

α_s and the τ hadronic width: fixed-order, contour-improved and higher-order perturbation theory

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ABSTRACT: The determination of α_s from hadronic τ decays is revisited, with a special emphasis on the question of higher-order perturbative corrections and different possibilities of resumming the perturbative series with the renormalisation group: fixed-order (FOPT) vs. contour-improved perturbation theory (CIPT). The difference between these approaches has evolved into a systematic effect that does not go away as higher orders in the perturbative expansion are added. We attempt to clarify under which circumstances one or the other approach provides a better approximation to the true result. To this end, we propose to describe the Adler function series by a model that includes the exactly known coefficients and theoretical constraints on the large-order behaviour originating from the operator product expansion and the renormalisation group. Within this framework we find that while CIPT is unable to account for the fully resummed series, FOPT smoothly approaches the Borel sum, before the expected divergent behaviour sets in at even higher orders. Employing FOPT up to the fifth order to determine α_s in the $\overline{\text{MS}}$ scheme, we obtain $\alpha_s(M_\tau) = 0.320^{+0.012}_{-0.007}$, corresponding to $\alpha_s(M_Z) = 0.1185^{+0.0014}_{-0.0009}$. Improving this result by including yet higher orders from our model yields $\alpha_s(M_\tau) = 0.316 \pm 0.006$, which after evolution leads to $\alpha_s(M_Z) = 0.1180 \pm 0.0008$. Our results are lower than previous values obtained from τ decays.

KEYWORDS: QCD, hadronic τ decays, perturbative series, renormalisation group.

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1. Introduction

Precision determinations of fundamental parameters within the Standard Model are of utmost importance in order to test its internal consistency or point towards physics which goes beyond it. In this respect the central parameter of the strong interaction sector is the strong coupling α_s , and until now tremendous efforts have been put into an ever better determination of α_s [1, 2].

One of the most precise determinations of α_s , competitive with the current world average, is provided by detailed investigations of the τ hadronic width

$$R_\tau \equiv \frac{\Gamma[\tau^- \rightarrow \text{hadrons } \nu_\tau(\gamma)]}{\Gamma[\tau^- \rightarrow e^- \bar{\nu}_e \nu_\tau(\gamma)]} = 3.640 \pm 0.010, \quad (1.1)$$

as well as invariant mass distributions [3, 4, 5, 6, 7]. The recent analyses of the ALEPH spectral function data [7, 8, 9] on the basis of the final full LEP data set yielded $\alpha_s(M_\tau) = 0.344 \pm 0.005_{\text{exp}} \pm 0.007_{\text{th}}$ which after evolution to the Z -boson mass scale results in $\alpha_s(M_Z) = 0.1212 \pm 0.0011$. The dominant quantifiable theory uncertainty resides in the contribution of as yet uncalculated higher-order perturbative QCD corrections and improvements of the perturbative series through renormalisation group methods.

Of particular interest for the α_s determination is the τ decay rate into light u and d quarks, $R_{\tau,V/A}$, which proceeds either through a vector or an axialvector current, since in this case power corrections are especially suppressed. Theoretically, $R_{\tau,V/A}$ can be expressed in the form [3, 10, 11, 12]

$$R_{\tau,V/A} = \frac{N_c}{2} S_{\text{EW}} |V_{ud}|^2 \left[1 + \delta^{(0)} + \delta'_{\text{EW}} + \sum_{D \geq 2} \delta_{ud,V/A}^{(D)} \right]. \quad (1.2)$$

Here, $S_{\text{EW}} = 1.0198 \pm 0.0006$ [13] and $\delta'_{\text{EW}} = 0.0010 \pm 0.0010$ [14] are electroweak corrections, $\delta^{(0)}$ comprises the perturbative QCD correction which will be our main interest in this work, and the $\delta_{ud,V/A}^{(D)}$ denote quark mass and higher D -dimensional operator corrections which arise in the framework of the operator product expansion (OPE). The higher-order OPE contributions are small and will only be considered towards the end of our work, when we present our determination of α_s .

A particular problem emerges from the observation that different ways of performing the renormalisation group resummation, namely fixed-order (FOPT) or contour-improved perturbation theory (CIPT) [15, 16], apparently lead to differing results. This is especially noteworthy as historically the values of α_s extracted from τ decays employing CIPT have always been on the high side of the world averages, and with the recent update $\alpha_s(M_Z) = 0.1185 \pm 0.0010$ [2] of the latter, this disparity is becoming significant.

CIPT is conventionally the method of choice, since the expansion of the running coupling $\alpha_s(\sqrt{s})$ in $\alpha_s(M_\tau)$ used in FOPT within a certain contour integral (see section 3) is near its radius of convergence and thus argued to lead to a poorly behaved fixed-order series. This argument, however, is not entirely compelling, since QCD perturbation series have zero radius of convergence anyway, and are asymptotic at best, no matter whether CIPT or FOPT is used. Indeed, in the large- β_0 approximation, which may be viewed as a toy model for the entire perturbation series, FOPT was identified to provide the better approximation to the full result [17]. With the recent calculation of the $\mathcal{O}(\alpha_s^4)$ term in the series expansion of the Adler function [18], the discrepancy between FOPT and CIPT appears to be the largest systematic theoretical uncertainty of the α_s determination, as it is evident that it does not go away by adding the presently known higher-order terms.

The following study is motivated by the need to resolve this discrepancy and to understand its origin. Previous investigations [17, 19] show that a preference for CIPT or FOPT may strongly depend on the assumptions made on higher-order terms in the perturbation expansion. Thus, in section 4, we study several toy models in order to address, for each model, the following questions:

- i) Are FO and CI perturbation theory seen to be compatible, once terms beyond the currently known coefficients of the perturbative series for $\delta^{(0)}$ are included?

- ii) How do FO and CI perturbation theory at a particular order compare to the true result for $\delta^{(0)}$, and is the closest approach to the true result related to the minimal terms in the respective series?
- iii) And finally, which of the two methods, FOPT or CIPT provides the closer approach to the true value at order $\mathcal{O}(\alpha_s^4)$, and in general?

Here our working assumption is that the true result is approximated with reasonable accuracy by the Borel sum of the model series, since the power corrections to R_τ are known to be small.

