identity

$$q^{-1+\epsilon} = \left(\frac{-t}{Q^2_{max}z_t}\right)^{-1+\epsilon}(1-z_t)^{-1+\epsilon}$$

In the case of two emissions, when both are set to be collinear the phase space can be decomposed as



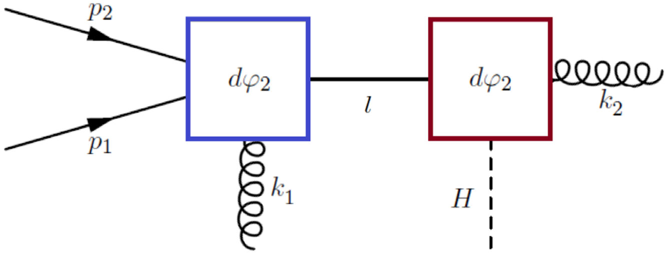
$$d\phi_3(p_1, p_2; k_1, k_2, p_H) = \int_0^{Q^2_{max}} \frac{dQ^2}{2\pi} d\phi_2(p_1 + p_2; p_H, Q) d\phi_2(Q; k_1, k_2)$$

This explicitly gives

$$\frac{d\sigma^{(2)}_{real}}{dy dp_T^2}(s, y, m_H^2, p_T^2) = \frac{1}{2s} \frac{1}{512\pi^3 \Gamma(1-\epsilon)} \left(\frac{4\pi}{\tilde{p}_T^2}\right)^\epsilon \left(\frac{4\pi}{Q^2}\right)^\epsilon \int d\Omega_\epsilon |\mathcal{M}(\Omega_\epsilon, s, y, m_H^2, Q^2)|^2$$

This is **by definition** the collinear scale

In the case of two emissions, when one is set to be soft the phase space can be decomposed as



$$d\phi_3(p_1+p_2;k_1,k_2,p_H)=\int\frac{dl^2}{2\pi}{\color{blue}d\phi_2(p_1+p_2;k_1,l)}{\color{red}d\phi_2(l;p_H,k_2)}$$

This explicitly gives

$$\frac{dC^{(2)}}{dy}=\Phi\mathcal{C}(\pi,\epsilon)\int_{l^2_{min}}^{l^2_{max}}\frac{dl^2}{\sqrt{l^2}}\left|\frac{dp_T^2}{dl^2}\right|\left(\frac{4\pi}{{\color{red}\tilde{p}_T^2}}\right)^{\epsilon}\frac{1}{s-l^2}\left({\color{blue}\frac{(s-l^2)^2}{s}}\right)^{1-\epsilon}\int d\Omega_i|M(\Omega,s,y,m_H^2,l^2)|^2$$

Since this scale depends on an integration variable, we write the integration in terms of a dimensionless parameter varying from the maximum and the minimum of the extrema

$$l^2=m^2\frac{u-m^2}{t-m^2}+w\left(s-m^2\frac{u-m^2}{t-m^2}\right)$$

$$\frac{(s-l^2)^2}{s}=s\,(1-w)^2\left(1-\frac{m^2}{s}\frac{u-m^2}{t-m^2}\right)^2$$

→ The singular part is t