FOPT [fixed-order perturbation theory] Analysis *)

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*) FOPT Analysis is trivial. Not so trivial: why this is (most likely) the right thing to do.

References:

MB, Matthias Jamin, JHEP 0809 (2008) 044, arXiv:0806.3156 [hep-ph];
MB, Diogo Boito, Matthias Jamin, work in progress

Notation

Adler function

$$D_V^{(1+0)}(s) \;=\; \frac{N_c}{12\pi^2}\sum_{n=0}^{\infty} a_\mu^n \sum_{k=1}^{n+1} k\, c_{n,k}\, \ln^{k-1}\frac{-s}{\mu^2} = \frac{N_c}{12\pi^2}\sum_{n=0}^{\infty} c_{n,1}\, a_Q^n$$

$c_{n,1}$ known to $n = 4$ [Baikov, Chetyrkin, Kühn; 2008].

FOPT
$$\delta_{\rm FO}^{(0)} = \sum_{n=1}^{\infty} a(M_\tau^2)^n \sum_{k=1}^{n} k\, c_{n,k}\, J_{k-1} \qquad J_l \equiv \frac{1}{2\pi i}\oint\limits_{|x|=1} \frac{dx}{x}\,(1-x)^3\,(1+x)\ln^l(-x)$$

CIPT
$$\delta_{\rm CI}^{(0)} = \sum_{n=1}^{\infty} c_{n,1}\, J_n^a(M_\tau^2) \qquad J_n^a(M_\tau^2) \equiv \frac{1}{2\pi i}\oint\limits_{|x|=1} \frac{dx}{x}\,(1-x)^3\,(1+x)\, a^n(-M_\tau^2 x)$$

The problem

Series expansions for $\alpha_s(M_\tau^2) = 0.34$:

$$
\begin{array}{rcl}
& & \quad \alpha_s^1 \qquad\quad \alpha_s^2 \qquad\quad \alpha_s^3 \qquad\quad \alpha_s^4 \qquad\qquad \alpha_s^5 \\
\delta_{\rm FO}^{(0)} & = & 0.1082 + 0.0609 + 0.0334 + 0.0174\,(+\,0.0088\,) \;=\; 0.2200\;(0.2288) \\
\delta_{\rm CI}^{(0)} & = & 0.1479 + 0.0297 + 0.0122 + 0.0086\,(+\,0.0038\,) \;=\; 0.1984\;(0.2021)
\end{array}
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[We will often use the estimate $c_{5,1} = 283$.]

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FO/CI difference *increases* by adding more orders.
Systematic problem.

Arguments for CIPT ...

- Better convergence, smaller scale dependence

  Scale error on $\alpha_s(M_\tau^2)$ from variation of μ in [1,2.5] GeV is ${}^{+0.010}_{-0.005}$ for FO and ${}^{+0.005}_{-0.002}$ for CI.

- Expansion of the running coupling on the circle as used in FO has only a finite radius of convergence [Le Diberder, Pich; 1992]

$$\alpha_s(M_\tau^2 e^{i\pi}) = \frac{\alpha_s(M_\tau^2)}{1 + \frac{\beta_0}{4\pi} i\pi\alpha_s(M_\tau^2)}$$

Actual $\alpha_s(M_\tau^2)$ is close.

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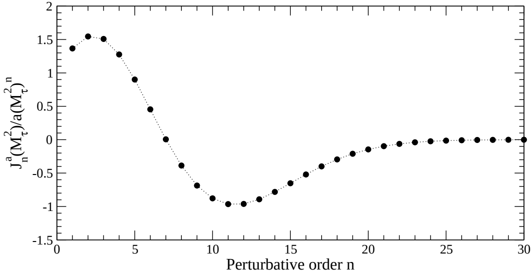
Not very compelling, because ...

Standard arguments for CIPT are not compelling (1/2)

$$\delta_{\rm CI}^{(0)} = \sum_{n=1}^{\infty} c_{n,1}\, a^n(M_\tau^2) \times [J_n^a(M_\tau^2)/a^n(M_\tau^2)]$$



Series must exhibit small terms at intermediate orders, because the effective coupling goes through zero. Small scale dependence and good convergence may be a spurious effect.

Standard arguments for CIPT are not compelling (2/2)

- QCD perturbation expansions are only asymptotic anyway.
  Zero radius of convergence.