The lessons learnt from the toy models lead us to proposing an ansatz for the Adler function series, more precisely its Borel transform, which incorporates the presently available terms in the perturbative expansion, as well as known features of renormalon singularities [20] determined solely by the operator product expansion and the renormalisation group. This ansatz is described and analysed in detail in section 6. While in the toy models discussed in section 4, one may obtain compatible descriptions of the perturbative series by FOPT or CIPT, or a preference for one of the two prescriptions, we find that the features favouring FOPT prevail in our ansatz for the physical case. For the physical model, CIPT never comes close to the result for the Borel sum. On the other hand, FOPT approaches this sum in a smooth fashion until its minimal term, after which the expected asymptotic (divergent) behaviour sets in.

We believe that these features are characteristic to R_τ and therefore argue that FOPT provides the better approximation to the perturbative series for $\delta^{(0)}$, both at $\mathcal{O}(\alpha_s^4)$ and in general. Based on this observation we proceed to determine the strong coupling α_s in section 7 in two ways: first, employing FOPT and an estimate of the $\mathcal{O}(\alpha_s^5)$ term suggested by several independent arguments; second, employing our ansatz for the entire series as discussed in section 6. Both approaches lead to values of $\alpha_s(M_Z)$ systematically lower than previous determinations from hadronic τ decays employing CIPT.

2. Theoretical framework

We briefly review the main theoretical expressions required in the analysis of the inclusive hadronic τ decay width. Further details and complete expressions can be found in the original works [3, 21, 22]. The central quantities in such an analysis are the two-point correlation functions

$$\Pi_{\mu\nu,ij}^{V/A}(p) \equiv i \int dx e^{ipx} \langle \Omega | T \{ J_{\mu,ij}^{V/A}(x) J_{\nu,ij}^{V/A}(0)^\dagger \} | \Omega \rangle, \quad (2.1)$$

where $|\Omega\rangle$ denotes the physical vacuum and the hadronic vector/axialvector currents are given by $J_{\mu,ij}^{V/A}(x) = [\bar{q}_j \gamma_\mu (\gamma_5) q_i](x)$. The indices i, j stand for the light quark flavours up, down and strange. The correlators $\Pi_{\mu\nu,ij}^{V/A}(p)$ have the Lorentz decomposition

$$\Pi_{\mu\nu,ij}^{V/A}(p) = (p_\mu p_\nu - g_{\mu\nu} p^2) \Pi_{ij}^{V/A,(1)}(p^2) + p_\mu p_\nu \Pi_{ij}^{V/A,(0)}(p^2), \quad (2.2)$$

where the superscripts denote the components corresponding to angular momentum $J = 1$ (transversal) and $J = 0$ (longitudinal) in the hadronic rest frame.

Experimentally, the hadronic decay rate of the τ lepton can be separated into the contributions of vector $R_{\tau,V}$ and axialvector $R_{\tau,A}$ components for the $(\bar{u}d)$ -quark current as well as the contribution with net-strangeness $R_{\tau,S}$, resulting from the $(\bar{u}s)$ -quark current,

$$R_\tau = R_{\tau,V} + R_{\tau,A} + R_{\tau,S}. \quad (2.3)$$

In the Cabibbo-suppressed $(\bar{u}s)$ sector, a separation of vector from axialvector contributions is problematic, since G -parity is not a good quantum number in modes with strange particles.¹ On the theory side, R_τ can be expressed as an integral of the spectral functions $\text{Im } \Pi^{(1)}(s)$ and $\text{Im } \Pi^{(0)}(s)$ over the invariant mass $s = p^2$ of the final state hadrons [23]:

$$R_\tau = 12\pi \int_0^{M_\tau^2} \frac{ds}{M_\tau^2} \left(1 - \frac{s}{M_\tau^2}\right)^2 \left[\left(1 + 2\frac{s}{M_\tau^2}\right) \text{Im } \Pi^{(1)}(s) + \text{Im } \Pi^{(0)}(s) \right]. \quad (2.4)$$

For simplicity, in the following, we shall omit the EW correction factor S_{EW} , but it will of course be included in our final numerical analysis for α_s . The appropriate combinations of the two-point correlation functions resulting from the weak decay through the W -boson are given by

$$\Pi^{(J)}(s) \equiv |V_{ud}|^2 \left[\Pi_{ud}^{V,(J)}(s) + \Pi_{ud}^{A,(J)}(s) \right] + |V_{us}|^2 \left[\Pi_{us}^{V,(J)}(s) + \Pi_{us}^{A,(J)}(s) \right], \quad (2.5)$$

with V_{ij} being the corresponding elements of the Cabibbo-Kobayashi-Maskawa quark-mixing matrix.

The *exact* (non-perturbative) correlation functions are analytic in the complex s -plane cut along the positive axis. Exploiting this property, eq. (2.4) can be expressed as a contour integral in the complex s -plane running counter-clockwise around the circle $|s| = M_\tau^2$ [3]

$$R_\tau = 6\pi i \oint_{|s|=M_\tau^2} \frac{ds}{M_\tau^2} \left(1 - \frac{s}{M_\tau^2}\right)^2 \left[\left(1 + 2\frac{s}{M_\tau^2}\right) \Pi^{(1)}(s) + \Pi^{(0)}(s) \right]. \quad (2.6)$$

The same analytic property holds to any finite order in perturbation theory in α_s (although the discontinuity is arbitrarily wrong at small s). Eqs. (2.4) and (2.6) are equivalent if the correlation functions are substituted either by the exact values or finite order perturbative expansions. The equivalence of eqs. (2.4) and (2.6) does not hold in renormalisation group improved perturbation theory due to the Landau pole singularity [17].

Whereas the correlators $\Pi^{(1)}(s)$ and $\Pi^{(0)}(s)$ themselves are not physical quantities in the sense that they contain renormalisation scale and scheme dependent subtraction constants and thus do not satisfy homogeneous renormalisation group equations, by means of partial integration, eq. (2.6) can be rewritten in terms of the physical correlation functions $D^{(1+0)}(s)$ and $D^{(0)}(s)$,

$$D^{(1+0)}(s) \equiv -s \frac{d}{ds} \left[\Pi^{(1+0)}(s) \right], \quad D^{(0)}(s) \equiv \frac{s}{M_\tau^2} \frac{d}{ds} \left[s \Pi^{(0)}(s) \right], \quad (2.7)$$

¹A small component with strange quarks but without net-strangeness also resides in $R_{\tau,V}$ and $R_{\tau,A}$, with the dominant decay channel being $\tau^- \rightarrow \pi^- K^0 \bar{K}^0 \nu_\tau$.

