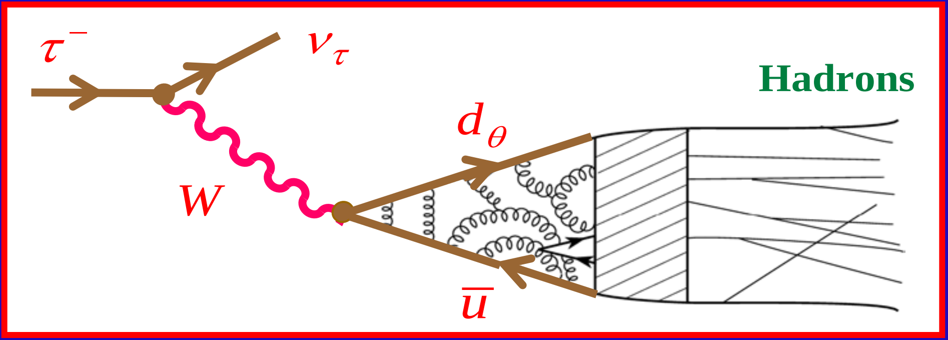
α_s Determination from Hadronic τ Decays

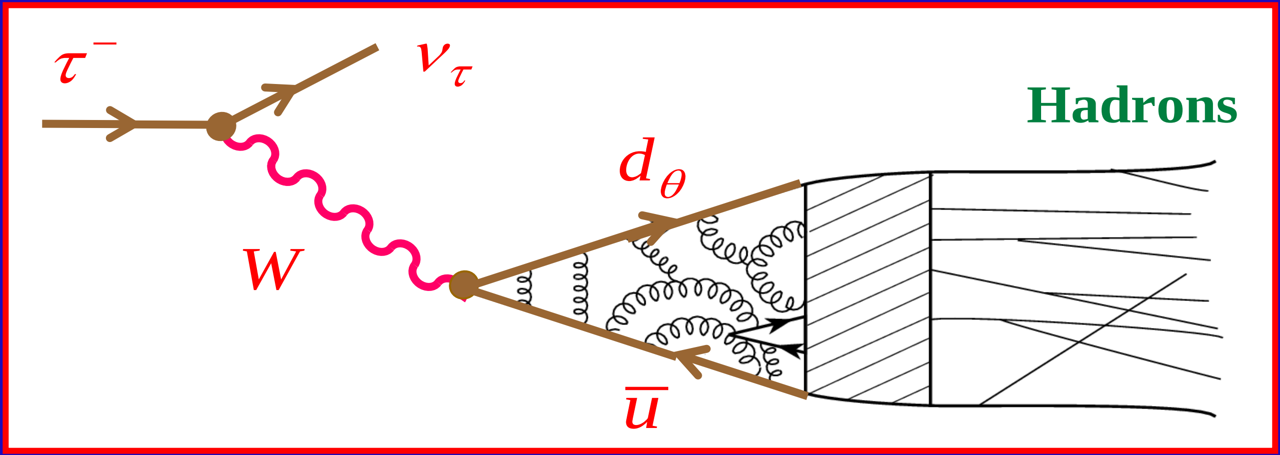
A. Pich

IFIC, Valencia



Workshop on Precision Measurements of α_s
MPI Munich, Germany, 9-11 February 2011

HADRONIC TAU DECAY

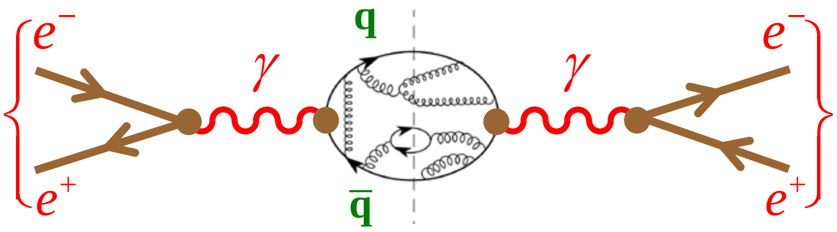


$$d_\theta = V_{ud}\; d + V_{us}\; s$$

Only lepton massive enough to decay into hadrons

$$R_\tau \equiv \frac{\Gamma\left(\tau^- \to \nu_\tau + \text{Hadrons}\right)}{\Gamma\left(\tau^- \to \nu_\tau\, e^-\, \overline{\nu}_e\right)} \approx N_C \qquad ; \qquad R_\tau = \frac{1 - B_e - B_\mu}{B_e} = 3.6291 \pm 0.0086$$

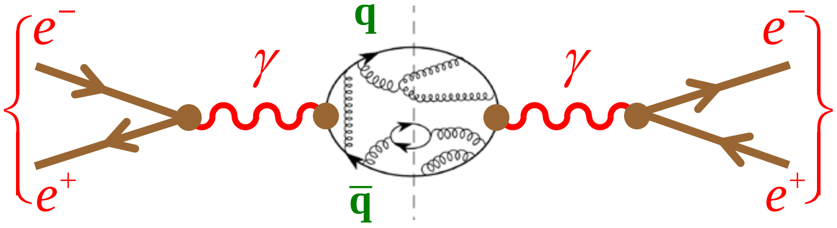
$\sigma \sim \mathrm{Im}$



$$\frac{\sigma(e^+e^- \to \mathrm{had})}{\sigma(e^+e^- \to \mu^+\mu^-)} = 12\pi \,\mathrm{Im}\,\Pi_{\mathrm{em}}(s)$$

$$\Pi^{\mu\nu}_{\mathrm{em}}(q) \equiv i \int d^4x \; e^{iqx} \,\langle 0 | T[J^{\mu}_{\mathrm{em}}(x)\, J^{\nu}_{\mathrm{em}}(0)] | 0 \rangle = \left(-g^{\mu\nu} q^2 + q^{\mu} q^{\nu}\right) \Pi_{\mathrm{em}}(q^2)$$