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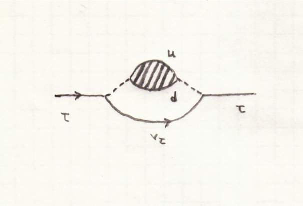
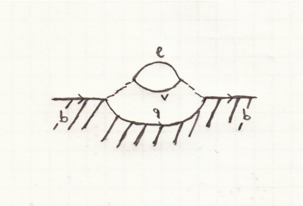
  Why then bother about a subset of terms that has finite radius of convergence?

- We don't use running couplings in CI-type prescriptions in related problems.
  Cf. semileptonic heavy quark decay, which may also be expressed as

$$\Gamma = \int^{m_Q^2} ds\, w(s) \Pi_{HQ}(s)$$





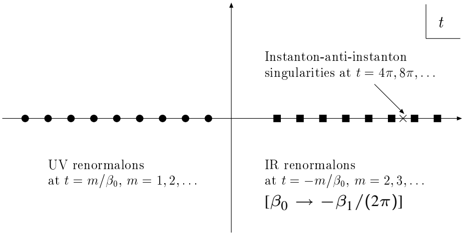
The FOPT/CIPT discrepancy *is* a problem.
What else do we know about the perturbation series?

Asymptotic behaviour

General structure of large-order behaviour is (believed to be) known.

$$B[F](t) = \sum_{n=0}^{\infty} r_n \, \frac{t^n}{n!}$$

$$F(\alpha) = \int_0^{\infty} dt \, e^{-t/\alpha} \, B[F](t)$$



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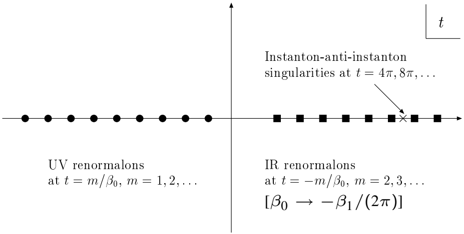

$$B[F_p](u) = \frac{c_p}{(p-u)^{1+\gamma}} \left[\, 1 + \tilde{b}_1(p-u) + \ldots \,\right] \qquad (u \equiv \beta_1 t/(2\pi)) \qquad \Leftrightarrow$$

$$F_p(a) = \frac{\pi c_p}{p^{1+\gamma}\,\Gamma(1+\gamma)} \sum_{n=0}^{\infty} \Gamma(n+1+\gamma) \left(\frac{\beta_1}{2p}\right)^n a^{n+1} \times \left[1 + \frac{p\gamma\tilde{b}_1}{(n+\gamma)} + \mathcal{O}\left(\frac{1}{n^2}\right)\right]$$

In principle the renormalon singularity structure is calculable, *except for c_p.*

General properties and the case of R_τ

UV renormalons

- Sign-alternating, singularity structure related to higher-dim operators in the cut-off QCD Lagrangian [Parisi, 1977; MB, Kivel, Braun 1997]
- Leading singularity for Adler function and R_τ, $u = -1$
- No sign-alternation seen. [In fact, no factorial behaviour of any form.]
  c_{-1} must be small in the $\overline{\rm MS}$ scheme – true for fermion loops ("large-β_0").

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IR renormalons

- Fixed-sign, singularity structure related to higher-dim operators in the OPE [Mueller, 1985; Zakharov, 1992; MB, 1993]
- No $u = 1$ singularity, $u = 2, 3, ...$ related to dim-4, 6, ... condensates.
- $u = 2$ especially simple, only one operator (gluon condensate), presumably numerically leading in intermediate orders; $u = 3$ complicated.
- *$1/n^2$ suppression of the $u = 2$ singularity in going from Adler function to R_τ, none for $u = 3$. [Related to weight function and anomalous dimension of the gluon condensate.]* [MB, 1993]

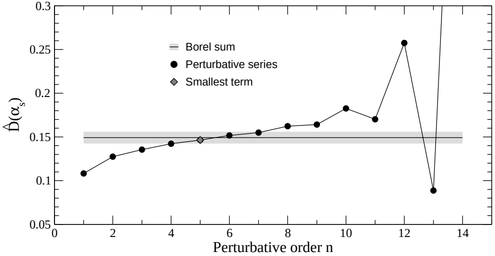
Incorporate this into an Ansatz for $c_{n,1}$, $n > 5$.
Then compare FOPT/CIPT.

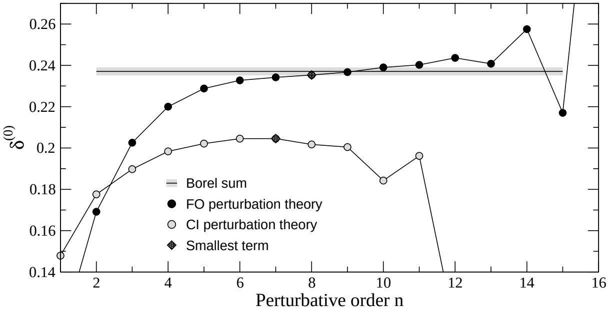
Assumption: series is asymptotic, sum to minimal term should be the Borel sum $\pm$ size of power corrections (small).