the first of which being the well-known Adler function [24]. The renormalisation dependent contributions to $\Pi^{(1)}(s)$ and $\Pi^{(0)}(s)$ drop out after contour integration, and in FOPT the perturbative expansions for R_τ based on $\Pi(s)$ or $D(s)$ are identical. However, after RG improvement in CIPT, for the first few terms the series displays a faster rate of convergence when employing the correlators $D^{(1+0)}(s)$ and $D^{(0)}(s)$. Thus in the present work we shall only consider an analysis of R_τ based on this choice. Utilising the dimensionless integration variable $x \equiv s/M_\tau^2$, eq. (2.6) then becomes

$$R_\tau = -i\pi \oint_{|x|=1} \frac{dx}{x} (1-x)^3 \left[3(1+x) D^{(1+0)}(M_\tau^2 x) + 4D^{(0)}(M_\tau^2 x) \right]. \quad (2.8)$$

For large enough negative s , the contributions to $D^{(J)}(s)$ can be organised in the framework of the operator product expansion in a series of local gauge-invariant operators of increasing dimension times appropriate inverse powers of s . This expansion is expected to be well behaved along the complex contour $|s| = M_\tau^2$, except close to the crossing point with the positive real axis [25]. As can be seen from eq. (2.8), however, the contribution near the physical cut at $s = M_\tau^2$ is strongly suppressed by a zero of order three. Therefore, uncertainties associated with the use of the OPE near the time-like axis are expected to be very small. Inserting the OPE series for $D^{(J)}(s)$ into (2.8), performing the contour integration, and extracting the terms proportional to $|V_{ud}|^2$, $R_{\tau,V/A}$ in the form of eq. (1.2) emerges.

The purely perturbative correction $\delta^{(0)}$ only receives contributions from the vector and axialvector correlation function in the chiral limit. Since in this limit vector and axialvector contributions coincide, and $D^{(0)}(s) = 0$, to investigate $\delta^{(0)}$ we can restrict ourselves to the study of the perturbative expansion of the vector correlator $\Pi_V^{(1+0)}(s)$ in the massless case. It exhibits the general structure

$$\Pi_V^{(1+0)}(s) = -\frac{N_c}{12\pi^2} \sum_{n=0}^{\infty} a_\mu^n \sum_{k=0}^{n+1} c_{n,k} L^k, \quad L \equiv \ln \frac{-s}{\mu^2}, \quad (2.9)$$

with $a_\mu \equiv a(\mu^2) \equiv \alpha_s(\mu)/\pi$ and μ the renormalisation scale. As was already remarked above, $\Pi_V^{(1+0)}(s)$ itself is not a physical quantity. However, the spectral function is as well as the Adler function $D_V^{(1+0)}(s)$, whose general expansion then takes the form:

$$D_V^{(1+0)}(s) = \frac{N_c}{12\pi^2} \sum_{n=0}^{\infty} a_\mu^n \sum_{k=1}^{n+1} k c_{n,k} L^{k-1}. \quad (2.10)$$

In this expression, only the coefficients $c_{n,1}$ have to be considered as independent. The coefficients $c_{n,k}$ with $k = 2, \dots, n+1$ can be related to the $c_{n,1}$ and β -function coefficients by means of the renormalisation group equation (RGE), while the coefficients $c_{n,0}$ do not appear in measurable quantities and $c_{n,n+1} = 0$ for $n \geq 1$. Up to order α_s^4 , the RG

constraints lead to:

$$\begin{aligned}
c_{2,2} &= -\frac{\beta_1}{4} c_{1,1}, & c_{3,3} &= \frac{\beta_1^2}{12} c_{1,1}, & c_{3,2} &= -\frac{1}{4} (\beta_2 c_{1,1} + 2\beta_1 c_{2,1}), \\
c_{4,4} &= -\frac{\beta_1^3}{32} c_{1,1}, & c_{4,3} &= \frac{\beta_1}{24} (5\beta_2 c_{1,1} + 6\beta_1 c_{2,1}), \\
c_{4,2} &= -\frac{1}{4} (\beta_3 c_{1,1} + 2\beta_2 c_{2,1} + 3\beta_1 c_{3,1}).
\end{aligned} \tag{2.11}$$

In our convention, the QCD β -function is defined as $\beta(a_\mu) \equiv -\mu da_\mu/d\mu = \sum_{k=1} \beta_k a_\mu^{k+1}$, with the first coefficient being $\beta_1 = 11N_c/6 - N_f/3$. Since the Adler function $D_V^{(1+0)}(s)$ satisfies a homogeneous RGE, the logarithms in eq. (2.10) can be summed with the choice $\mu^2 = -s \equiv Q^2$, leading to the simple expression:

$$D_V^{(1+0)}(Q^2) = \frac{N_c}{12\pi^2} \sum_{n=0}^{\infty} c_{n,1} a_Q^n, \tag{2.12}$$

where $a_Q \equiv \alpha_s(Q)/\pi$.

Until recently, the independent coefficients $c_{n,1}$ were known analytically up to order α_s^3 [26, 27]. At $N_c = 3$ in the $\overline{\text{MS}}$ -scheme [28] they read:

$$c_{0,1} = c_{1,1} = 1, \quad c_{2,1} = \frac{365}{24} - 11\zeta_3 - \left(\frac{11}{12} - \frac{2}{3}\zeta_3\right)N_f = 1.640, \tag{2.13}$$

$$c_{3,1} = \frac{87029}{288} - \frac{1103}{4}\zeta_3 + \frac{275}{6}\zeta_5 - \left(\frac{7847}{216} - \frac{262}{9}\zeta_3 + \frac{25}{9}\zeta_5\right)N_f + \left(\frac{151}{162} - \frac{19}{27}\zeta_3\right)N_f^2 = 6.371,$$

where numerical values are given at $N_f = 3$. For the next five- and six-loop coefficients $c_{4,1}$ and $c_{5,1}$, estimates employing principles of ‘‘minimal sensitivity’’ (PMS) or ‘‘fastest apparent convergence’’ (FAC) [29, 30], together with known terms of order $\alpha_s^4 N_f^2$, exist, which for $N_f = 3$ yield [31, 32]:

$$c_{4,1} = 27 \pm 16, \quad c_{5,1} = 145 \pm 100. \tag{2.14}$$

However, as of this year, the complete result for the $\mathcal{O}(\alpha_s^4)$ coefficient $c_{4,1}$ is available [18], which greatly helps in our analysis. At $N_f = 3$ it reads:

$$c_{4,1} = \frac{78631453}{20736} - \frac{1704247}{432}\zeta_3 + \frac{4185}{8}\zeta_3^2 + \frac{34165}{96}\zeta_5 - \frac{1995}{16}\zeta_7 = 49.076. \tag{2.15}$$

Since this result turns out to be larger than the estimate presented in eq. (2.14), we shall not use the PMS/FAC prediction for $c_{5,1}$ of (2.14). Instead, we attempt to estimate this coefficient based either on a uniform convergence rate of the series, or on our model.

