Technical University of Munich
Department of Physics

False vacuum decay of excited states from finite-time instantons

IMPRS EPP Recruiting Workshop

Nils Wagner

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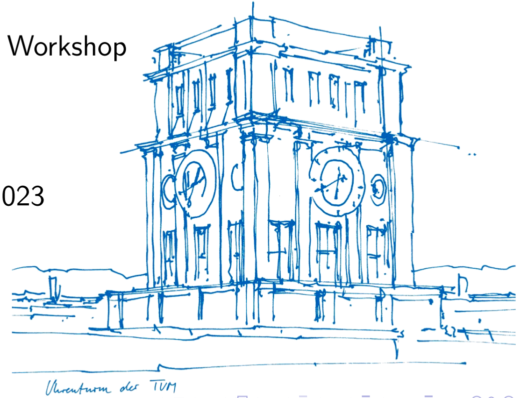


Supervisor: Prof. Dr. Björn Garbrecht
Research Group: T70 (TUM)

Outline

Outline

Motivation

[1] Sakharov (1967), *Pisma Zh. Eksp. Teor. Fiz.* vol. *5*
[2] Garbrecht (2020), *Prog. Part. Nucl. Phys.* vol. *110*

Phase transitions in the early universe could provide an explanation for the observed baryon asymmetry [1, 2]. Such an investigation requires intricate knowledge of

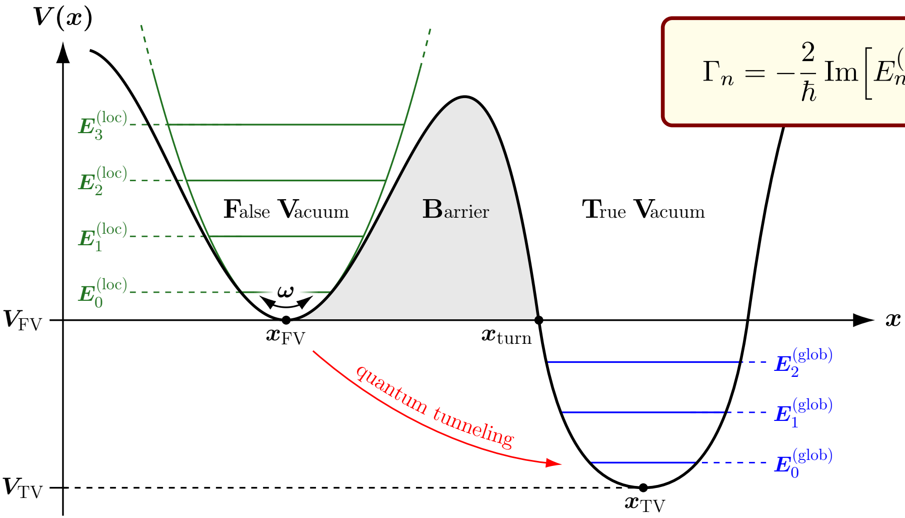
(non-perturbative) tunneling phenomena in QFT at finite temperature .

usual instanton method is not fully understood

requires us to incorporate the decay of excited states

Goal: Examine the decay of excited states using functional methods.

Generic tunneling potential



$$\Gamma_n = -\frac{2}{\hbar}\,\mathrm{Im}\left[E_n^{(\mathrm{loc})}\right]$$

Extracting decay rates

[3] Gamow (1928), *Z. Physik* vol. *51*(3)
[4] Bender & Wu (1973), *PRD* vol. *7*(6)
[5] Callan & Coleman (1977), *PRD* vol. *16*(6)
[6] Ai, Garbrecht & Tamarit (2019), *JHEP* vol. *12*

There exist numerous methods of attributing a meaningful imaginary part to the local energies $E_n^{(\mathrm{loc})}$, fitting into roughly two categories:

Wave function techniques based on (approximate) solutions to the Schrödinger equation [3, 4, etc.]