Note: Borel sum is the same in the FO and CI approach.

$$B[D](u) \;=\; B[D_1^{\rm UV}](u) + B[D_2^{\rm IR}](u) + B[D_3^{\rm IR}](u) + d_0^{\rm PO} + d_1^{\rm PO} u\,,$$

- Ansatz for the Adler function that reproduces known $c_{4,1}$ and $c_{5,1} = 283$.
- Fit constants c_p for $u = -1, 2, 3$ to $c_{3,1}$, $c_{4,1}$ and $c_{5,1}$, and adjust $d_{0,1}^{\rm PO}$ to reproduce $c_{1,1}$ and $c_{2,1}$.
- Find $d_1^{\rm UV} = -1.56 \cdot 10^{-2}$, $d_2^{\rm IR} = 3.16$, $d_3^{\rm IR} = -13.5$, $d_0^{\rm PO} = 0.781$, $d_1^{\rm PO} = 7.66 \cdot 10^{-3}$.
- Pole ansatz works well already at $n = 2$ ($d_1^{\rm PO}$ small).





- FO converges to Borel sum
- CI smoother at low orders (better convergence, smaller scale dependence), but never reaches the Borel sum (vanishing of J_n^a).
- At $n = 4, 5$ FO is close to the true result, CI too small $\Rightarrow \alpha_s$ from CI too large.
(A similar observation has been made in the large-β_0 approximation [Ball, MB, Braun, 1995].)

Explanation?
Model dependence?

Cancellations at large orders and CIPT

$$\delta^{(0)}_{\rm FO} = \sum_{n=1}^{\infty} \left[c_{n,1} + g_n\right] a(M_\tau^2)^n \qquad g_n = \sum_{k=2}^{n} k\, c_{n,k} J_{k-1}$$

- g_n from integration on the circle, $c_{n,1}$ from Adler function
- For the leading IR contribution ($u = 2$) there are *important cancellations:*

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- $c_{n,k}$ depends on $c_{m,1}$, β_m up to $m = n - k + 1$, e.g. $c_{4,2} = -\frac{1}{4}\left(\beta_3\, c_{1,1} + 2\beta_2\, c_{2,1} + 3\beta_1\, c_{3,1}\right)$

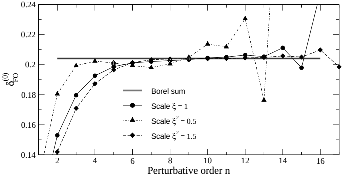
α_s^n	$c_{1,1}$	$c_{2,1}$	$c_{3,1}$	$c_{4,1}$	$c_{5,1}$	$c_{6,1}$	$c_{7,1}$	$c_{8,1}$	g_n	$c_{n,1} + g_n$
2	3.56	1.64							3.56	5.20
4	−20.6	30.5	68.1	49.1					78.0	127.1
6	−2924	−2858	−2280	2214	5041	3275			−807	2468
8	14652	−29552	−145846	−502719	−393887	260511	467787	388442	−329054	59388

CI at order n sums the first n columns to all orders. Destroys cancellations, running coupling effects are only dominant at $n < 5$, then factorial behaviour is more important.

Model dependence

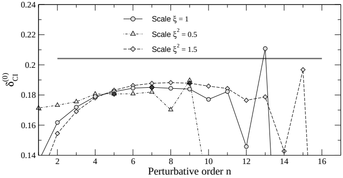
[MB, Jamin, 2008; Descotes-Genon, Malaescu, 2010]

- Qualitative features independent of
  - reference scale choice $\mu = \xi M_\tau$
  - including estimate of $c_{5,1}$, or adding $u = 4$ instead of $d_1^{\rm PO}$ in the fit.
  - variation of $\delta c_5 = (c_{5,1} - 283)/283$ between -1 and 1.



- Significant differences are obtained if $d_2^{\rm PO} \sim 1$ is added and the remaining parameters fit as before.

$$d_2^{IR} = 3.16 - 4.66\, d_2^{\rm PO} + 0.44\, \delta c_5$$
$$d_3^{IR} = -13.5 + 163.3\, d_2^{\rm PO} + 21.7\, \delta c_5$$
$$d_1^{\rm PO} = 0.007 + 3.84\, d_2^{\rm PO} + 0.75\, \delta c_5$$



But $d_2^{\rm PO} \sim 1$ produces a model with large intrinsic cancellations between $u = 2$ and $u = 3$ and in low orders.