3. Renormalisation group summation

We now discuss the renormalisation group improvement of the purely perturbative correction $\delta^{(0)}$ to R_τ by means of resummation of the logarithms appearing in eq. (2.10).

Returning to eq. (2.8) and inserting the general expansion (2.10) for $D_V^{(1+0)}(s)$, $\delta^{(0)}$ is found to take the form

$$\delta^{(0)} = \sum_{n=1}^{\infty} a_{\mu}^n \sum_{k=1}^n k c_{n,k} \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x} (1-x)^3 (1+x) \ln^{k-1} \left(\frac{-M_{\tau}^2 x}{\mu^2} \right), \quad (3.1)$$

where the identical contribution from the axialvector correlator has already been taken into account.

As discussed above, the Adler function $D_V^{(1+0)}(s)$ and therefore also $\delta^{(0)}$ satisfy a homogeneous RGE. In fixed-order perturbation theory (FOPT) the logarithms in eq. (3.1) are summed by setting $\mu^2 = M_{\tau}^2$, leading to

$$\delta_{\text{FO}}^{(0)} = \sum_{n=1}^{\infty} a(M_{\tau}^2)^n \sum_{k=1}^n k c_{n,k} J_{k-1}. \quad (3.2)$$

The contour integrals J_l in eq. (3.2) are defined by

$$J_l \equiv \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x} (1-x)^3 (1+x) \ln^l(-x) = \frac{1}{2\pi} \left[I_{l,0} + 2 I_{l,1} - 2 I_{l,3} - I_{l,4} \right]. \quad (3.3)$$

The integrals $I_{l,m}$ are given by

$$\begin{aligned} I_{l,m} &\equiv i \oint_{|x|=1} dx (-x)^{m-1} \ln^l(-x) = i^l \int_{-\pi}^{+\pi} d\alpha \alpha^l e^{im\alpha} = i \left(\frac{-1}{m} \right)^{l+1} \Gamma(l+1, -i\alpha m) \Big|_{-\pi}^{+\pi} \\ &= (-1)^{l+m} \frac{2l!}{m^{l+2}} \sum_{k=1}^{[(l+1)/2]} (-1)^k \frac{m^{2k} \pi^{2k-1}}{(2k-1)!}, \end{aligned} \quad (3.4)$$

where $\Gamma(l+1, z)$ is the incomplete Γ -function, $[n]$ denotes the integer part of n and $m \geq 1$. For $m = 0$, one obtains $I_{l,0} = i^l [1 + (-1)^l] \pi^{l+1} / (l+1)$. The first few of the integrals J_l , which are needed up to order α_s^4 , read:

$$J_0 = 1, \quad J_1 = -\frac{19}{12}, \quad J_2 = \frac{265}{72} - \frac{1}{3} \pi^2, \quad J_3 = -\frac{3355}{288} + \frac{19}{12} \pi^2, \quad (3.5)$$

in agreement with ref. [16].

At order α_s^n FOPT contains unsummed logarithms of order $\ln^l(-x) \sim \pi^l$ with $l < n$ related to the contour integrals J_l . Contour-improved perturbation theory (CIPT) sums these logarithms with the choice $\mu^2 = -M_{\tau}^2 x$ in eq. (3.1), which yields

$$\delta_{\text{CI}}^{(0)} = \sum_{n=1}^{\infty} c_{n,1} J_n^a(M_{\tau}^2) \quad (3.6)$$

in terms of the contour integrals $J_n^a(M_{\tau}^2)$ over the running coupling, defined as:

$$J_n^a(M_{\tau}^2) \equiv \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x} (1-x)^3 (1+x) a^n(-M_{\tau}^2 x). \quad (3.7)$$

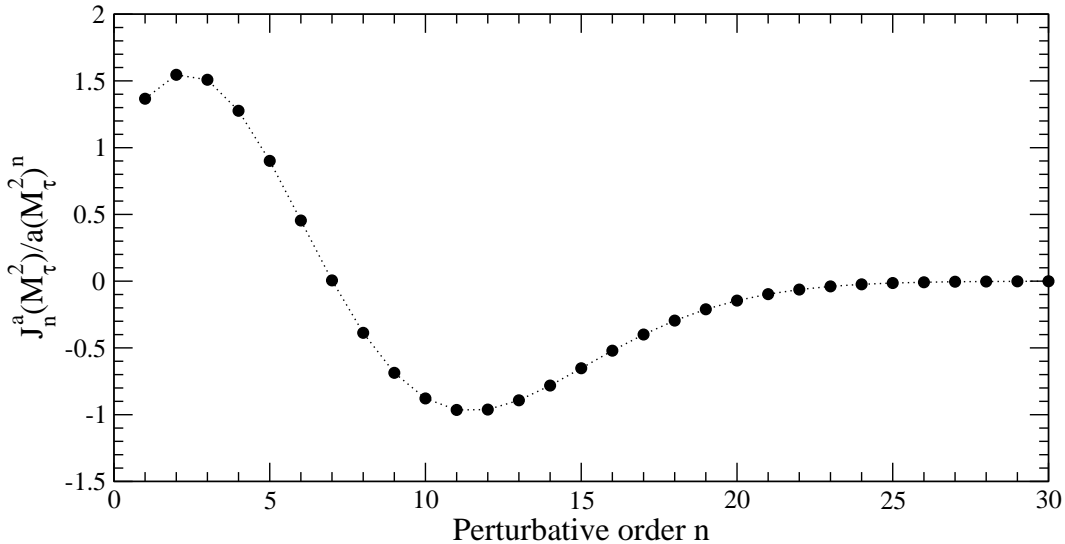


Figure 1: Contour integrals $J_n^a(M_\tau^2)$ of eq. (3.7) required for CIPT as a function of the perturbative order n , computed with an initial value $\alpha_s(M_\tau) = 0.34$ and 4-loop running.

In contrast to FOPT, for CIPT each order n just depends on the corresponding coefficient $c_{n,1}$. Thus, all contributions proportional to the coefficient $c_{n,1}$ which in FOPT appear at all perturbative orders equal or greater than n are resummed into a single term. This is related to the fact that CIPT resums the running of the QCD coupling along the integration contour in the complex s -plane as can be derived directly from eq. (2.12).

