α from ALPHA

Rainer Sommer

Precision "Measurements" of α_s, February 2011, München

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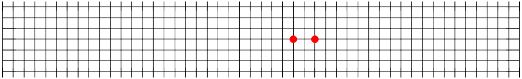
- This applies to the use of **perturbation theory**

  Alternative [Lüscher, Weisz, Wolff;]:
  Connect low and high energies non-perturbatively (lattice)

  but there is a particular lattice issue ...

A particular lattice issue (α_{qq} as an example)



$$\underset{\uparrow}{L} \gg \underset{\uparrow}{\frac{1}{0.2\mathrm{GeV}}} \gg \frac{1}{\mu} \sim \frac{1}{10\mathrm{GeV}} \gg \underset{\uparrow}{a}$$

box size confinement scale, m_π spacing

$$\Downarrow$$

$$L/a \gg 50$$

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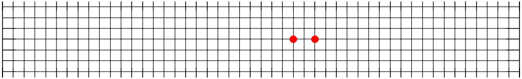

$$\begin{array}{ccccccccc} L & \gg & \frac{1}{0.2\mathrm{GeV}} & \gg & \frac{1}{\mu} & \sim & \frac{1}{10\mathrm{GeV}} & \gg & a \\ \uparrow & & \uparrow & & & & & & \uparrow \\ \text{box size} & & \text{confinement scale, } m_\pi & & & & & & \text{spacing} \end{array}$$

$$\Downarrow$$

$$L/a \gg 50$$

Solution: $\boxed{L = 1/\mu}$ $\longrightarrow$ left with $L/a \gg 1$ [Lüscher, Weisz, Wolff]

Finite size effect as a physical observable; finite size scaling!

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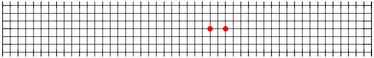

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HPQCD
solution: $\boxed{a \sim 1/\mu}$

cutoff effects and perturbative corrections are intertwined
bare perturbation theory (PT at the cutoff) [HPQCD, Phys.Rev.Lett. 95 (2005) 052002]

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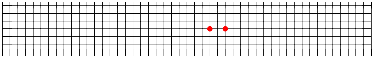

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HPQCD solution: $\boxed{a \sim 1/\mu}$

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- $\log W_{12} = -5.551\alpha_V(3/a) \times [1 - 0.86\alpha + 1.72\alpha^2 \underbrace{-5(2)}_{\uparrow}\alpha^3 \underbrace{-1(2)}_{\uparrow}\alpha^4 + \ldots]$

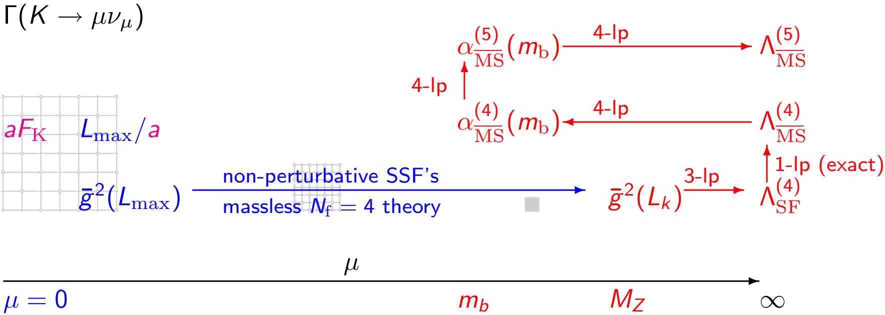
  fitted

  $\alpha = 0.21 \ldots 0.29$
- $\log\left(\frac{W_{13}}{W_{22}}\right) = -1.323\,\alpha(1.21/a)\times[1-0.39(1)\,\alpha+0.3(2)\,\alpha^2-2(1)\,\alpha^3+0(2)\,\alpha^4 \ \ldots]$