$\sigma \sim \mathrm{Im}$



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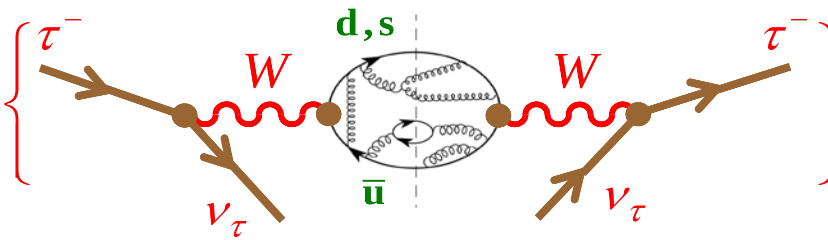
$\Gamma_{\tau\to\nu_\tau+\mathrm{had}} \sim \mathrm{Im}$

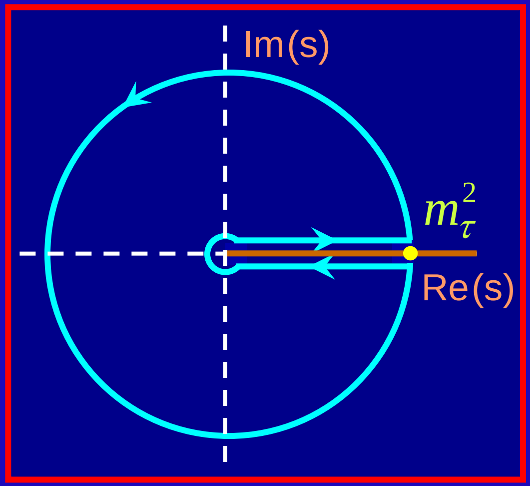


$$R_\tau \equiv \frac{\Gamma(\tau^-\to\nu_\tau+\mathrm{had})}{\Gamma(\tau^-\to\nu_\tau e^-\bar{\nu}_e)} = 12\pi\int_0^{m_\tau^2} dx\left(1-\frac{s}{m_\tau^2}\right)^2\left[\left(1+2\frac{s}{m_\tau^2}\right)\mathrm{Im}\,\Pi^{(1)}(s)+\mathrm{Im}\,\Pi^{(0)}(s)\right]$$

$$\Pi^{(J)}(s) \equiv |V_{ud}|^2\left[\Pi^{(J)}_{ud,V}(s)+\Pi^{(J)}_{ud,A}(s)\right] + |V_{us}|^2\left[\Pi^{(J)}_{us,V}(s)+\Pi^{(J)}_{us,A}(s)\right]$$

$$\Pi^{\mu\nu}_{ij,J}(q) \equiv i\int d^4x\ e^{iqx}\,\langle 0|T[J^{\mu}_{ij}(x)\,J^{\nu}_{ij}(0)^{\dagger}]|0\rangle = \left(-g^{\mu\nu}q^2+q^{\mu}q^{\nu}\right)\Pi^{(1)}_{ij,J}(q^2)+q^{\mu}q^{\nu}\,\Pi^{(0)}_{ij,J}(q^2)$$

$$R_\tau \equiv \frac{\Gamma(\tau^- \to \nu_\tau + \text{had})}{\Gamma(\tau^- \to \nu_\tau e^- \bar{\nu}_e)} = 12\pi \int_0^1 dx\,(1-x)^2 \left[(1+2x)\,\text{Im}\,\Pi^{(1)}(x m_\tau^2) + \text{Im}\,\Pi^{(0)}(x m_\tau^2)\right]$$



$$R_\tau = 6\pi i \oint_{|x|=1} dx\,(1-x)^2 \left[(1+2x)\,\Pi^{(0+1)}(x m_\tau^2) - 2x\,\Pi^{(0)}(x m_\tau^2)\right]$$

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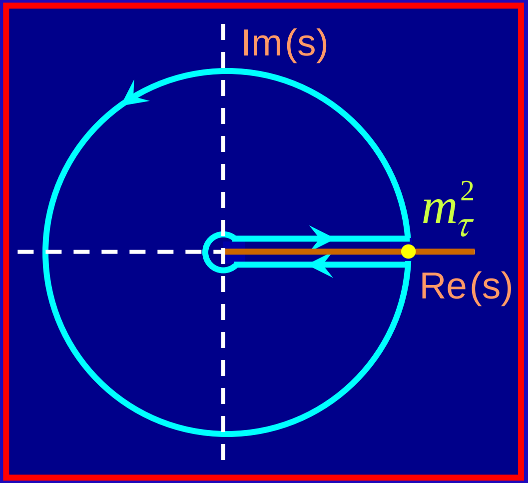

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$$\Pi^{(J)}(s) = \sum_{D=2n} \frac{C_D^{(J)}(s,\mu)\,\langle O_D(\mu)\rangle}{(-s)^{D/2}}$$