In view of our numerical analysis below, a few additional remarks are in order. Although from the form of eq. (3.6), the contour integrals $J_n^a(M_\tau^2)$ could be considered as effective couplings, they have a rather non-trivial dependence on the perturbative order n . This can be seen from figure 1, in which we display $J_n^a(M_\tau^2)/a(M_\tau^2)^n$ for an initial value $\alpha_s(M_\tau) = 0.34$ [7] as a function of n up to $n = 30$. Up to the 7th order, $J_n^a(M_\tau^2)$ is positive and then above the 7th order turns negative. This implies that $J_7^a(M_\tau^2)$ is rather small which is later reflected in the fact that with four-loop running of α_s always the 7th term in the CIPT series is found smallest. This observation already casts some doubts on the approach of treating the CIPT series in the sense of an asymptotic series for which quite often the optimal truncation is provided by breaking the series at the smallest term [33].

We now recall (as is well-known, see for instance [18]) that the two approaches lead to significant numerical differences. Using the analytically known coefficients of eqs. (2.13) as well as (2.15) and $\alpha_s(M_\tau) = 0.34$ in eqs. (3.2) and (3.6), we obtain:

$$\alpha_s^1 \quad \alpha_s^2 \quad \alpha_s^3 \quad \alpha_s^4 \quad \alpha_s^5$$

$$\delta_{\text{FO}}^{(0)} = 0.1082 + 0.0609 + 0.0334 + 0.0174 (+0.0088) = 0.2200 (0.2288), \quad (3.8)$$

$$\delta_{\text{CI}}^{(0)} = 0.1479 + 0.0297 + 0.0122 + 0.0086 (+0.0038) = 0.1984 (0.2021). \quad (3.9)$$

The CI series displays a faster convergence, but the two series do not appear to approach a common value as successive terms are added. Summing both series up to order α_s^4 , the

difference between FO and CI perturbation theory amounts to 0.0216. The size of this difference is of the order of the last included term in the FO series and about a factor of 2.5 times the corresponding CIPT term. This apparent disparity in the perturbative prediction at the moment represents the dominant theoretical uncertainty in the extraction of the strong coupling α_s from the hadronic τ decay rate. Thus, it is legitimate to ask how the series for $\delta_{\text{CI}}^{(0)}$ and $\delta_{\text{FO}}^{(0)}$ behave if even higher-order perturbative coefficients are included. On the one hand both results would be expected to be compatible, if an all-order result were available. On the other hand, re-expansion of the contour-improved integrals $J_n^a(M_\tau^2)$ into the fixed-order series in $a(M_\tau^2)$ results in a series that is barely convergent for realistic values of $a(M_\tau^2)$ [16], so one might question the validity of FOPT altogether. In the following sections we examine these issues assuming different behaviours of the higher-order terms in the series.

An immediate question which arises is, if, on the basis of the series up to the fourth order, we are in a position to say something about the next coefficient $c_{5,1}$? As a somewhat naive guess, we might assume that the size of the fifth order should be at most of the size of the previous term and larger than zero, which is to say that the asymptotic (divergent, sign-alternating) behaviour has not yet set in at the fifth order. Applying this criterion to the CI series, one arrives at the estimate $0 < c_{5,1} < 642$. A slightly more elaborate estimate can be based on the striking feature that the convergence rate of the FOPT series is found to be very uniform. Each new term is rather closely half of the preceding one, and the slight dependence on the order can even be nicely fit to a linear behaviour. Assuming that this property persists also at the fifth order, we arrive at the estimate²

$$c_{5,1} \approx 283. \tag{3.10}$$

This value lies close to the centre of the range given above, and in section 6 it will be seen that eq. (3.10) is corroborated by our model of higher order coefficients. Interestingly, it is also close to the update of the FAC estimate to account for the newly available exact $c_{4,1}$, which yields $c_{5,1} = 275$ [18]. Including the estimate (3.10) in the series for $\delta_{\text{FO}}^{(0)}$ and $\delta_{\text{CI}}^{(0)}$, the numbers in brackets given in eqs. (3.8) and (3.9) are obtained. Now, the difference $\delta_{\text{FO}}^{(0)} - \delta_{\text{CI}}^{(0)} = 0.0267$ is increased even further, and found much larger than the last included summands.

4. Higher orders: toy models

To acquire some feeling of what can happen to $\delta^{(0)}$ in FO and CI perturbation theory when higher terms in the perturbative series are included, in this section we exhibit a few toy models. Let us emphasise that we do not believe that these models have much in common with the true QCD case (with the exception, perhaps, of the large- β_0 approximation). Rather our concern is to find out which features of the higher-order series determine whether CIPT or FOPT represents a better approximation to the true result. Inspired by what we

²Postdicting $c_{4,1}$ in this way, we find $c_{4,1} = 52!$ The first few terms of the FO series for $\delta^{(0)}$ are very nearly geometric.

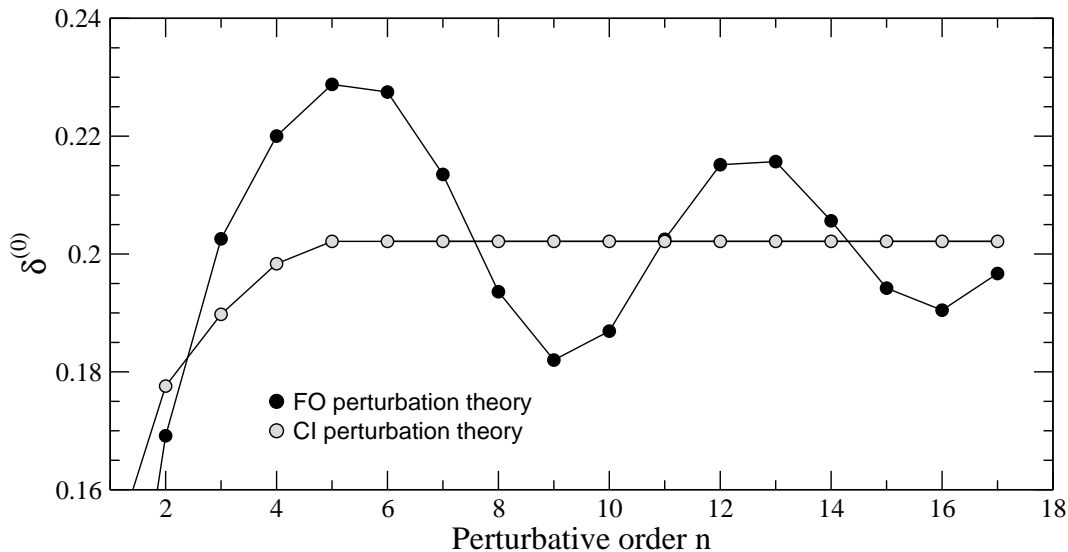


Figure 2: Results for $\delta_{\text{FO}}^{(0)}$ (full circles) and $\delta_{\text{CI}}^{(0)}$ (grey circles) at $\alpha_s(M_\tau) = 0.34$, setting the higher-order coefficients $c_{n,1} = 0$ for $n \geq 6$. The results up to the fifth order coincide with eqs. (3.8) and (3.9) respectively.

learn from the models, we are led to the construction of a realistic ansatz which will be discussed in detail in the following sections.

