

Recent progress in hadronic τ decays

Matthias Jamin^a

^a Institució Catalana de Recerca i Estudis Avançats (ICREA),
 Institut de Física d'Altes Energies (IFAE), Campus UAB, E-08193 Bellaterra, Barcelona, Spain

The determination of α_s from hadronic τ decays is impeded by the fact that two choices for the renormalisation group resummation, namely fixed-order (FOPT) and contour-improved perturbation theory (CIPT), yield systematically differing results. On the basis of a model for higher-order terms in the perturbative series, which incorporates well-known structure from renormalons, it is found that FOPT smoothly approaches the Borel sum for the τ hadronic width, while CIPT is unable to account for the resummed series. An example for the behaviour of QCD spectral function moments, displaying a similar behaviour, is presented as well.

1. INTRODUCTION

Hadronic decays of the τ lepton provide an excellent playground for the study of QCD at low energies. Its mass of $M_\tau \approx 1.8$ GeV is in an energy region where perturbation theory is still applicable, but also non-perturbative effects come into play and need to be included. These may arise from vacuum-condensate terms in the framework of the operator product expansion (OPE) and from so-called *duality-violating* contributions close to the physical, Minkowskian energy axis.

In the seminal article [1], the strategy for a precise determination of the QCD coupling α_s from the total τ hadronic width

$$R_\tau \equiv \frac{\Gamma[\tau^- \rightarrow \text{hadrons } \nu_\tau(\gamma)]}{\Gamma[\tau^- \rightarrow e^- \bar{\nu}_e \nu_\tau(\gamma)]} = 3.640(10), \quad (1)$$

was developed, while in the subsequent years moments of spectral τ -decay distributions were incorporated into the analyses as well [2,3,4].

The analytical computation of the perturbative order α_s^4 correction [5] has recently revived the interest in α_s analyses from hadronic τ decays which after evolution to the Z boson mass scale resulted in the following determinations:

$$\alpha_s(M_Z^2) = \begin{cases} 0.1202 (6)_{\text{exp}}(18)_{\text{th}} & [5], \\ 0.1212 (5)_{\text{exp}}(9)_{\text{th}} & [6], \\ 0.1180 (4)_{\text{exp}}(7)_{\text{th}} & [7], \\ 0.1187 (6)_{\text{exp}}(15)_{\text{th}} & [8]. \end{cases} \quad (2)$$

The dispersion in these results dominantly originates from different treatments of the renormalisation group (RG) resummation of the perturbative series, namely fixed-order perturbation theory (FOPT), or contour-improved perturbation theory (CIPT) [9,10], being systematically larger than the last included term in the expansion, which often in asymptotic series provides an estimate of the uncertainty due to higher-order terms not included in the partial sum. This points to the necessity of investigating the influence of different RG resummations in more detail, which can for example be performed within exactly resumable models for the perturbative series.

Most suitable for the α_s determination is the τ decay rate into light u and d quarks $R_{\tau,V/A}$ via a vector or axialvector current, since in this case power corrections are especially suppressed. Theoretically, $R_{\tau,V/A}$ takes the form [1]

$$R_{\tau,V/A} = \frac{N_c}{2} S_{\text{EW}} |V_{ud}|^2 \left[1 + \delta^{(0)} + \delta'_{\text{EW}} + \sum_{D \geq 2} \delta_{ud,V/A}^{(D)} \right], \quad (3)$$

where $S_{\text{EW}} = 1.0198(6)$ [11] and $\delta'_{\text{EW}} = 0.0010(10)$ [12] are electroweak corrections, $\delta^{(0)}$ comprises the perturbative QCD correction, and the $\delta_{ud,V/A}^{(D)}$ denote quark mass and higher D -dimensional operator corrections which arise in the framework of the OPE.

2. PERTURBATIVE CORRECTION $\delta^{(0)}$

Below, only the purely perturbative correction $\delta^{(0)}$ shall be considered, which gives the dominant contribution to $R_{\tau,V/A}$. In FOPT it takes the general form

$$\delta_{\text{FO}}^{(0)} = \sum_{n=1}^{\infty} a(M_{\tau}^2)^n \sum_{k=1}^n k c_{n,k} J_{k-1}, \quad (4)$$

where $a(\mu^2) \equiv a_{\mu} \equiv \alpha_s(\mu)/\pi$, and $c_{n,k}$ are the coefficients which appear in the perturbative expansion of the vector correlation function,

$$\Pi_V(s) = -\frac{N_c}{12\pi^2} \sum_{n=0}^{\infty} a_{\mu}^n \sum_{k=0}^{n+1} c_{n,k} \ln^k\left(\frac{-s}{\mu^2}\right). \quad (5)$$