OPE

Braaten-Narison-Pich

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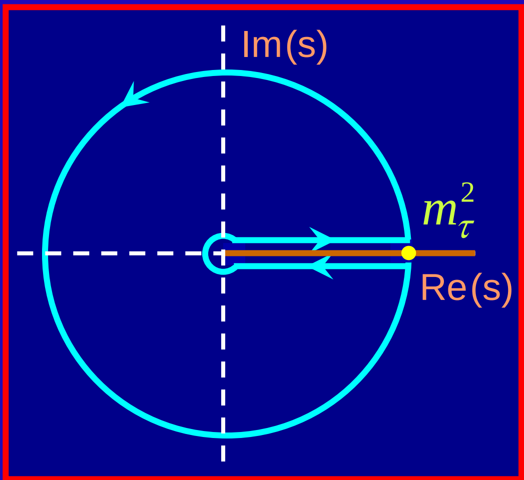

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OPE

$$R_\tau = N_C\, S_{\mathrm{EW}} \left(1 + \delta_{\mathrm{P}} + \delta_{\mathrm{NP}}\right) = R_{\tau,V} + R_{\tau,A} + R_{\tau,S}$$

$S_{\mathrm{EW}} = 1.0201\,(3)$; $\delta_{\mathrm{NP}} = -0.0059 \pm 0.0014$

Marciano-Sirlin, Braaten-Li, Erler

Fitted from data (Davier et al)

$\delta_{\mathrm{P}} = a_\tau + 5.20\, a_\tau^2 + 26\, a_\tau^3 + \ldots \approx 20\%$; $a_\tau \equiv \alpha_s(m_\tau)/\pi$

Perturbative: $(m_q=0)$

$K_4 = 49.07570$ (Baikov-Chetyrkin-Kühn '08)

$$-s\,\frac{d}{ds}\,\Pi^{(0+1)}(s) = \frac{1}{4\pi^2}\sum_{n=0} K_n \left(\frac{\alpha_s(-s)}{\pi}\right)^n \quad ; \quad K_0 = K_1 = 1 \quad , \quad K_2 = 1.63982 \quad , \quad K_3 = 6.37101$$

$$\delta_P = \sum_{n=1} K_n\, A^{(n)}(\alpha_s) = a_\tau + 5.20\; a_\tau^2 + 26\; a_\tau^3 + 127\; a_\tau^4 + \cdots$$

Le Diberder- Pich '92

$$A^{(n)}(\alpha_s) \equiv \frac{1}{2\pi i}\oint_{|x|=1}\frac{dx}{x}\left(1-2x+2x^3-x^4\right)\left(\frac{\alpha_s(-s)}{\pi}\right)^n = a_\tau^n + \cdots \quad ; \quad a_\tau \equiv \alpha_s(m_\tau)/\pi$$

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Power Corrections:

$$\Pi^{(0+1)}_{\mathrm{OPE}}(s) \approx \frac{1}{4\pi^2}\sum_{n\ge 2}\frac{C_{2n}\langle O_{2n}\rangle}{(-s)^n}$$

Braaten-Narison-Pich '92

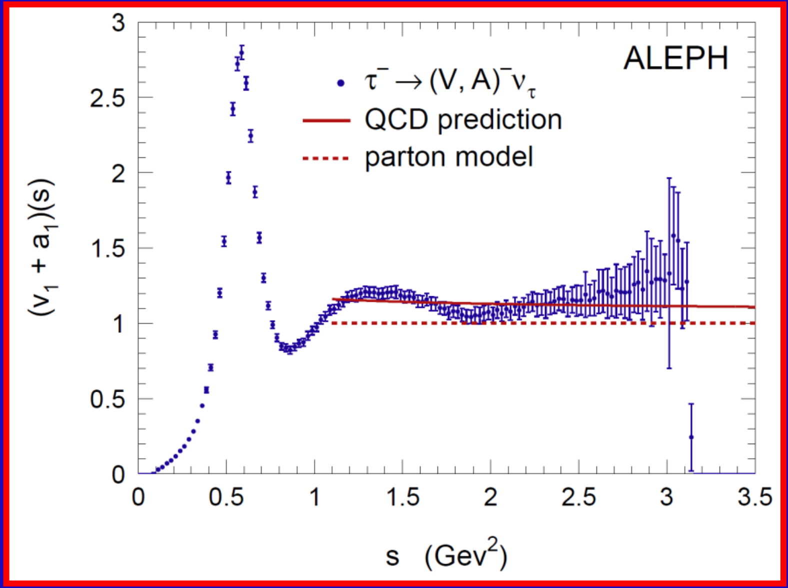
$$C_4\langle O_4\rangle \approx \frac{2\pi}{3}\langle 0|\alpha_s G^{\mu\nu}G_{\mu\nu}|0\rangle$$

$$\delta_{\mathrm{NP}} \approx \frac{-1}{2\pi i}\oint_{|x|=1} dx\;(1-3x^2+2x^3)\sum_{n\ge 2}\frac{C_{2n}\langle O_{2n}\rangle}{(-x m_\tau^2)^n} = -3\,\frac{C_6\langle O_6\rangle}{m_\tau^6} - 2\,\frac{C_8\langle O_8\rangle}{m_\tau^8}$$

Suppressed by m_τ^6 [additional chiral suppression in $C_6\langle O_6\rangle^{V+A}$]

Similar predictions for $R_{\tau,V}$, $R_{\tau,A}$, $R_{\tau,S}$ and the moments

$$R_\tau^{kl}(s_0) \equiv \int_0^{s_0} ds \left(1-\frac{s}{s_0}\right)^k \left(\frac{s}{m_\tau^2}\right)^l \frac{dR_\tau}{ds}$$