4.1 Truncated Adler function

Let us begin by considering the case in which all perturbative coefficients $c_{n,1}$ are set to zero for $n \geq 6$. This model has already been investigated in ref. [19], in order to see how FOPT and CIPT might be compatible even though at the fifth order their difference appears rather dramatic. A graphical display of the model is shown in figure 2. The result for $\delta_{\text{FO}}^{(0)}$ is given as the full circles and $\delta_{\text{CI}}^{(0)}$ as the grey circles, as a function of the order up to which the perturbative series has been summed. To guide the eye, we have also connected the points by straight line segments. Since at each order $\delta_{\text{CI}}^{(0)}$ only depends on the explicit coefficient $c_{n,1}$, starting from the order where the coefficients are taken to be zero, CIPT becomes exact (and thus $\delta_{\text{CI}}^{(0)}$ constant). On the contrary, in FOPT, besides the contribution from the $c_{n,1}$, at each order we also have contributions involving all lower $c_{k,1}$ with $k < n$, which are due to the running of α_s along the complex contour. In FOPT these latter terms are present even if the higher $c_{n,1}$ are set to zero, and entail that for higher orders $\delta_{\text{FO}}^{(0)}$ oscillates around the constant $\delta_{\text{CI}}^{(0)}$. It is evident that for the present example FOPT represents a rather poor approximation to the exact result up to very high n .

Quite generally, we can write $\delta_{\text{FO}}^{(0)}$ in the form

$$\delta_{\text{FO}}^{(0)} = \sum_{n=1}^{\infty} [c_{n,1} + g_n] a(M_\tau^2)^n, \quad (4.1)$$

where the $c_{n,1}$ series is simply the Adler function series, while the g_n series represents the additional contribution from the contour-integral of the Adler function series. By

comparison with eq. (3.2), $g_n = \sum_{k=2}^n k c_{n,k} J_{k-1}$, hence depending only on $c_{k,1}$ with $k < n$ and β -function coefficients as remarked above. With only a few values of $c_{n,1}$ given, the g_n series has a finite radius of convergence [16]. For $\alpha_s(M_\tau)$ slightly larger than 0.34 the series becomes divergent due to large running coupling effects along the circular contour – the amplitude of oscillations of FOPT around the exact result grows and FOPT is never a good approximation. This is the reason why CIPT is usually argued to provide the more reliable approximation to R_τ .

The general picture observed in this special case also translates to other models in which the coefficients $c_{n,1}$ are small compared to the running effects along the complex contour. However, the real series expansion of R_τ is not of this general form. Rather, both the c_n and g_n series are divergent with zero radius of convergence, and can at best assumed to be asymptotic. Moreover, systematic cancellations are predicted between the c_n and g_n terms, when n is sufficiently large.

4.2 The “large- β_0 ” approximation

We take a first look at the issues that arise when the series have zero radius of convergence in another toy-model, the so-called “large- β_0 ” approximation.³ An analytic result for the Borel transform of the Adler function and the corresponding R_τ is available [17, 20, 34, 35, 36] in this approximation, and thus the perturbative coefficients $c_{n,1}$ are known to all orders.

Let us briefly review the results for $\delta^{(0)}$ in the large- β_0 approximation. To make contact to the notation employed in the original works on renormalons in connection to the Adler function and R_τ [17, 36] (for a review see [20]), it is convenient to define the new function $\widehat{D}(s)$ by

$$\frac{12\pi^2}{N_c} D_V^{(1+0)}(s) \equiv 1 + \widehat{D}(s) \equiv 1 + \sum_{n=0}^{\infty} r_n \alpha_s(\sqrt{s})^{n+1}. \quad (4.2)$$

Then the expansion coefficients of $D_V^{(1+0)}(s)$ and $\widehat{D}(s)$ are related by $c_{n,1} = \pi^n r_{n-1}$. Next, the Borel-transform of $\widehat{D}(s)$ is defined by

$$B[\widehat{D}](t) \equiv \sum_{n=0}^{\infty} r_n \frac{t^n}{n!}. \quad (4.3)$$

If $B[\widehat{D}](t)$ has no singularities for real positive t (which is not the case for the Adler function) and does not increase too rapidly at positive infinity, one can define the Borel integral (α positive) as

$$\widehat{D}(\alpha) \equiv \int_0^{\infty} dt e^{-t/\alpha} B[\widehat{D}](t), \quad (4.4)$$

³For historical reasons, we shall speak about the “large- β_0 ” approximation, although in the notation employed in this work, the leading coefficient of the β -function is termed β_1 . The “large- β_0 ” approximation uses only the term with the highest power in the number of light flavours, N_f , in $c_{n,1}$, and replaces N_f by $-3\beta_1$. Correspondingly, in the evolution of α_s , only one-loop running is taken into account.

which has the same series expansion in α as $\widehat{D}(s)$ does in $\alpha_s(\sqrt{s})$. The integral $\widehat{D}(\alpha)$, if it exists, gives the Borel sum of the original divergent series. Calculating so-called bubble-chain diagrams, it was found that the Borel-transformed Adler function $B[\widehat{D}](t)$ obtains infrared (IR) and ultraviolet (UV) renormalon poles at positive and negative integer values of the variable $u \equiv \beta_1 t / (2\pi)$, respectively [34, 35]. (With the exception of $u = 1$.) While the IR renormalons are related to power corrections in the operator product expansion, the leading UV renormalon, being closest to $u = 0$, dictates the large-order behaviour of the perturbative expansion.