Functional techniques based on the (Euclidean) propagator, employing path integrals [5, 6, etc.]

directly extendable to field theory

Outline

Method of steepest descent

[7] Pham (1983), *Singularities*, Proc. Symp. Pure Math. vol. *40*(2)
[8] Witten (2011), *AMS/IP Stud. Adv. Math.* vol. *50*

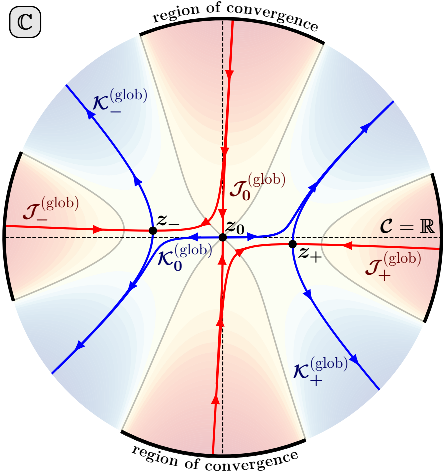
Want to approximate the parameter integral

$$F(\hbar) = \int_{-\infty}^{\infty} \exp\left[-\frac{f(x)}{\hbar}\right] \mathrm{d}x$$

$$\sim \sum_{x_{\min}} \underbrace{\sqrt{\frac{2\pi\hbar}{f''_{\exp}(x_{\min})}}}_{\text{fluctuations around minima}} \underbrace{\exp\left[-\frac{f(x_{\min})}{\hbar}\right]}_{\text{leading contribution from minima}}$$

in the limit $\hbar \to 0$.

For complex f one decomposes the integration contour $\mathcal{C}$ into parts on which $\mathrm{Im}(f)$ is constant
$\longrightarrow$ **Picard-Lefschetz theory** [7, 8, etc.]



Laplace method for the Euclidean propagator

In a similar manner, one can express the leading order behavior for the Euclidean propagator

$$K_{\mathrm{E}}\big(x_T, T; x_0, 0\big) = \int_{x(0)=x_0}^{x(T)=x_T} \mathcal{D}_{\mathrm{E}}[\![x]\!] \, \exp\left\{ -\frac{1}{\hbar} \underbrace{\int_0^T \frac{m\dot{x}(\tau)^2}{2} + V\big[x(\tau)\big] \, \mathrm{d}\tau}_{\text{Euclidean action } S_{\mathrm{E}}[\![x]\!]} \right\} .$$

In the absence of (quasi-)zero modes one finds

$$\boxed{K_{\mathrm{E}}\big(x_T, T; x_0, 0\big) \sim \sqrt{\frac{m}{\pi\hbar}} \sum_{x_{\min}(\tau)} \det\nolimits_{\zeta}\left\{ -\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} + \frac{V''\big[x_{\min}(\tau)\big]}{m} \right\}^{-\frac{1}{2}} \exp\left(-\frac{S_{\mathrm{E}}[\![x_{\min}]\!]}{\hbar} \right) .}$$

Structure: zero mode factor $\mathscr{Z} = 1$, fluctuation factor, leading exponential contribution

Outline

Late-time behavior of the Euclidean propagator

[5] Callan & Coleman (1977), *PRD* vol. *16*(6)
[9] Schwartz, et al. (2017), *PRD* vol. *95*(8)

Observe that one can project out the ground state energy from the late-time behavior of the Euclidean propagator