Sensitivity to power corrections through k , l

The non-perturbative contribution to R_τ can be obtained from the invariant-mass distribution of the final hadrons:

$$\delta_{NP} = -0.0059 \pm 0.0014$$

Davier et al. (ALEPH data)

Recent $\alpha_s(m_\tau)$ Analyses

Reference	Method	δ_P	$\alpha_s(m_\tau)$	$\alpha_s(m_Z)$
Baikov et al	CIPT, FOPT	0.1998 (43)	**0.332 (16)**	**0.1202 (19)**
Davier et al	CIPT	0.2066 (70)	**0.344 (09)**	**0.1212 (11)**
Beneke-Jamin	BSR + FOPT	0.2042 (50)	**0.316 (06)**	**0.1180 (08)**
Maltman-Yavin	PWM + CIPT	–	**0.321 (13)**	**0.1187 (16)**
Menke	CIPT, FOPT	0.2042 (50)	**0.342 (11)**	**0.1213 (12)**
Narison	CIPT, FOPT	–	**0.324 (08)**	**0.1192 (10)**
Caprini-Fischer	BSR + CIPT	0.2042 (50)	**0.321 (10)**	–
Cvetič et al	β_{exp} + CIPT	0.2040 (40)	**0.341 (08)**	**0.1211 (10)**
Pich	**CIPT**	**0.1997 (35)**	**0.338 (12)**	**0.1209 (14)**

CIPT: Contour-improved perturbation theory
FOPT: Fixed-order perturbation theory
BSR: Borel summation of renormalon series
CIPTm: Modified CIPT (conformal mapping)
βexp: Expansion in derivatives of the coupling (β function)
PWM: Pinched-weight moments

Perturbative Uncertainty on $\alpha_s(m_\tau)$

$$-s \frac{d}{ds} \Pi^{(0+1)}(s) = \frac{1}{4\pi^2} \sum_{n=0} K_n \; a(-s)^n$$

$$\delta_P = \underbrace{\sum_{n=1} K_n \; A^{(n)}(\alpha_s)}_{\text{CIPT}} = \underbrace{\sum_{n=0} r_n \; a_\tau^n}_{\text{FOPT}}$$

$$r_n = K_n + g_n$$

$$A^{(n)}(\alpha_s) \equiv \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x} (1 - 2x + 2x^3 - x^4) \, a_\tau(-x m_\tau^2)^n = a_\tau^n + \cdots \quad ; \quad a_\tau \equiv \alpha_s(m_\tau)/\pi$$

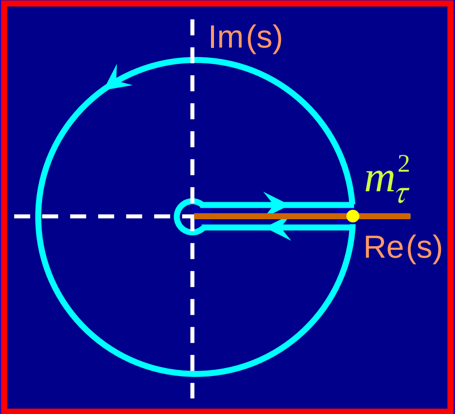
n	1	2	3	4	5
K_n	1	1.6398	6.3710	49.0757	
g_n	0	3.5625	19.9949	78.0029	307.78
r_n	1	5.2023	26.3659	127.079	

The dominant corrections come from the contour integration

Le Diberder- Pich 1992

Large running of α_s along the circle $s = m_\tau^2 \; e^{i\varphi}$, $\varphi \in [0 , 2\pi]$

$$A^{(n)}(a_\tau) \equiv \frac{1}{2\pi i}\oint_{|x|=1}\frac{dx}{x}(1-2x+2x^3-x^4)\, a_\tau(-x m_\tau^2)^n = a_\tau^n + \cdots \quad ; \quad a_\tau \equiv \alpha_s(m_\tau)/\pi$$



$$A^{(1)}(a_\tau) = a_\tau - \frac{19}{24}\beta_1 a_\tau^2 + \left[\beta_1^2\left(\frac{265}{288}-\frac{\pi^2}{12}\right) - \frac{19}{24}\beta_2\right] a_\tau^3 + \cdots$$

$$a(-s) \simeq \frac{a_\tau}{1-\frac{\beta_1}{2}a_\tau \log\left(-s/m_\tau^2\right)} = \frac{a_\tau}{1-i\frac{\beta_1}{2}a_\tau\phi} = a_\tau \sum_n \left(i\frac{\beta_1}{2}a_\tau\phi\right)^n \quad ; \quad \phi\in[0,2\pi]$$

FOPT expansion only convergent if $a_\tau < 0.14$ (0.11) [at 1 (3) loops]

Experimentally $a_\tau \approx 0.11$ ➡ **FOPT should not be used** (divergent series)

FOPT suffers a large renormalization-scale dependence (Le Diberder- Pich , Menke)