The central result of refs. [34, 35] is that the Borel transform of the Adler function in the large- β_0 approximation (see also eq. (5.10) of ref. [20]) can be expressed as [35]

$$B[\widehat{D}](u) = \frac{32}{3\pi} \frac{e^{-Cu}}{(2-u)} \sum_{k=2}^{\infty} \frac{(-1)^k k}{[k^2 - (1-u)^2]^2}, \quad (4.5)$$

where C is a scheme-dependent constant which cancels the scheme dependence of α in eq. (4.4), such that $\widehat{D}(s)$ is independent of this choice. In the $\overline{\text{MS}}$ -scheme it takes the value $C = -5/3$. Taylor expanding the Borel transform in the variable u and performing the Borel integral (4.4) term by term, the perturbative coefficients $c_{n,1}$ for the large- β_0 approximation can be deduced.⁴ Numerical values for the first 12 coefficients in the $\overline{\text{MS}}$ -scheme are presented in table 1. One observes that the dominance of the leading UV renormalon at $u = -1$, and the corresponding sign alternating behaviour of the series, in the large- β_0 approximation sets in at the 6th order.

The perturbative coefficients of table 1 can now be employed to calculate $\delta^{(0)}$ in the large- β_0 approximation. For consistency the running of α_s was also implemented at the one-loop order. (We shall soon see that this plays an important role.) A graphical account of our results at the physical coupling $\alpha_s(M_\tau) = 0.34$ is displayed in figure 3. Again, the full circles correspond to $\delta_{\text{FO}}^{(0)}$ in FOPT while the grey circles provide $\delta_{\text{CI}}^{(0)}$ in the large- β_0 approximation. In both cases, the grey diamonds represent the order at which the FO and CI series have their smallest terms, before the asymptotic behaviour sets in. The qualitative picture is rather different from the previous case of the truncated Adler function displayed in figure 2. Both series appear to reach a plateau before the divergence sets in, though the plateau is reached at higher order for FOPT in agreement with the earlier analysis [17]. However, the difference in the plateau values of $\delta^{(0)}$ is far larger than the minimal terms of both series, hence FOPT and CIPT seem to give incompatible results within the conventional uncertainty estimates.

⁴A particularly effective way of analytically generating the large- β_0 coefficients $c_{n,1}$ can be derived from eq. (26) of ref. [37].

$c_{1,1}$	$c_{2,1}$	$c_{3,1}$	$c_{4,1}$	$c_{5,1}$	$c_{6,1}$
1	1.5565	15.711	24.832	787.83	-1991.4
$c_{7,1}$	$c_{8,1}$	$c_{9,1}$	$c_{10,1}$	$c_{11,1}$	$c_{12,1}$
$9.857 \cdot 10^4$	$-1.078 \cdot 10^6$	$2.775 \cdot 10^7$	$-5.388 \cdot 10^8$	$1.396 \cdot 10^{10}$	$-3.598 \cdot 10^{11}$

Table 1: Perturbative coefficients $c_{n,1}$ in the large- β_0 approximation up to 12th order.

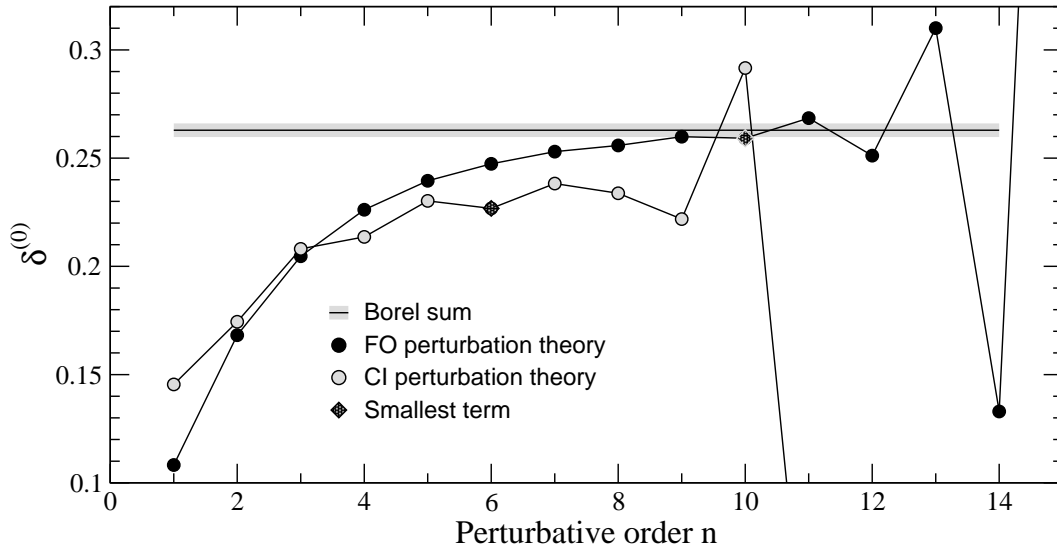


Figure 3: Results for $\delta_{\text{FO}}^{(0)}$ (full circles) and $\delta_{\text{CI}}^{(0)}$ (grey circles) at $\alpha_s(M_\tau) = 0.34$, employing the higher-order coefficients $c_{n,1}$ of table 1 obtained in the large- β_0 approximation, as a function of the order n up to which the terms in the perturbative series have been summed. The straight line represents the result for the Borel sum of the series, and the shaded band provides an error estimate inferred from the complex ambiguity.

Which is the better approximation to the “true” result? Since we now have a closed expression for the Borel transform of the Adler function, we can compute the Borel integral and compare the perturbative series to this result. We do not expect the Borel integral to correspond exactly to the “true” result. First, the “true” result receives condensate corrections from higher-dimensional operators in the OPE expansion. Second, when the Borel transform has poles on the positive t -axis, the Borel integral must be defined by an arbitrary deformation of the contour into the complex plane, which introduces an ambiguity, whose dependence on α matches the condensate corrections. The two are closely related (see the review [20]), and it has been observed [17] that the size of the ambiguity divided by π is indeed of the order of non-perturbative corrections in the OPE. We thus conclude that we expect the “true” result to coincide with the principal value of the Borel integral within an accuracy set by about the ambiguity of Borel integral (divided by π), and certainly not parametrically larger.

In figure 3 the horizontal line represents the value of the Borel integral when the series is summed via the principal-value prescription, and the shaded region marks the size of the complex ambiguity in the Borel integral.⁵ Details on the analytical calculation of the Borel transform have been relegated to appendix A. A good approximation of the “true” result should approach the shaded region smoothly and diverge eventually. Figure 3 shows the remarkable result, already observed in [17], that in the large- β_0 approximation FOPT approaches the Borel sum in a rather monotonous fashion until the 10th order after which the sign-alternating divergent behaviour of the series sets in, while CIPT never comes close

⁵To obtain the maximal complex ambiguity the IR renormalon poles are circled in such a way that they all contribute with the same sign. The modulus of the imaginary part thus obtained is then divided by π .

to the “true” result due to an earlier onset of the divergence. Clearly FOPT represents the better approximation here, even at $n = 4$, in stark contrast to the previous example of the truncated Adler function series.