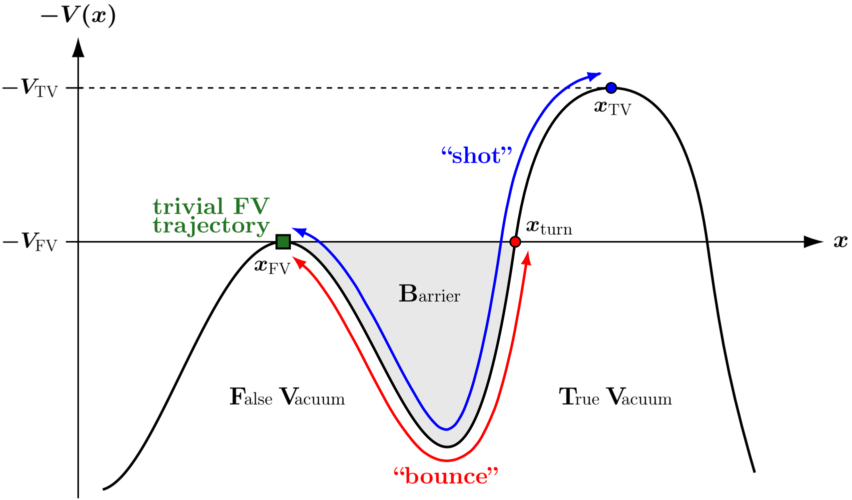
$$K_{\mathrm{E}}\left(x_T, T; x_0, 0\right) = \sum_{n=0}^{\infty} \overline{\psi_n^{(\mathrm{glob})}(x_0)}\, \psi_n^{(\mathrm{glob})}(x_T)\, \exp\left[-\frac{E_n^{(\mathrm{glob})} T}{\hbar}\right] ,$$

which leads to the exact relation

$$\boxed{E_0^{(\mathrm{glob})} = -\hbar \lim_{T\to\infty}\left\{T^{-1} \ln\left[K_{\mathrm{E}}\left(x_{\mathrm{FV}}, T; x_{\mathrm{FV}}, 0\right)\right]\right\} .}$$

One hereby chooses $x_0 = x_T = x_{\mathrm{FV}}$ for convenience [5, 9].

Critical trajectories for large T



Coleman's conjecture

[6] Ai, Garbrecht & Tamarit (2019), *JHEP* vol. *12*
[9] Schwartz, et al. (2017), *PRD* vol. *95*(8)

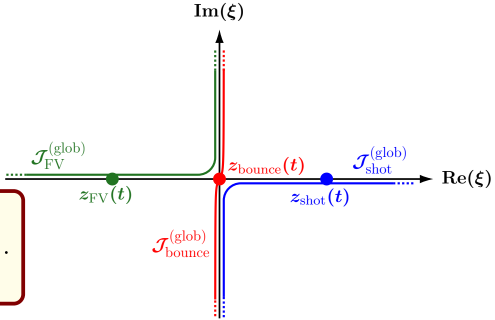
We require information about $E_0^{(\mathrm{loc})}$, so how is the relation

$$E_0^{(\mathrm{glob})} = -\hbar \lim_{T\to\infty}\left\{T^{-1}\ln\left[\underbrace{K_{\mathrm{E}}^{(\mathrm{shot})}(T)}_{\text{dominant}} + \underbrace{K_{\mathrm{E}}^{(\mathrm{FV})}(T) + K_{\mathrm{E}}^{(\mathrm{bounce})}(T)}_{\text{exponentially suppressed}}\right]\right\}$$

of any use?

Dropping the steepest descent contour associated with the shot solution [6, 9], one conjectures the identity

$$\boxed{E_0^{(\mathrm{loc})} = -\hbar \lim_{T\to\infty}\left\{T^{-1}\ln\left[K_{\mathrm{E}}^{(\mathrm{FV})} + \frac{1}{2}K_{\mathrm{E}}^{(\mathrm{bounce})}\right]\right\}.}$$



Final expression

Taking care of certain caveats yields the ground state decay width Γ_0 as

$$\Gamma_0 = \sqrt{\frac{S_\mathrm{E}\left[\!\left[x^{(T=\infty)}_{\mathrm{bounce}}\right]\!\right]}{2\pi\hbar}} \left| \frac{\det'_\zeta\left\{-\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} + V''\left[x^{(T=\infty)}_{\mathrm{bounce}}(\tau)\right]\right\}}{\det_\zeta\left\{-\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} + V''\left[x_{\mathrm{FV}}(\tau)\right]\right\}} \right|^{-\frac{1}{2}} \exp\left\{-\frac{1}{\hbar} S_\mathrm{E}\left[\!\left[x^{(T=\infty)}_{\mathrm{bounce}}\right]\!\right]\right\}.$$

Structure: zero mode factor, determinant ratio, leading exponential term

Important: By virtue of analogy, this formula can be transferred to field theory!