  $\alpha = 0.33 \ldots 0.68$

The master formula of the ALPHA Collaboration strategy

$$\frac{\Lambda^{(5)}_{\overline{\rm MS}}}{F_{\rm K}} = \frac{1}{F_{\rm K}L_{\rm max}} \times \frac{L_{\rm max}}{L_k} \times L_k\Lambda^{(4)}_{\overline{\rm MS}} \times \frac{\Lambda^{(5)}_{\overline{\rm MS}}}{\Lambda^{(4)}_{\overline{\rm MS}}}$$



$\Gamma(K \to \mu\nu_\mu)$

$aF_{\rm K}$ $\quad L_{\rm max}/a$

$$\bar{g}^2(L_{\rm max}) \xrightarrow[\text{massless } N_{\rm f}=4 \text{ theory}]{\text{non-perturbative SSF's}} \bar{g}^2(L_k) \xrightarrow{\text{3-lp}} \Lambda^{(4)}_{\rm SF}$$

$$\begin{tikzcd}
\alpha^{(5)}_{\overline{\rm MS}}(m_{\rm b}) \arrow[r, "\text{4-lp}"] & \Lambda^{(5)}_{\overline{\rm MS}} \\
\alpha^{(4)}_{\overline{\rm MS}}(m_{\rm b}) \arrow[u, "\text{4-lp}"] & \Lambda^{(4)}_{\overline{\rm MS}} \arrow[l, "\text{4-lp}"'] \\
 & \Lambda^{(4)}_{\rm SF} \arrow[u, "\text{1-lp (exact)}"']
\end{tikzcd}$$

μ

$\mu = 0$ $\quad m_b$ $\quad M_Z$ $\quad \infty$

The master formula of the ALPHA Collaboration strategy

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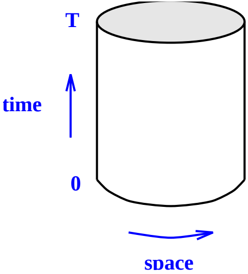
- simple strategy
- minimal assumptions
  - continuum limit of lattice theory
  - asymptotic freedom; NP "corrections" vanish quickly as $\mu \to \infty$
  - simulations are correct
- practical difficulties and limitations
  $\longrightarrow$ need to be a bit more technical

Definition of $\bar{g}_{\rm SF}$



$$\begin{aligned}
\exp\{-\Gamma\} &= \int \mathrm{D}[U,\overline{\psi},\psi]\exp\left\{-S[U,\overline{\psi},\psi]\right\} \\
U(x,k)|_{x_0=0} &= \exp\{aC_k(\eta)\}, \\
U(x,k)|_{x_0=T} &= \exp\{aC'_k(\eta)\}
\end{aligned}$$

$$\Gamma' = \left.\frac{\partial\Gamma}{\partial\eta}\right|_{\eta=0} = \frac{k}{\bar{g}^2(L)} \propto \langle\, F_{0k}|_{\rm boundary}\rangle$$

Why so complicated?

- leading term in observable $\propto \alpha$
- one can simulate massless quarks (gap in $\not{D}$)
- 3-loop β-function has been computed in the SF-scheme
- good statistical precision
- ◦ (non-perturbative way to introduce a background field)

The step scaling function

- … is a discrete *beta* function:

$$\sigma(s,\bar{g}^2(L)) = \bar{g}^2(sL) \qquad \text{mostly } s=2$$

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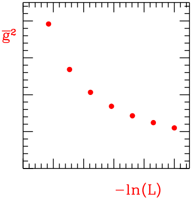
$$\sigma(s, \bar{g}^2(L)) = \bar{g}^2(sL) \qquad \text{mostly } s = 2$$

- determines the non-perturbative running:



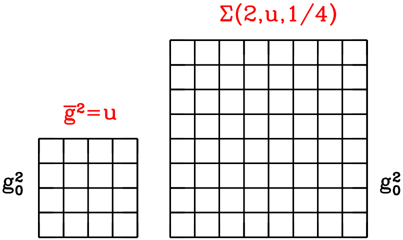
$$\begin{array}{rcl} u_0 & = & \bar{g}^2(L_{\max}) \\ & \downarrow & \\ \sigma(2, u_{k+1}) & = & u_k \\ & \downarrow & \\ u_k & = & \bar{g}^2(2^{-k} L_{\max}) \end{array}$$