The difference between FOPT and CIPT grows at higher orders

CIPT gives rise to a well-behaved perturbative series:

$$A^{(n)}(a_\tau) \equiv \frac{1}{2\pi i}\oint_{|x|=1}\frac{dx}{x}(1-2x+2x^3-x^4)\, a(-s)^n \;=\; a_\tau^n + \cdots$$

a = 0.11	$A^{(1)}(a)$	$A^{(2)}(a)$	$A^{(3)}(a)$	$A^{(4)}(a)$	δ_P
$\beta_{n>1} = 0$	0.14828	0.01925	0.00225	0.00024	**0.20578**
$\beta_{n>2} = 0$	0.15103	0.01905	0.00209	0.00020	**0.20537**
$\beta_{n>3} = 0$	0.15093	0.01882	0.00202	0.00019	**0.20389**
$\beta_{n>4} = 0$	0.15058	0.01865	0.00198	0.00018	**0.20273**
$O(a^4)$	0.16115	0.02431	0.00290	0.00015	0.22665

Uncertainty only related to the unknown K_n $(n \geq 5)$ coefficients

Renormalons

$$D(s) \equiv -s \frac{d}{ds} \Pi^{(0+1)}(s) = \frac{1}{4\pi^2} \sum_{n=0} K_n \; a(-s)^n$$

Asymptotic series

Borel Summation:

$$B(t) \equiv \sum_{n=0} K_{n+1} \frac{t^n}{n!} \quad \Longrightarrow \quad D(s) = \frac{1}{4\pi^2} \left\{ 1 + \int_0^\infty dt\, e^{-t/a(-s)}\, B(t) \right\}$$

However, $B(t)$ **has pole singularities at**

- $u \equiv -\beta_1 t/2 = +n \qquad (n \geq 2)$ **Infrared Renormalons**
- $u \equiv -\beta_1 t/2 = -n \qquad (n \geq 1)$ **Ultraviolet Renormalons**

IR - n Renormalon $\Longrightarrow$ **Ambiguity:** $\delta D(s) \sim \left(\frac{\Lambda^2}{-s} \right)^n$

Renormalon Hypothesis: Asymptotics already reached

Modelling a better behaved FOPT

(Beneke – Jamin)

- **Large higher-order K_n corrections could cancel the g_n ones**

  Happens in the "large-β_0" approximation (UV renormalon chain)

- D = 4 corrections very suppressed in R_τ

  ➡ **n = 2 IR renormalons can do the job ($K_n \approx - g_n$)**

- No sign of renormalon behaviour in known coefficients

  ➡ **n = -1,2,3 renormalons + linear polynomial**

  5 unknown constants fitted to K_n ($2 \le n \le 5$). $K_5 = 283$ assumed

- **Borel summation:** large renormalon contributions. Smaller α

Nice model of higher orders. But too many different possibilities …

(Descotes-Genon – Malaescu)

Renormalon Hypothesis: Asymptotics already reached

(Caprini – Fischer)

1) Optimal Conformal Mapping in the Borel Plane

$$w(u) \equiv \frac{\sqrt{1+u} - \sqrt{1-u/2}}{\sqrt{1+u} + \sqrt{1-u/2}} \quad , \qquad (1+w)^{2\gamma_1} (1-w)^{2\gamma_2} \, B(u) = \sum_{n=0} c_n \, w^n$$

Maps the *u*-plane into the unit disk: $|w| \le 1$. Converges in $|w| < 1$

2) CIPT properly implemented within the Borel transform

3) Assume $u = -1$ and $u = 2$ renormalons dominate at $O(a^4)$

- $K_{n\geq 5}$ predicted ($K_5 = 256$, $K_6 = 2929$ ...)
- Positive contribution to δ_P
- Smaller value of α_s (in agreement with Beneke-Jamin)

Renormalon Hypothesis: Asymptotics already reached

HOWEVER

- **No sign of renormalon behaviour in $K_{n\leq 4}$**

  Sing-alternating series expected from dominance of n=-1 UVR

- **Same procedure would also work (nicely) assuming asymptotics reached at n=7**

  Assuming arbitrary values of K_5 and K_6 , different results would be obtained

Alternative Estimate of Higher Orders

Reshuffling of perturbative series through the β function and its derivatives + CIPT (Cvetič et al)

$$D(s) = \frac{1}{4\pi^2} \left\{ 1 + \sum_{n=1} \tilde{c}_n \; \tilde{a}_n(-s) \right\} \qquad , \qquad \tilde{a}_{n+1}(-s) \equiv \frac{1}{n!\,\beta_1^n} \frac{d^n a}{d(\log(-s))^n}$$

Very small renormalization-scale dependence

$K_{n\leq 4}$, $K_5 = 0,\ 275$ ➡ $\alpha_s(m_\tau) = 0.341\ (8)$

Non-perturbative contributions

$$R_\tau = N_C \, S_{EW} \left(1 + \delta_P + \delta_{NP}\right)$$

	δ_{NP}	
Davier et al ‘08	**– 0.0059 ± 0.0014**	ALEPH data
ALEPH ‘05	– 0.0043 ± 0.0019	
OPAL ‘99	– 0.0024 ± 0.0025	
CLEO ‘95		
Maltman-Yavin ‘08	**+ 0.012 ± 0.018**	Phenom. analysis
Braaten et al ‘92	– 0.009 ± 0.005	Theory estimate
Beneke-Jamin ‘08	– 0.007 ± 0.003	Theory estimate