Outline

Generalized ansatz

[10] Liang & Müller-Kirsten (1994), *PRD* vol. *50*(10)
[11] Liang & Müller-Kirsten (1995), *PRD* vol. *51*(2)

Proceed similarly as in the previous case by considering $E_n^{(\mathrm{glob})}$ given by

$$E_n^{(\mathrm{glob})} = -T^{-1}\hbar\,\ln\left[\int_{\mathbb{R}^2} \overline{\psi_n^{(\mathrm{glob})}(x_T)}\,\psi_n^{(\mathrm{glob})}(x_0)\,K_{\mathrm{E}}\big(x_T, T; x_0, 0\big)\,\mathrm{d}x_0\,\mathrm{d}x_T\right].$$

To extract $E_n^{(\mathrm{loc})}$, one employs the following two substitutions [10, 11]:

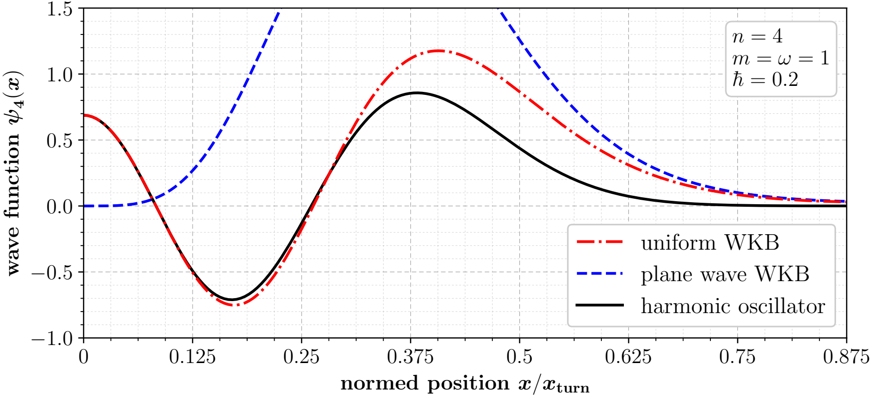
1. ... omit shot-like contributions.
2. ... replace $\psi_n^{(\mathrm{glob})}(x)$ by $\psi_n^{(\mathrm{loc})}(x)$.

$$\boxed{E_n^{(\mathrm{loc})} = -\frac{\hbar}{T}\,\ln\left\{\int_{\mathbb{R}^2} \overline{\psi_n^{(\mathrm{loc})}(x_T)}\,\psi_n^{(\mathrm{loc})}(x_0)\,\left[K_{\mathrm{E}}^{(\mathrm{FV})}\big(x_T, T; x_0, 0\big) + \frac{1}{2}\,K_{\mathrm{E}}^{(\mathrm{bounce\text{-}like})}\big(x_T, T; x_0, 0\big)\right]\mathrm{d}x_0\,\mathrm{d}x_T\right\}.}$$

Local wave function

Two naturally arising candidates for $\psi_n^{(\mathrm{loc})}(x)$:

1. Harmonic oscillator states — good approximation near x_{FV}
2. Traditional WKB ansatz — correct estimate inside the barrier



Methods of evaluation

There are two ways of computing the expression

$$\int_{\mathbb{R}^2} \overline{\psi_n^{(\mathrm{loc})}(x_T)}\, \psi_n^{(\mathrm{loc})}(x_0) \left[\int_{x(0)=x_0}^{x(T)=x_T} \mathcal{D}_{\mathrm{E}}[\![x]\!] \exp\left(-\frac{S_{\mathrm{E}}[\![x]\!]}{\hbar}\right)\right]_{\mathcal{J}^{(\mathrm{loc})}_{\mathrm{FV/bounce}}} \mathrm{d}x_0 \mathrm{d}x_T\,.$$