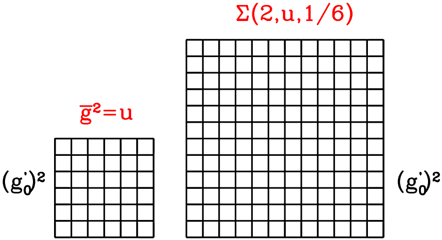
The step scaling function: $\sigma(s,u) = \bar{g}^2(sL)$ with $u = \bar{g}^2(L)$

On the lattice:
additional dependence on the resolution a/L

g_0 fixed, L/a fixed:

$$\begin{aligned} \bar{g}^2(L) = u, &\qquad \bar{g}^2(sL) = u'\,, \\ \Sigma(s,u,a/L) \quad = &\quad u' \end{aligned}$$





continuum limit:

$$\Sigma(s,u,a/L) = \sigma(s,u) + \mathrm{O}(a/L)$$

in the following always $s = 2$

everywhere: $m = 0$ (PCAC mass defined in $(L/a)^4$ lattice)

Improvement: very important for $\bar{g}^2_{\rm SF}$ [ALPHA]

- Standard $\mathrm{O}(a)$ improvement: $c_{\rm sw}(g_0)$
  $N_{\rm f} = 0$ [Lüscher, Sint, S, Weisz, '95], $N_{\rm f} = 2$ [Jansen, S], $N_{\rm f} = 4$ [Tekin, S., Wolff, 2009]
- boundary $\mathrm{O}(\mathrm{a})$-terms, e.g. $c_t(g_0) a \int \mathrm{d}^3\mathbf{x} F_{0k} F_{0k}$ at $x_0 = 0$ and $x_0 = T$
  $c_{\rm t}$ to two-loops for Wilson gauge [Bode, Weisz & Wolff, 1999]
  check of remaining uncertainty for $N_{\rm f} = 2$

  !
- remaining cutoff effects of SSF:

$$\delta(u, a/L) = \frac{\Sigma(u,a/L)-\sigma(u)}{\sigma(u)} = \delta_1(a/L)u + \delta_2(a/L)u^2 + \ldots$$

$$\delta_1(a/L) = \delta_{10}(a/L) + \delta_{11}(a/L)N_{\rm f}$$ [ALPHA Collaboration, 1993-1997]

$$\delta_2(a/L) = \delta_{20}(a/L) + \delta_{21}(a/L)N_{\rm f} + \delta_{22}(a/L)N_{\rm f}^2$$ [Bode, Weisz & Wolff, 1999]

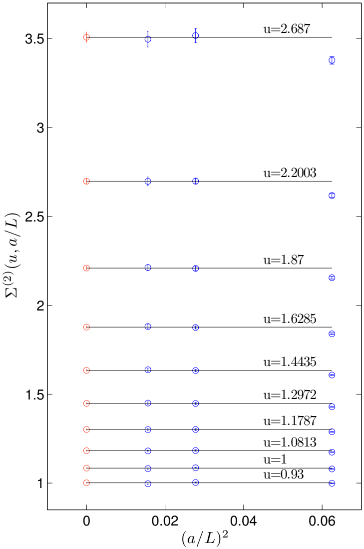
seen to be small

- Observable improvement
  [De Divitiis et al., 1993]
  improved step scaling function:

$$\frac{\Sigma(u, a/L)}{1 + \delta_1(a/L)u + \delta_2(a/L)u^2} = \sigma(u) + \mathrm{O}(u^4\, a/L)$$

	Wilson gauge		Iwasaki gauge	
L/a	δ_1	δ_2	δ_1	δ_2
4	−0.0103	−0.0001	−0.0351	0.0137
6	−0.0042	−0.0041	−0.0291	0.0083
8	−0.0021	−0.0030	−0.0224	0.0049

Continuum limit ($N_\mathrm{f} = 4$)