$\delta_{NP} = -\,0.0059 \pm 0.0014$ ➡ $\delta_P = 0.1997 \pm 0.0035$

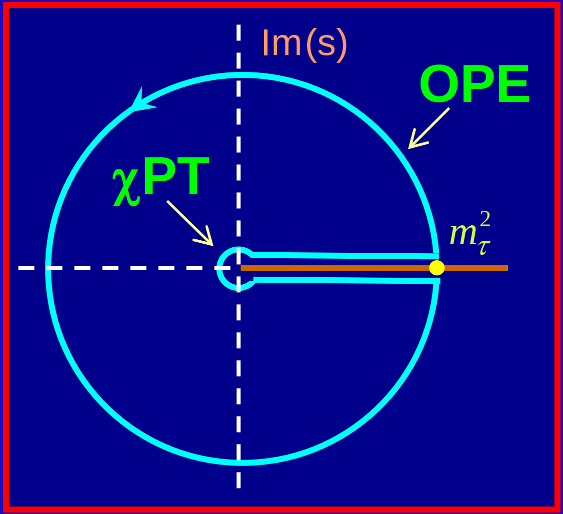
Small changes in δ_{NP} imply corresponding shifts in δ_P

"Duality Violations" ≡ OPE uncertainties

Shifman '00, Cata–Golterman–Peris '08, Davier et al '08

$$\delta_{\rm DV} = 2\pi i \oint_{|x|=1} dx\,(1-x)^2(1+2x)\left[\,\Pi^{(0+1)}(x m_\tau^2) - \Pi^{(0+1)}_{\rm OPE}(x m_\tau^2)\right]$$

- Suppressed in R_τ because of the $(1-x)^2$ factor (double zero)
- Smaller than errors in δ_{NP} (which are subdominant with respect δ_P)
- Can be studied in $\Pi_{VV}(s)-\Pi_{AA}(s)$ where perturbative contributions cancel



$$\lim_{s\to\infty} s^2\,\Pi(s) = 0 \qquad\rightarrow\qquad \Pi^{\rm OPE}(s) = -\frac{O_6}{s^3} + \frac{O_8}{s^4} - \cdots$$

González-Prades-Pich '10, Catà-Golterman-Peris '05

$$\int_{s_{\rm th}}^{s_0} {\rm d}s\; w(s)\,\rho(s) \;+\; \frac{1}{2\pi i}\oint_{|s|=s_0} {\rm d}s\; w(s)\,\Pi^{\rm OPE}(s) \;+\; {\rm DV}[w(s),s_0]$$
$$=\; 2 f_\pi^2\, w(m_\pi^2) \;+\; \underset{s=0}{\rm Res}\,[w(s)\,\Pi(s)]$$