Sequential semi-classical evaluation of all integrals involved	Rewriting the expression into a single composite path integral

Critical paths

Sequential evaluation

$$\int_{\mathbb{R}^2} \overline{\psi(x_T)}\psi(x_0)\left[\int_{x_0}^{x_T} \mathcal{D}[\![x]\!]\exp\left(-\frac{S_{\mathrm{E}}[\![x]\!]}{\hbar}\right)\right]\mathrm{d}x_0\mathrm{d}x_T$$

- Obtain the family $x_{\text{classical}}^{(x_0,x_T,T)}(\tau)$ of critical trajectories to the Dirichlet boundary conditions of the propagator
- Additional conditions on the endpoints $x_{0,T}^{(\text{crit})}$ from weight functions $\psi(x)$

Composite path integral

$$\int_{\mathcal{C}^\infty([0,T])} \mathcal{D}[\![x]\!]\,\overline{\psi[\![x(T)]\!]}\,\psi[\![x(0)]\!]\exp\left(-\frac{S_{\mathrm{E}}[\![x]\!]}{\hbar}\right)$$

- The stationarity condition of the full exponent reads
$$\frac{\delta}{\delta x(t)}\left[\psi_{\exp}^{(\text{loc})}[\![x(0)]\!]+\psi_{\exp}^{(\text{loc})}[\![x(T)]\!]+S_{\mathrm{E}}[\![x]\!]\right]\overset{!}{=}0$$
- $t=0,T$ yield endpoint restrictions

Bounce-like critical trajectory satisfies $E_{\text{crit}}=0$, thus one can show

$$\text{exponent} = \psi_{\exp}^{(\text{loc})}\left[x_0^{(\text{crit})}\right]+\psi_{\exp}^{(\text{loc})}\left[x_T^{(\text{crit})}\right]+S_{\mathrm{E}}\left[\!\!\left[x_{\text{bounce}}^{(T)}\right]\!\!\right]=S_{\mathrm{E}}\left[\!\!\left[x_{\text{bounce}}^{(T=\infty)}\right]\!\!\right].$$

Final result

[12] Gel'fand & Yaglom (1960), *J. Math. Phys.* vol. *1*(1)
[13] Kirsten & McKane (2004), *J. Phys. A* vol. *37*
[14] Levit, Negele, & Paltiel (1980), *PRC* vol. *22*(5)
[15] Weiss & Häffner (1983), *PRD* vol. *27*(12)

Sequential evaluation

$$\int_{\mathbb{R}^2} \overline{\psi(x_T)}\psi(x_0)\left[\int_{x_0}^{x_T} \mathcal{D}[\![x]\!] \exp\left(-\frac{S_{\mathrm{E}}[\![x]\!]}{\hbar}\right)\right] \mathrm{d}x_0 \mathrm{d}x_T$$

- We have three Gaussian integrations, thus three fluctuation factors.
- Soft and negative modes get traded between the different integrals.

Composite path integral

$$\int_{\mathcal{C}^\infty([0,T])} \mathcal{D}[\![x]\!]\, \overline{\psi[\![x(T)]\!]}\, \psi[\![x(0)]\!] \exp\left(-\frac{S_{\mathrm{E}}[\![x]\!]}{\hbar}\right)$$

- Only a single determinant factor **with altered boundary conditions**. This is easily encompassed by the Gel'fand-Yaglom theorem [12, 13].