- Case 1: $u = 2$ singularity is dominant (as expected)

  Cancellations are important

  FOPT preferred

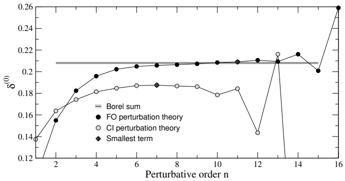
- Case 2: Models with $u = 2$ singularity artificially set to zero or suppressed

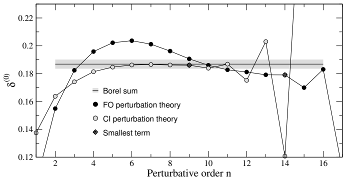
  No cancellations, factorial behaviour suppressed relative to running coupling-effects [cf. Jamin. 2005]

  CIPT preferred

- Case 3: Models with (unnaturally) large cancellations between $u = 2$ and $u = 3$ and/or polynomial coefficients.

  Anything can happen.



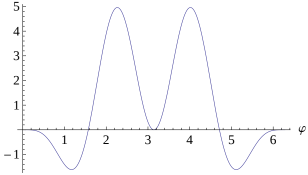


(Plots for $\alpha_s(M_\tau) = 0.3186$,
from Jamin, arXiv:1101.0681 [hep-ph].)

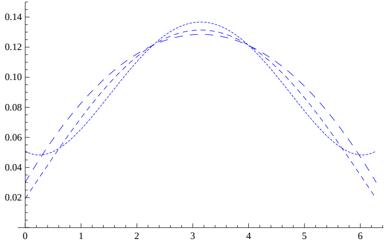
The Adler function along the circle in the complex plane (1/3)

$$\delta^{(0)} \propto \oint\limits_{|x|=1} \frac{dx}{x}\,(1-x)^3\,3\,(1+x)\,D^{(1+0)}(M_\tau^2 x)$$

$$f(x) = (1-e^{i\phi})^3(1+e^{i\phi})$$



Adler function on the circle at 4th, 5th, 7th (decreasing dashes) order for FO (blue, left) and CI (black, right).



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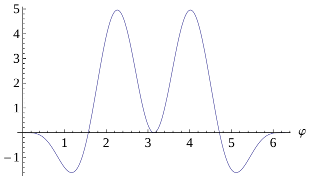

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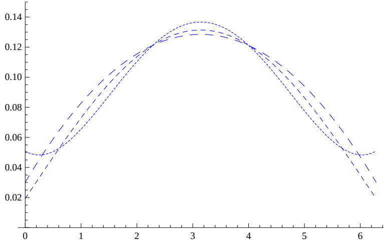
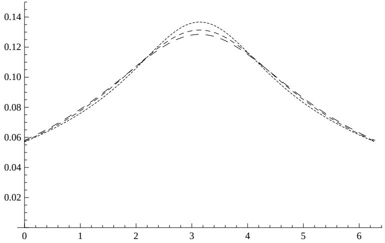
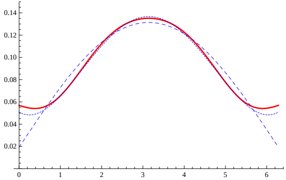
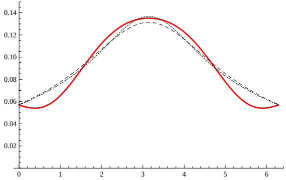




Case 1 (left): Ansatz from [MB, Jamin, 2008]

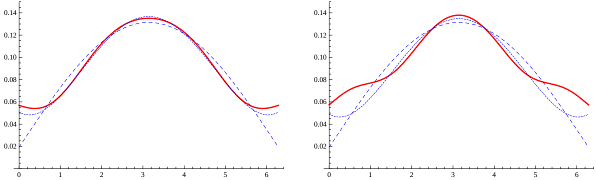


solid red: Borel sum



The Adler function along the circle in the complex plane (2/3)

Case 1 (left): Ansatz from [MB, Jamin, 2008]
Case 2 (middle): Model with $u = 2$ singularity set to zero by hand