Summary: My Numerical Estimate

1) $\delta_P = \mathbf{0.1997\ (35)}$

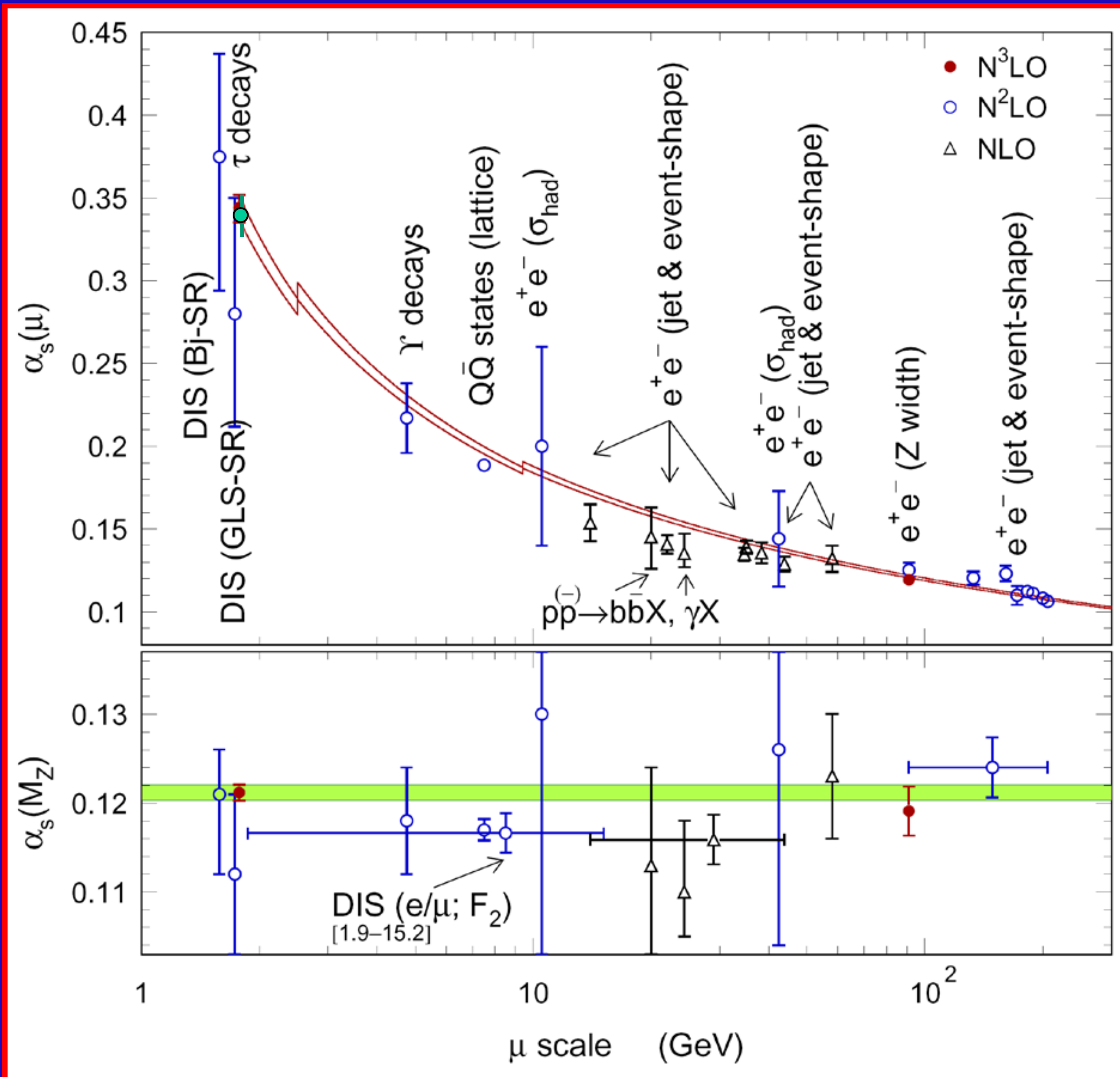
2) **CIPT**

3) $K_5 = 275 \pm 400$

4) $\beta_5 = \pm\, \beta_4^2 / \beta_3 = \pm\, 443$

5) $\mu^2 / (-s) \in [0.5, 1.5]$

$\Rightarrow$ $\alpha_s(m_\tau) = 0.338\ \ (12)$



$$\alpha_s(m_\tau^2) = 0.338 \pm 0.012$$

$$\alpha_s(M_Z^2) = 0.1209 \pm 0.0014$$

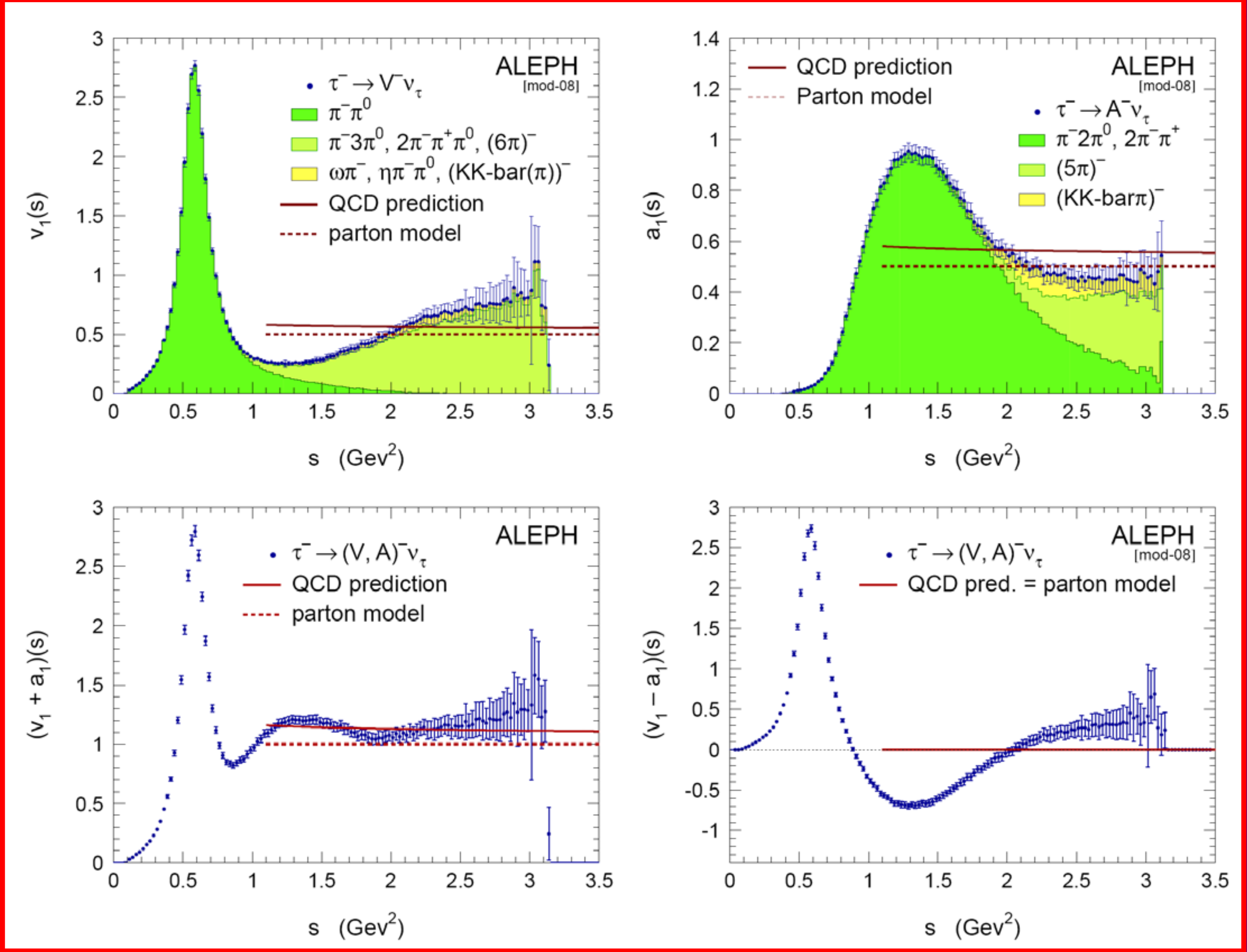
$$\alpha_s(M_Z^2)_{Z\,\mathrm{width}} = 0.1190 \pm 0.0027$$