Both procedures reproduce the known result [14, 15]

$$\Gamma_n = -\frac{2}{\hbar}\,\mathrm{Im}\left[E_n^{(\mathrm{loc})}\right] = \frac{1}{n!}\left(\frac{2m\omega\mathcal{A}^2}{\hbar}\right)^n \sqrt{\frac{m\omega^3\mathcal{A}^2}{\pi\hbar}}\, \exp\left\{-\frac{1}{\hbar}\, S_{\mathrm{E}}\left[\!\left[x_{\mathrm{bounce}}^{(T=\infty)}\right]\!\right]\right\}.$$

Concluding remarks

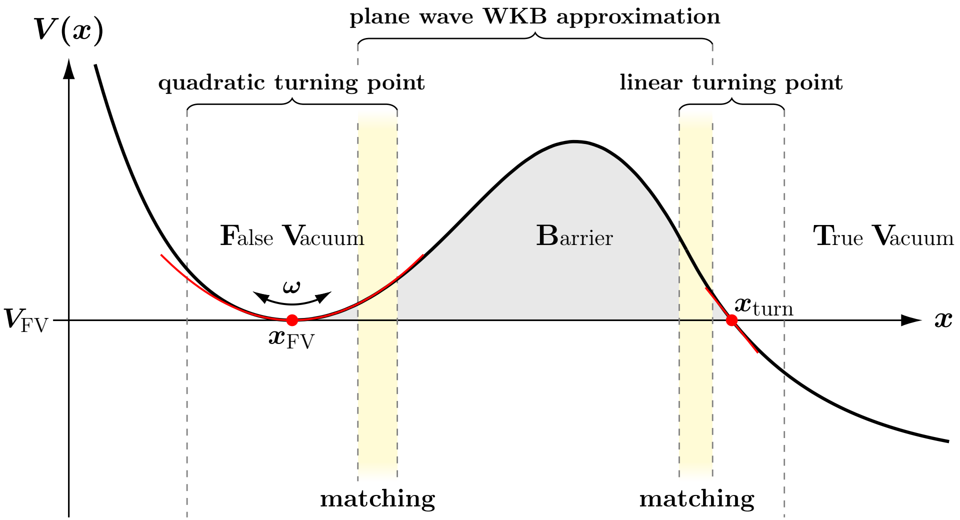
Key conclusion: Solving a composite path integral is in many situations advantageous to sequentially approximating the involved integrals.

The former computation should serve as a first step towards resolving the role of instantons in tunneling:

real time dynamics of a particle trapped inside a meta-stable FV region	$\xleftrightarrow[\textbf{?}]{\text{direct relation}}$	instanton solutions in **imaginary time**

Thanks for your attention!

Traditional WKB procedure



Direct method

Use the probability $P_{\rm TV}(t) \sim 1 - e^{-\Gamma t}$ of the particle to be located inside the true vacuum region to deduce

$$\Gamma = -\frac{\mathrm{d}}{\mathrm{d}t}\Big\{ \ln\Big[1 - P_{\rm TV}(t)\Big]\Big\} \approx \frac{P_{\rm TV}(t)}{t} \qquad \text{for } t \ll \Gamma^{-1} \, .$$

Now one can employ the (endpoint-weighted) path integral representation

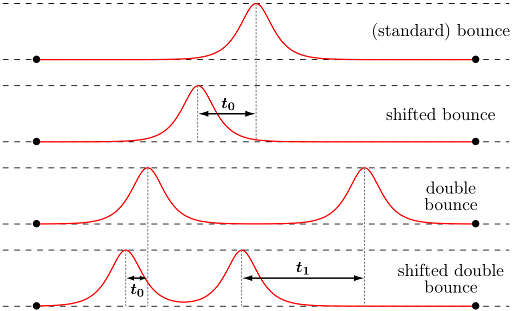
$$P_{\rm TV}(t) = \Bigg[\int_{\mathcal{C}^\infty([-\tau,\tau'])} \mathcal{D}_\theta[\![z]\!]\, \overline{\psi_{t=0}[\![z(\tau')]\!]}\, \psi_{t=0}[\![z(-\tau)]\!] \\ \times \exp\left(\frac{i S_\theta[\![z]\!]}{\hbar}\right) \Theta\Big[\![z(0) - x_{\rm turn}\Big]\!]\Bigg]_{\substack{\tau,\tau' > 0 \\ \tau = +e^{i\theta} t \\ \tau' = -e^{i\theta} t}} .$$

What makes the bounce special?