- *Constant fit:*

  $$\Sigma^{(2)}(u,a/L) = \sigma(u)$$

  for $L/a = 6,8$

- *Global fit:*

  $$\Sigma^{(2)}(u,a/L) = \sigma(u) + \rho\, u^4\, (a/L)^2$$

  for $L/a = 6,8$
  $\rightarrow \rho = 0.007(85)$

- $L/a = 8$ *data:*

  $$\sigma(u) = \Sigma^{(2)}(u,1/8)$$

Continuum SSF

u		$\sigma(u)$	
	constant fit	global fit	$L/a = 8$ data
0.9300	1.002 (3)	1.002 (3)	0.997 (5)
1.0000	1.084 (3)	1.084 (3)	1.081 (4)
1.0813	1.182 (3)	1.182 (4)	1.181 (5)
1.1787	1.301 (4)	1.301 (5)	1.301 (6)
1.2972	1.448 (5)	1.448 (7)	1.450 (7)
1.4435	1.634 (5)	1.634(10)	1.637 (8)
1.6285	1.877 (7)	1.877(16)	1.880(11)
1.8700	2.209(10)	2.207(27)	2.212(17)
2.2003	2.698(14)	2.694(49)	2.697(24)
2.6870	3.507(30)	3.50 (10)	3.496(44)

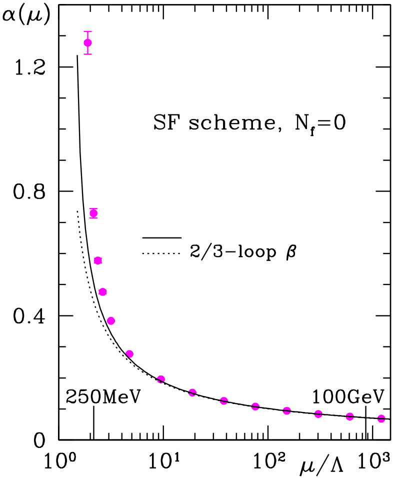
↑ **result**

Recursive reconstruction of $\bar{g}_{\mathrm{SF}}(L)$

$$\begin{aligned} u_i &\equiv \bar{g}^2\left(L_{\mathrm{max}}/2^i\right) \\ u_i &= \sigma(u_{i+1}), \quad i=0,\ldots,n, \quad u_0 = u_{\mathrm{max}} = \bar{g}^2\left(L_{\mathrm{max}}\right), \end{aligned}$$

solve for u_{i+1}, $i = 0 \ldots n = 10$

Non-perturbative running of α in the SF scheme



[ALPHA Collaboration , 2001]

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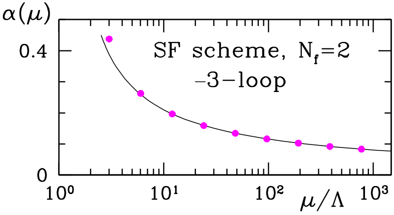

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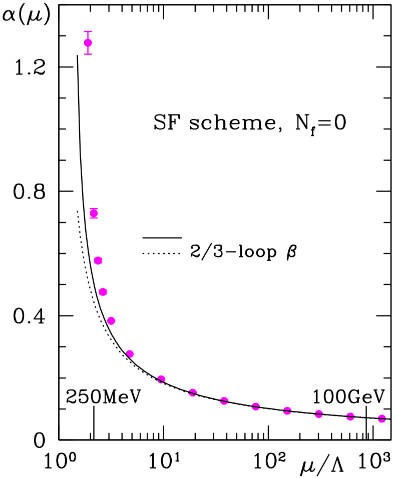

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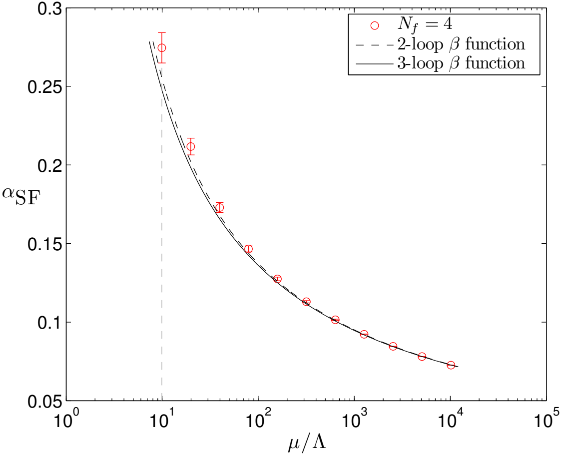


[ALPHA Collaboration , 2010]

energies μ

$0.9\,\mathrm{GeV}$

.