At each perturbative order, the coefficients $c_{n,1}$ can be considered independent, while all other $c_{n,k}$ with $k \geq 2$ are calculable from the RG equation. Further details can for example be found in ref. [7]. Finally, the J_l are contour integrals in the complex s -plane, which are defined by

$$J_l \equiv \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x} (1-x)^3 (1+x) \ln^l(-x). \quad (6)$$

The first three, being required up to $\mathcal{O}(\alpha_s^3)$, take the numerical values

$$J_0 = 1, \quad J_1 = -\frac{19}{12}, \quad J_2 = \frac{265}{72} - \frac{1}{3}\pi^2. \quad (7)$$

At order α_s^n FOPT contains unsummed logarithms of order $\ln^l(-x) \sim \pi^l$ with $l < n$ related to the contour integrals J_l . CIPT sums these logarithms, which yields

$$\delta_{\text{CI}}^{(0)} = \sum_{n=1}^{\infty} c_{n,1} J_n^a(M_{\tau}^2) \quad (8)$$

in terms of the contour integrals $J_n^a(M_{\tau}^2)$ over the running coupling, defined as:

$$J_n^a(M_{\tau}^2) \equiv \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x} (1-x)^3 (1+x) a^n(-M_{\tau}^2 x). \quad (9)$$

In contrast to FOPT, for CIPT each order n just depends on the corresponding coefficient $c_{n,1}$.

Thus, all contributions proportional to the coefficient $c_{n,1}$ which in FOPT appear at all perturbative orders equal or greater than n are resummed into a single term.

Numerically, the two approaches lead to significant differences. Employing the recent average $\alpha_s(M_Z) = 0.1184$ [13], leading to $\alpha_s(M_{\tau}) = 0.3186$, in eqs. (4) and (8), one finds

$$\delta_{\text{FO}}^{(0)} = 0.1959 (0.2022), \quad (10)$$

$$\delta_{\text{CI}}^{(0)} = 0.1814 (0.1847), \quad (11)$$

where the first number in both cases employs the known coefficients up to $\mathcal{O}(\alpha_s^4)$ [5] and the numbers in brackets include an estimate of the $\mathcal{O}(\alpha_s^5)$ term with $c_{5,1} \approx 283$ [7]. Inspecting the individual contributions from each order, up to $\mathcal{O}(\alpha_s^5)$ the CIPT series appears to be better convergent. However, around the seventh order, the contour integrals $J_n^a(M_{\tau}^2)$ change sign and thus at this order the contributions are bound to become small. Therefore, the faster approach to the minimal term does not necessarily imply that CIPT gives the closer approach to the true result for the resummed series.

3. A PHYSICAL MODEL

To investigate whether FOPT or CIPT results in a better approximation to $\delta^{(0)}$, one requires a physically motivated model for its series. Such a model was constructed in ref. [7] and is based on the Borel transform of the Adler function $D_V(s)$:

$$D_V(s) \equiv -s \frac{d}{ds} \Pi_V(s) \equiv \frac{N_c}{12\pi^2} [1 + \widehat{D}(s)]. \quad (12)$$

In the following discussion it is slightly more convenient to utilise the related function $\widehat{D}(s)$. Its Borel transform $B[\widehat{D}](t)$ is defined by the relation

$$\widehat{D}(\alpha) \equiv \int_0^{\infty} dt e^{-t/\alpha} B[\widehat{D}](t). \quad (13)$$

The integral $\widehat{D}(\alpha)$, if it exists, gives the Borel sum of the original divergent series. It was found that the Borel-transformed Adler function $B[\widehat{D}](t)$ obtains infrared (IR) and ultraviolet (UV) renor-

malon poles at positive and negative integer values of the variable $u \equiv 9t/(4\pi)$, respectively [14,15,16]. (With the exception of $u = 1$.)