The bounce entails three caveats, two of them are tied to the fluctuation determinant

$$\det_\zeta\left\{-\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}+\frac{V''\left[x_{\text{bounce}}(\tau)\right]}{m}\right\}^{-\frac{1}{2}} \overset{\zeta}{=} \prod_{\mu=0}^{\infty} \lambda_\mu^{-\frac{1}{2}} \,.$$

1. The smallest eigenvalue is negative.
2. The next-larger eigenvalue is positive, but exponentially small $\lambda \sim e^{-\omega T/2}$.
   $\longrightarrow$ related to approximate time-translation symmetry
3. For large times T, multi-bounce configurations become dominant.



Critical paths

In both cases, the conditions for the endpoints $x_{0,T}^{(\mathrm{crit})}$ evaluate to

$$\underbrace{\begin{aligned}\sqrt{2mV\left[x_0^{(\mathrm{crit})}\right]} \\ \sqrt{2mV\left[x_T^{(\mathrm{crit})}\right]}\end{aligned}}_{\text{wave function}} \begin{aligned}&\overset{!}{=} \\ &\overset{!}{=}\end{aligned} \underbrace{\begin{aligned}&\mathrm{sign}\left[\dot{x}^{(\mathrm{crit})}(0)\right]\sqrt{2m\left\{V\left[x_0^{(\mathrm{crit})}\right]+E_{\mathrm{crit}}\right\}}\,, \\ -&\mathrm{sign}\left[\dot{x}^{(\mathrm{crit})}(T)\right]\sqrt{2m\left\{V\left[x_T^{(\mathrm{crit})}\right]+E_{\mathrm{crit}}\right\}}\,.\end{aligned}}_{\text{derivative of } S_{\mathrm{E}}(x_0,x_T,T)}$$

Any critical path $x_{\mathrm{crit}}(\tau)$ immediately requires $E_{\mathrm{crit}} \overset{!}{=} 0$. Moreover, there are two cases:

1. **Path has no turning point**
   Then the only solution is $x(\tau) = x_{\mathrm{FV}}$, i.e. the trivial false vacuum trajectory.
2. **Path has a single turning point (necessarily x_{turn})**
   Critical trajectories are bounce-like paths, possessing an exact zero-mode.

Exponent for bounce-like trajectories

An important observation at this point is that the full exponent evaluated on the one-parameter family of bounce solutions is:

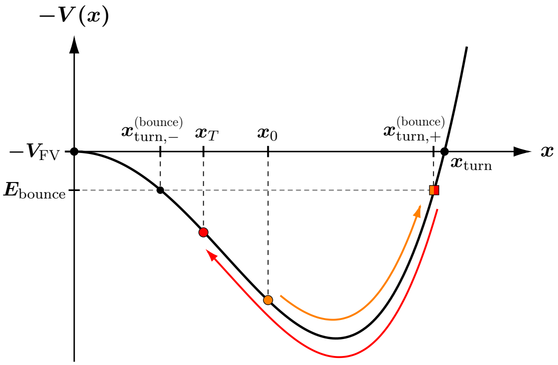
$$\text{exponent} = \psi^{(\text{loc})}_{\text{exp}}\left[x_0^{(\text{crit})}\right] + \psi^{(\text{loc})}_{\text{exp}}\left[x_T^{(\text{crit})}\right] + S_{\text{E}}\left[x_0^{(\text{crit})}, x_T^{(\text{crit})}, T\right]$$