.

.

$900\,\mathrm{GeV}$

Lambda parameter

for large i, small $u_i = \bar{g}^2(L_i)$:

$$L_i\Lambda = \left[b_0\bar{g}^2(L_i)\right]^{-\frac{b_1}{2b_0^2}} \exp\left\{-\frac{1}{2b_0\bar{g}^2(L_i)}\right\} \times$$
$$\exp\left\{-\int_0^{\bar{g}(L_i)} dx \left[\frac{1}{\beta_{3-\mathrm{loop}}(x)} + \frac{1}{b_0x^3} - \frac{b_1}{b_0^2 x}\right]\right\}$$

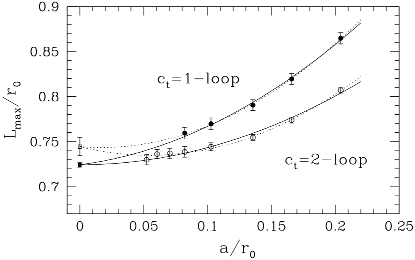
	constant fit		global fit		$L/a = 8$ data	
i	u_i	$\ln(\Lambda L_{\rm max})$	u_i	$\ln(\Lambda L_{\rm max})$	u_i	$\ln(\Lambda L_{\rm max})$
0	3.45	−2.028	3.45	−2.028	3.45	−2.028
1	2.660(14)	−2.074(17)	2.666(46)	−2.066(56)	2.660 (21)	−2.073(26)
2	2.173(13)	−2.117(24)	2.179(45)	−2.105(83)	2.173 (20)	−2.116(37)
3	1.842(11)	−2.155(28)	1.847(37)	−2.141(97)	1.842 (17)	−2.153(44)
4	1.6013(90)	−2.188(32)	1.606(30)	−2.17 (10)	1.602 (14)	−2.185(50)
5	1.4187(78)	−2.217(35)	1.422(25)	−2.20 (11)	1.419 (13)	−2.213(56)
6	1.2748(70)	−2.241(39)	1.278(20)	−2.23 (11)	1.275 (11)	−2.238(63)
7	1.1583(63)	−2.263(43)	1.161(17)	−2.25 (12)	1.159 (10)	−2.259(70)
8	1.0620(58)	−2.282(47)	1.064(15)	−2.27 (12)	1.0626(95)	−2.278(76)
9	0.9809(53)	−2.299(50)	0.982(13)	−2.29 (12)	**0.9815(87)**	**−2.294(83)**
10	0.9117(49)	−2.315(54)	0.913(11)	−2.30 (12)	0.9122(81)	−2.309(89)

A future error budget: 1. Where is the limit?

The only real source of errors are discretization effects: $\propto a,\ a^2$

note: CPU-time $\propto a^{-6}$ (estimated)

Continuum extrapolations with $c_{\rm t} = $ 2-loop have always looked excellent (Wilson gauge action), but the difference to 1-loop is not always that small ...



Here $N_{\rm f} = 0$

(same as $L_{\rm max} F_{\rm K}$)

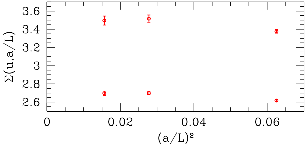
Estimate that the limit is at a similar precision: **2–3** %. (seems conservative)

For the SSF's we have not found significant differences between 1-loop and 2-loop c_t. This can and should be checked more precisely in the future.