Apart from very low orders, where a dominance of renormalon poles close to $u = 0$ has not yet set in, intermediate orders should be dominated by the leading IR renormalon poles, while the leading UV renormalon, being closest to $u = 0$, dictates the large-order behaviour of the perturbative expansion. Assuming that only the first two orders are not yet dominated by the lowest IR renormalons, one is led to the ansatz

$$B[\widehat{D}](u) = B[\widehat{D}_1^{\text{UV}}](u) + B[\widehat{D}_2^{\text{IR}}](u) + B[\widehat{D}_3^{\text{IR}}](u) + d_0^{\text{PO}} + d_1^{\text{PO}}u, \quad (14)$$

which includes one UV renormalon at $u = -1$, the two leading IR renormalons at $u = 2$ and $u = 3$, as well as polynomial terms for the two lowest perturbative orders. Explicit expressions for the UV and IR renormalon pole terms $B[\widehat{D}_p^{\text{UV}}](u)$ and $B[\widehat{D}_p^{\text{IR}}](u)$ can be found in section 5 of ref. [7].

Apart from the residues d_p^{UV} and d_p^{IR} , the full structure of the renormalon pole terms is dictated by the OPE and the RG. Therefore, the model (14) depends on five parameters, the three residues d_1^{UV} , d_2^{IR} and d_3^{IR} , as well as the two polynomial parameters d_0^{PO} and d_1^{PO} . These parameters can be fixed by matching to the perturbative expansion of $\widehat{D}(s)$ up to $\mathcal{O}(\alpha_s^5)$. Thereby, also the estimate for $c_{5,1}$ is used. The parameters of the model (14) are then found to be:

$$d_1^{\text{UV}} = -1.56 \cdot 10^{-2}, \quad d_2^{\text{IR}} = 3.16, \quad d_3^{\text{IR}} = -13.5,$$

$$d_0^{\text{PO}} = 0.781, \quad d_1^{\text{PO}} = 7.66 \cdot 10^{-3}. \quad (15)$$

The fact that the parameter d_1^{PO} turns out to be small implies that the coefficient $c_{2,1}$ is already reasonably well described by the renormalon pole contribution, although it was not used to fix the residues. Therefore, one could set $d_1^{\text{PO}} = 0$ and actually work with a model which only has four parameters. The predicted value $c_{5,1} = 280$ then turns out very close to the estimate, which can be viewed as one test of the stability of the model.

This is also corroborated in table 1, where the relative contributions of a certain renormalon

	$c_{2,1}$	$c_{3,1}$	$c_{4,1}$	$c_{5,1}$	$c_{6,1}$
IR ₂	-77.8	82.4	100.4	135.9	97.5
IR ₃	152.0	28.7	-10.0	-20.2	-13.3
UV ₁	22.5	-11.2	9.7	-15.6	15.8

Table 1

Relative contributions (in %) of the different IR and UV renormalon poles to the Adler-function coefficients $c_{2,1}$ to $c_{6,1}$ for the Borel model (14).

pole to the coefficients $c_{2,1}$ to $c_{6,1}$ is tabulated. The sum of the contributions to $c_{2,1}$ is close to 100%, implying again that the polynomial term is already small. Then, from $c_{3,1}$ to $c_{6,1}$ the leading IR pole at $u = 2$ is dominating the coefficients, before the leading UV pole at $u = -1$ takes over at even higher orders. (For the central model (14) this happens around the 10th order [7].)

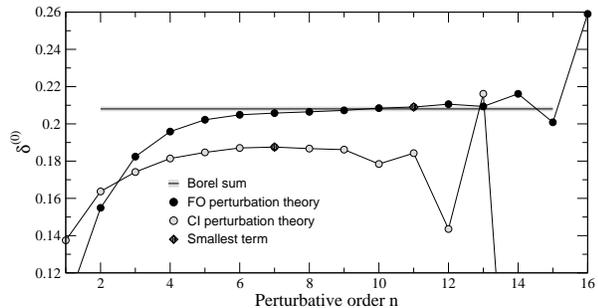


Figure 1. Results for $\delta_{\text{FO}}^{(0)}$ (full circles) and $\delta_{\text{CI}}^{(0)}$ (grey circles) at $\alpha_s(M_\tau) = 0.3186$, employing the model (14), as a function of the order n up to which the terms in the perturbative series have been summed. The straight line represents the result for the Borel sum of the series.