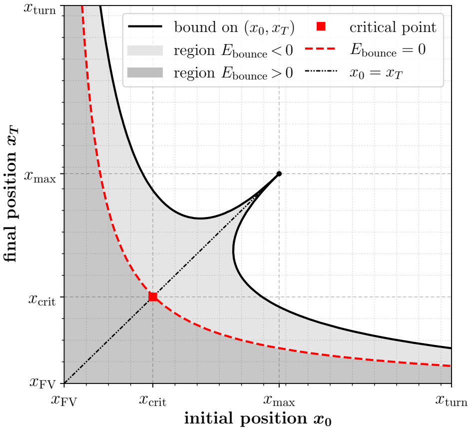
$$= \int_0^{x_0^{(\text{crit})}} \sqrt{2mV(\xi)}\,\mathrm{d}\xi + \int_{x_0^{(\text{crit})}}^{x_{\text{turn}}} \sqrt{2mV(\xi)}\,\mathrm{d}\xi + \left(x_0^{(\text{crit})} \leftrightarrow x_T^{(\text{crit})}\right)$$

$$= 2\int_0^{x_{\text{turn}}} \sqrt{2mV(\xi)}\,\mathrm{d}\xi = S_{\text{E}}\left[\!\left[x^{(T=\infty)}_{\text{bounce}}\right]\!\right] .$$

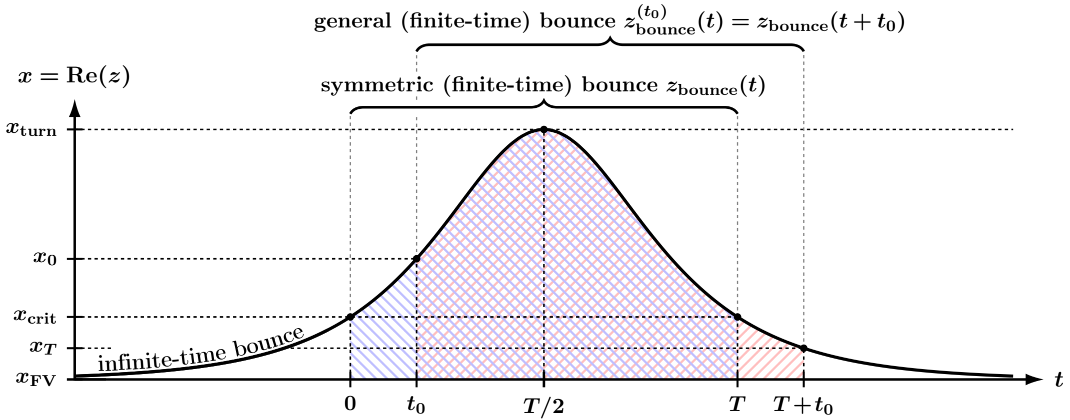
This ensures the correct exponential suppression for arbitrary parameter T.

Bounce-like trajectories (illustrations)





Collective coordinate bounce



Ways to compute a determinant

For homogeneous Dirichlet boundary conditions $e_\mu(0) = e_\mu(T) = 0$ one can employ the formulas:

$$\det_\zeta\left\{-\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} + \frac{V''\left[x_{\min}(\tau)\right]}{m}\right\} = 2\kappa(0)\kappa(T)\int_0^T \frac{1}{\kappa(\tau)^2}\,\mathrm{d}\tau \qquad \text{Shifting method}$$

$$= -2m\left[\frac{\partial^2 S_{\mathrm{E}}\left(x_0, x_T, T\right)}{\partial x_0 \partial x_T}\right]^{-1} \qquad \text{Van Vleck-determinant}$$

$$= 2y^{(2)}_{\lambda=0}(T) \qquad \text{Gel'fand Yaglom}$$

The Gel'fand Yaglom method can be generalized to deal with eigenvalue problems possessing generalized boundary conditions

$$\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}\begin{pmatrix} e_\mu(0) \\ p(0)\,\dot{e}_\mu(0) \end{pmatrix} + \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix}\begin{pmatrix} e_\mu(T) \\ p(T)\,\dot{e}_\mu(T) \end{pmatrix} = 0\,.$$