The most precise test of Asymptotic Freedom

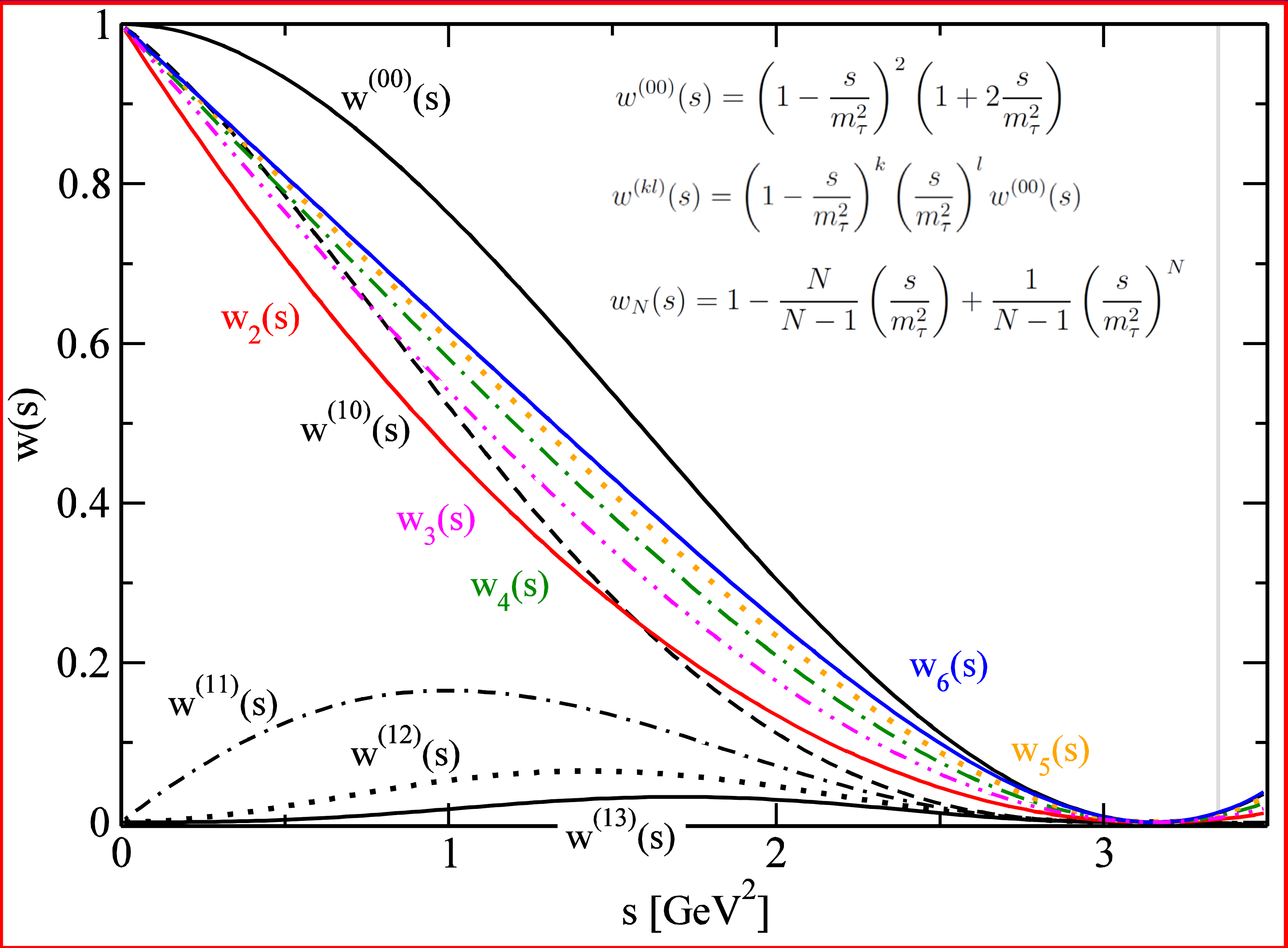
$$\alpha_s^\tau(M_Z^2) - \alpha_s^Z(M_Z^2) = 0.0019 \pm 0.0014_\tau \pm 0.0027_Z$$

SPECTRAL FUNCTIONS

$$v_1(s) = 2\pi \, \mathrm{Im}\, \Pi^{(0+1)}_{ud,V}(s) \qquad a_1(s) = 2\pi \, \mathrm{Im}\, \Pi^{(0+1)}_{ud,A}(s)$$



Davier et al '08



$w^{(00)}(s)$
$w^{(00)}(s) = \left(1 - \frac{s}{m_\tau^2}\right)^2 \left(1 + 2\frac{s}{m_\tau^2}\right)$
$w^{(kl)}(s) = \left(1 - \frac{s}{m_\tau^2}\right)^k \left(\frac{s}{m_\tau^2}\right)^l w^{(00)}(s)$
$w_N(s) = 1 - \frac{N}{N-1}\left(\frac{s}{m_\tau^2}\right) + \frac{1}{N-1}\left(\frac{s}{m_\tau^2}\right)^N$
$w_2(s)$
$w^{(10)}(s)$
$w_3(s)$
$w_4(s)$
$w_6(s)$
$w_5(s)$
$w^{(11)}(s)$
$w^{(12)}(s)$
$w^{(13)}(s)$
w(s)
s [GeV^2]
0
0.2
0.4
0.6
0.8
1
0
1
2
3