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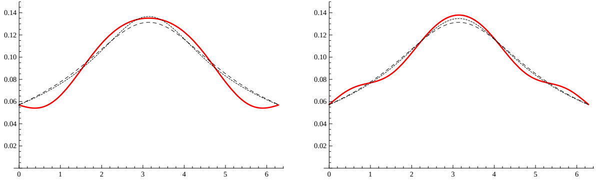
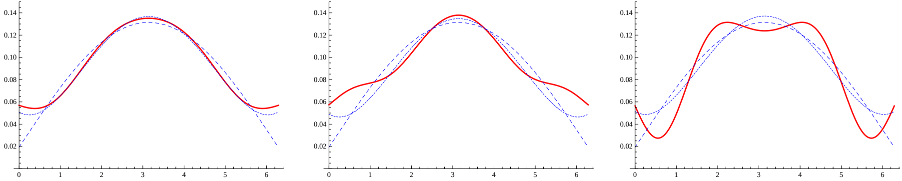


Case 1 (left): Ansatz from [MB, Jamin, 2008]
Case 2 (middle): Model with $u = 2$ singularity set to zero by hand
Case 3 (right): Model with large residues and cancellations between $u = 2$ and $u = 3$.
($d_2^{\rm PO} = -1$ from [Descotes-Genon, Malaescu, 2010])



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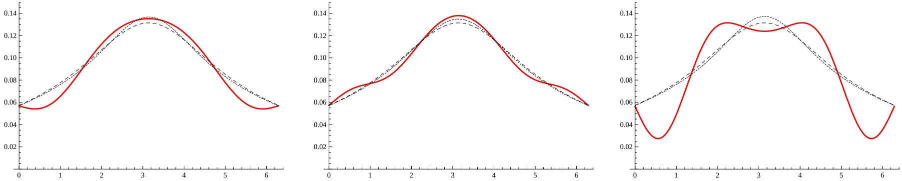

The Adler function along the circle in the complex plane (3/3)

- Did not find models where the shape of CI is close to the Borel sum on the circle.

- Models with large cancellations yield shapes of the Borel sum that cannot be reproduced by FO or CI in orders where the series is asymptotic.

- Implications for moments!
  ALEPH moments are strongly oscillating along the circle.

Finally: α_s

α_s analysis

- $R^{ud}_{\tau,V+A} = 3.4678 \pm 0.0090$ (HFAG), $|V_{ud}| = 0.97425 \pm 0.00022$, (updated to HFAG compared to [MB, Jamin, 2008]), standard estimates of EW corrections
- $\delta_{\rm PC} \;=\; (-\,6.8 \pm 3.5) \cdot 10^{-3}$ (slightly updated compared to [MB, Jamin, 2008])

$$\delta^{(0)}_{\rm phen} \;=\; 0.2000 \pm 0.0032_{\rm exp} \pm 0.0037_{\rm PC} \;=\; 0.2000 \pm 0.0049$$

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$$\delta^{(0)}_{\rm phen} = 0.2000 \pm 0.0032_{\rm exp} \pm 0.0037_{\rm PC} = 0.2000 \pm 0.0049$$

	$\alpha_s(M_\tau)$
FOPT	$\mathbf{0.3167 \pm 0.0027_{exp} \pm 0.0032_{PC} \pm 0.0026_{c_{5,1}}\ ^{+0.0100}_{-0.0049}}$ (scale)
BS	$0.3122 \pm 0.0025_{\rm exp} \pm 0.0030_{\rm PC} \pm 0.0024_{c_{5,1}} \pm 0.0011_{\beta_4=0}\ ^{+0.0032}_{-0.0027}$ (scale)
CIPT	$0.3373 \pm 0.0039_{\rm exp} \pm 0.0046_{\rm PC} \pm 0.0047_{c_{5,1}}\ ^{+0.0049}_{-0.0020}$ (scale)
	$\alpha_s(M_Z)$
FOPT	$\mathbf{0.1181 \pm 0.0003_{exp}\ ^{+0.0012}_{-0.0007}}$ (th) $\mathbf{\pm 0.0002_{evol}}$
BS	$0.1175 \pm 0.0003_{\rm exp} \pm 0.0006({\rm th}) \pm 0.0002_{\rm evol}$
CIPT	$0.1205 \pm 0.0004_{\rm exp}\ ^{+0.0009}_{-0.0008}\ ({\rm th}) \pm 0.0002_{\rm evol}$

Recommendation: FOPT

Conclusion

- FOPT/CIPT plausibly resolved in favour of FOPT by including generic information on large orders
- Preference for FOPT due to specific properties of the inclusive width. May be different for moments.
- Strong coupling:

$$\alpha_s(M_Z) = 0.1181 \pm 0.0003_{\rm exp} \,{}^{+\,0.0012}_{-\,0.0007}\,(\rm th) \pm 0.0002_{\rm evol}$$

Standard FO analysis. th error reduced if one takes large-order information at face value. Probably more conservative not to.