The implications of the model (14) for $\delta^{(0)}$ in FOPT and CIPT is graphically represented in figure 1. The full circles denote the result for $\delta_{\text{FO}}^{(0)}$ and the grey circles the one for $\delta_{\text{CI}}^{(0)}$, as a function of the order n up to which the perturbative series has been summed. The straight line corresponds to the principal value Borel sum of the series, $\delta_{\text{BS}}^{(0)} = 0.2080$, and the shaded band provides an error estimate based on its imaginary

part divided by π . The order at which the series have their smallest terms is indicated by the grey diamonds. As is obvious from figure 1, FOPT displays the behaviour expected from an asymptotic series: the terms decrease up to a certain order around which the closest approach to the resummed result is found, and for even higher orders, the divergent large-order behaviour of the series sets in. For CIPT, on the other hand, the asymptotic behaviour sets in earlier, and the series is never able to come close to the Borel sum.¹

	$c_{2,1}$	$c_{3,1}$	$c_{4,1}$	$c_{5,1}$	$c_{6,1}$
IR ₃	-743.3	-140.5	49.1	98.9	99.1
IR ₄	662.8	244.2	47.7	6.3	-7.2
UV ₁	7.5	-3.7	3.2	-5.2	8.1

Table 2

Relative contributions (in %) of the different IR and UV renormalon poles to the Adler-function coefficients $c_{3,1}$ to $c_{6,1}$ for the Borel model with $d_2^{\text{IR}} = 0$.

As the behaviour of CIPT versus FOPT hinges on the contribution of the leading IR renormalon at $u = 2$, in principal also models can be constructed for which CIPT provides a better account of the Borel sum. These would generally be models where d_2^{IR} is much smaller than the value quoted in eq. (15). While such models can at present not be excluded, the pattern of the individual contributions appears more unnatural than in the main model (14): the known $c_{n,1}$ can only be reproduced when one allows for large cancellations between the individual terms. Thus, the behaviour generally expected from the presence of renormalon poles, namely dominance of leading IR poles at intermediate orders, would be lost.

This is apparent from table 2, which is the analog of table 1, but for a model where d_2^{IR} is forced to be zero and an additional IR pole at $u = 4$ is added, in order to be able to reproduce the known Adler function coefficients. On the one

¹The same conclusions had already been drawn in ref. [17] on the basis of the large- β_0 approximation for $\Pi_V(s)$.

hand for low coefficients there are huge cancellations between the IR renormalon poles and also $c_{2,1}$ is not well described at all. This entails that a large, additional polynomial term is required. Besides, it appears unnatural that the residue of the first IR renormalon pole at $u = 2$ is small, and still there should be a natural size contribution of the gluon condensate to the Adler function.

A graphical account of the model with $d_2^{\text{IR}} = 0$ is presented in figure 2. As anticipated, now CIPT provides a good description of the Borel sum, while FOPT is able to come reasonably close to it around its minimal term, but generally is rather badly behaved.² Nonetheless, again, the behaviour observed in table 2 and figure 2 appears unnatural from the perspective of the structure of the Borel transform of the Adler function and should be considered less likely than the behaviour of the model (14) with the residues (15).

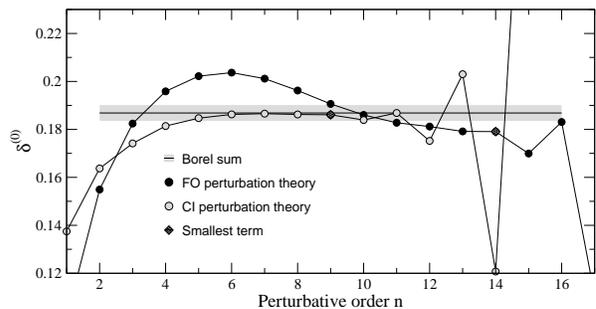


Figure 2. Results for $\delta_{\text{FO}}^{(0)}$ (full circles) and $\delta_{\text{CI}}^{(0)}$ (grey circles) at $\alpha_s(M_\tau) = 0.3186$, employing the model (14) with $d_2^{\text{IR}} = 0$ and an additional IR pole at $u = 4$, as a function of the order n up to which the terms in the perturbative series have been summed. The straight line represents the result for the Borel sum of the series.

In the standard α_s determinations from hadronic τ decays [2,3], besides the total decay rate also moments of the spectral decay distributions are employed. Historically, the so-called

²A similar behaviour was found in ref. [18] in models where the higher-order Adler function coefficients were assumed to be small, in contrast to the expected asymptotic behaviour of this series in QCD.

(k, l) -moments were used, for which a polynomial $(1-x)^k x^l$ is multiplied to the kinematical weight function from phase-space. In the analyses [2,3], moments with $k = 1$ and $l = 0, 1, 2, 3$ were taken into account, such that also condensate contributions up to dimension-8 could be extracted in addition to α_s . Therefore, it is of interest to analyse the behaviour of these moments in models of higher orders of perturbation theory as well.