A future error budget: 2. Plans ($N_\mathrm{f} = 4$) until 2014

- replace

$$L/a = 4, 6, 8$$
$$\rightarrow 8, 12, 16$$



by

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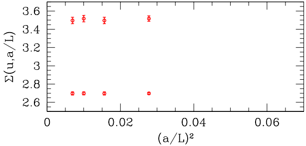

- double statistics

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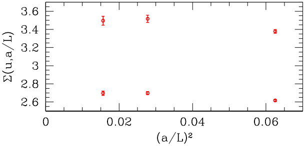

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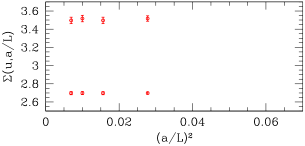

- double statistics
- tests of universality / increased precision with different discretizations?
  - staggered
  - chirally twisted boundary conditions

3. Estimated error budget

2014

	error sources		$\delta\Lambda/\Lambda$	Reference	machine
now	running	$N_\mathrm{f} = 4$	0.07	[S, Tekin, Wolff]	2TFlop / 2
future	running	$N_\mathrm{f} = 4$	0.04	[Marinkovic et al]	500TFlop / 10
now	$L_\mathrm{max} F_\mathrm{K}$	$N_\mathrm{f} = 2+1+1$	unknown		
	$L_\mathrm{max} F_\mathrm{K}$	$N_\mathrm{f} = 0$	0.02	[Necco, S; Garden et al.]	
	$L_\mathrm{max} F_\mathrm{K}$	$N_\mathrm{f} = 2$	$\approx$ 0.04	[ALPHA Collaboration ,CLS]	1000 TFlop / 50
future	$L_\mathrm{max} F_\mathrm{K}$	$N_\mathrm{f} = 2+1+1$	0.03	make use of work by others ?	
	$N_\mathrm{f} = 4 \longrightarrow N_\mathrm{f} = 5$		0.02	[Bernreuther & Wetzel; . . .] [Chetyrkin,Kühn,Sturm]	PT, 4-loop
	$\mathrm{O}(\alpha_{em})$		0.01		

- A total error of $\delta\Lambda/\Lambda = 0.05$ is achievable in a few years
- with minimal assumptions
  - nature of continuum limit
  - perturbative conversion $N_\mathrm{f} = 4 \longrightarrow N_\mathrm{f} = 5$ [some NP check ?]
- with conservative error estimates
- most difficult to predict is the progress in $L_\mathrm{max} F_\mathrm{K}$ (large volume $N_\mathrm{f} = 2+1+1$, tuning $m_\mathrm{s}, m_\mathrm{c}$ to their physical values)

4. Estimated error budget

for 2025

predictions become harder

	error sources		$\delta\Lambda/\Lambda$	Reference	machine
future	running	$N_\mathrm{f} = 4$	0.02	[?]	50000TFlop / 10
future	$L_\mathrm{max} F_\mathrm{K}$	$N_\mathrm{f} = 2+1+1$	0.01	[?]	
	$N_\mathrm{f} = 4 \longrightarrow N_\mathrm{f} = 5$		0.02	[Bernreuther & Wetzel; ...]	PT
	$\mathrm{O}(\alpha_{em})$		0.005		
	total		0.030		

- $\delta\alpha(M_Z) = 0.00045$
- no innovations assumed

For those interested: some numbers ($N_\mathrm{f} < 4$)

- N_f dependence of $\Lambda_{\overline{\mathrm{MS}}} r_0$ and comparison to phenomenology ($r_0 = 0.475\,\mathrm{fm}$)

N_f:	0	2	4	5
[“ALPHA Collaboration”, 2005]	0.60(5)	0.62(4)(4)		
[Bethke, L&L 2004] “experiment”			0.71(10)	0.52(8)
[Blümlein,Böttcher,Guffanti, L&L 2004] DIS NNLO			0.54(8)	

Here ALPHA Collaboration came from the use of r_0/a for $N_\mathrm{f} = 2$ from a different collaboration.

A calculation by ourselves gives [Knechtli & Leder 2010]

N_f:	0	2	4	5
[ALPHA Collaboration, 2010]	0.60(5)	0.73(3)(4)		
[Bethke, L&L 2004] “experiment”			0.71(10)	0.52(8)
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