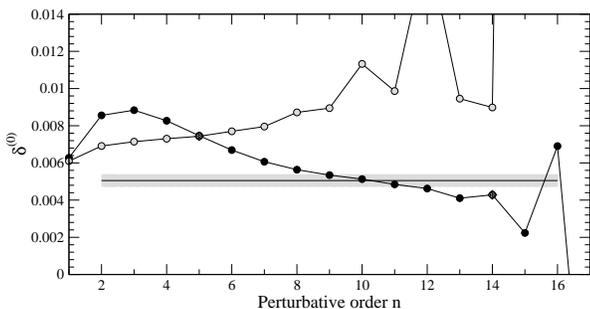


Figure 3. Results for $(1, 2)$ moments $\delta_{\text{FO}}^{(0,12)}$ (full circles) and $\delta_{\text{CI}}^{(0,12)}$ (grey circles) at $\alpha_s(M_\tau) = 0.3186$, employing the model (14), as a function of the order n up to which the terms in the perturbative series have been summed. The straight line represents the result for the Borel sum of the series.

An example of such an analysis is shown in figure 3 for the moment $(1, 2)$. It is observed that, like for $\delta^{(0)}$, CIPT is unable to provide a reasonable account of the full Borel sum. In contrast, FOPT comes close to the resummed result for perturbative orders around the minimal term, though it is obvious that this particular moment displays a very bad behaviour of the asymptotic series, with only unsatisfactory convergence up to its minimal term. The example of the $(1, 2)$ -moment should motivate an exhaustive investigation of the moments employed in previous α_s analyses from hadronic τ decays, which will be presented in the near future.

4. CONCLUSIONS

Models of higher orders of perturbation theory allow for the study of different resummation

prescriptions in the computation of the total τ hadronic width, as well as related moments of τ decay spectral distributions. The most prominent methods are fixed-order perturbation theory (FOPT) and the so-called contour-improved perturbation theory (CIPT), which performs a partial resummation of running effects of the QCD coupling α_s in the integration along the complex contour in the s -plane.

A physically motivated model for the higher-order behaviour of the Adler function was presented in ref. [7], and is given in eq. (14). The model was based on the general structure of the Borel transform of the Adler function and the renormalisation group equation. Furthermore, knowledge on the first four analytically available coefficients $c_{1,1}$ to $c_{4,1}$, as well as an estimate for the fifth coefficient $c_{5,1}$, were incorporated.

Results for $\delta^{(0)}$ in the main model (14) were displayed in figure 1, and it is observed that while FOPT provides a good account of the full Borel summation, CIPT is never able to come close to the resummed value.³ This general behaviour hinges on the size of the residue of the first IR renormalon pole at $u = 2$. In a model in which this residue is set to zero by hand, on the contrary CIPT well describes the Borel sum, whereas FOPT, though approaching the resummed value around its minimal term, generally is rather badly behaved. Models in which $d_2^{\text{IR}} \approx 0$, however, are only able to reproduce the known Adler function coefficients through large cancellations between different IR renormalon contributions, which appears unnatural.

In the standard experimental extractions of α_s from hadronic τ decays, also moments of the decay spectra are employed [2,3]. An example of the behaviour of such a moment was shown in figure 3. The repeatedly unacceptable behaviour of CIPT in this case and the also unsatisfactory convergence of FOPT, suggest that such moments should be investigated systematically, before their

³The general behaviour of the Borel model is also supported by an independent approach where the perturbative series is conformally transformed into a series which displays better convergence properties than the original series in powers of α_s [19], though in this case a modified CIPT better converges towards the full result.

usefulness in α_s determinations from hadronic τ decays is corroborated.

A final topic, not touched upon at all so far, are violations of quark-hadron duality [20,21,22]. Like the perturbative expansion in α_s , also the OPE in inverse powers of s may be only asymptotic, and thus exponentially suppressed terms could be relevant close to the Minkowskian axis where the bound states are situated. This was investigated in a model in [21] and the possible influence of duality violations in τ decay spectra was studied in refs. [22]. Also here a more systematic investigation seems to be in order which was initiated in [23] and will be continued in the future.

Hadronic τ decays have proven to be a very fruitful laboratory for the study of low-energy QCD and the extractions of fundamental QCD parameters like the coupling α_s . However, there remain unresolved theoretical issues which taint the precision of these determinations. Numerically the two most relevant appear to be the resummation of QCD running effects in the computation of the τ hadronic width and related decay moments as well as duality violations. As far as the former topic is concerned, a physically motivated model of the QCD Adler function favours the use of FOPT, since generally it is better able to represent the fully resummed series.

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