Let us try to gain a better insight as to why the behaviour in the large- β_0 approximation is so different from our first model. An immediate observation that can be made is that the $c_{n,1}$ have to be of a similar size as the running effects g_n from the contour, such that strong cancellations between these two effects take place. To make this more precise, consider the simple relation between $B[\widehat{D}]$ and the correspondingly defined Borel transform of $\delta^{(0)}$ expanded in $\alpha_s(M_\tau)$ given by [38]

$$B[\delta^{(0)}](u) = B[\widehat{D}](u) \sin(\pi u) \left[\frac{1}{\pi u} + \frac{2}{\pi(1-u)} - \frac{2}{\pi(3-u)} + \frac{1}{\pi(4-u)} \right], \quad (4.6)$$

which is valid in the large- β_0 limit.⁶ We are then in a position to derive the large-order behaviour. Applying the decomposition (4.1), we obtain

$$\begin{aligned} c_{n+1,1} &= \left(\frac{\beta_1}{2}\right)^n n! \left[\frac{4}{9} e^{-5/3} (-1)^n \left(n + \frac{7}{2}\right) + \frac{e^{10/3}}{2^n} + \dots \right], \\ g_{n+1} &= \left(\frac{\beta_1}{2}\right)^n n! \left[-\frac{4}{9} e^{-5/3} (-1)^n \left(n + \frac{16}{5}\right) - \frac{e^{10/3}}{2^n} + \dots \right], \end{aligned} \quad (4.7)$$

where the dots denote the less important poles beyond the leading ultraviolet renormalon pole ($u = -1$) and the leading infrared one ($u = 2$). This shows explicitly the strong cancellations that take place between the Adler function series and the extra terms generated by the integration along the circle. The suppression of the large- n divergence allows FOPT to approach the Borel sum smoothly. We also see why CIPT fails in this case: when such strong cancellations between the $c_{n,1}$ and g_n series are present, it is mandatory to combine the two series at the same order. However, CIPT uses the $c_{n,1}$ up to some finite order, while summing the g_n to all orders, thus missing the cancellation [17, 20]. The result is that CIPT runs earlier into the leading UV renormalon divergence, as seen in figure 3, though the divergence is damped by the suppression of the effective couplings $J_n^a(M_\tau^2)$ as discussed in section 3.

There are some lessons that can be drawn from our observations which are valid beyond the large- β_0 toy model. First, since the known exact coefficients of the Adler function series show no interference of a sign-alternating component, we expect the leading IR renormalon at $u = 2$ to be the most relevant contribution, before the eventual sign-alternation takes over, just as above. Second, the leading IR renormalon contribution will no longer cancel completely as it does in eq. (4.7). However, it is true in general that it is suppressed by a factor $1/n^2$ in the sum $c_{n,1} + g_n$ relative to $c_{n,1}$ alone [38], since this follows from the OPE and the anomalous dimension of the gluon condensate. Thus, we expect some of the features of the large- β_0 toy model to survive in the realistic QCD case.

⁶To derive this result, insert eq. (4.4) into eq. (2.8) and perform the contour integral employing the one-loop expression for $\alpha_s(\sqrt{s})$.

Next, we corroborate these findings with two further simplistic models, which will also uncover another crucial ingredient for the building of a more realistic model for higher order coefficients.

4.3 Single pole models

In order to separate the effects of a specific renormalon pole, and to give support to our previous findings, we investigate a single IR renormalon pole at position p , for which we shall consider the particular cases $p = 2$ and $p = 3$. Explicitly, our single pole model for the Borel transform of the Adler function series takes the form:

$$B[\widehat{D}_p](u) \equiv \frac{d_p^{\text{IR}}}{(p-u)} + \sum_{i=0}^3 d_i^{\text{PO}} u^i. \quad (4.8)$$

We fix the residue d_p^{IR} such that the perturbative coefficient $c_{5,1} = 283$, and the polynomial terms d_i^{PO} are adjusted such as to reproduce the lower order coefficients according to eqs. (2.13) and (2.15). The reason behind employing $c_{5,1}$ to fix the residue is that we like to work with a perturbative order at which a dominance of the leading IR renormalon to the coefficient is expected, but this assumption is not crucial for our argument.

The numerical results for the model (4.8) with $p = 2$ are shown in figure 4. The upper plot corresponds to $\delta_{\text{FO}}^{(0)}$, the lower one to $\delta_{\text{CI}}^{(0)}$. Let us first concentrate on the triangles which are connected by dashed line segments, and for which the running of α_s at 1-loop has been employed. This case is rather similar to the large- β_0 approximation. FOPT approaches the Borel sum (the horizontal dashed line) well, although here in a slightly oscillatory manner, and up to the order shown in the plot the divergent behaviour has not yet set in. Conversely, like for the large- β_0 approximation above, for CIPT the divergent behaviour sets in much earlier and the series never comes close to the Borel sum.

The picture drastically changes when the running of the coupling in the contour integration is implemented at four loops. This case is shown as the full and grey diamonds in figure 4, which are connected by dash-dotted lines. The corresponding Borel sum is the straight dash-dotted line. $\delta_{\text{FO}}^{(0)}$ first overshoots the “true” result by a large amount, then displays a large oscillation similar to figure 2, before it starts to diverge, while CIPT appears more like the 1-loop case. Still, with 4-loop running, neither FOPT nor CIPT approach the Borel sum in a sensible fashion. The reason for this unexpected behaviour can be traced back to the fact that our model (4.8) only contains a simple pole. Such a simple pole is the correct structure of a renormalon pole in the large- β_0 limit, but when higher terms in the β -function are to be included, the renormalon pole structure gets more complicated, also involving cuts, whose gross features are determined by the OPE [20]. Hence, to construct a consistent model which aims to use a 4-loop running coupling, also the renormalon cut structure has to be incorporated at the same order. The results required at four loops will be derived in the next section. Once four-loop running is consistently included in the Adler function Borel transform eq. (4.8) and the contour integration, we obtain the circles connected by solid lines in figure 4. Now we find again that FOPT for an IR pole at $u = 2$ smoothly approaches the Borel sum, while CIPT fails.

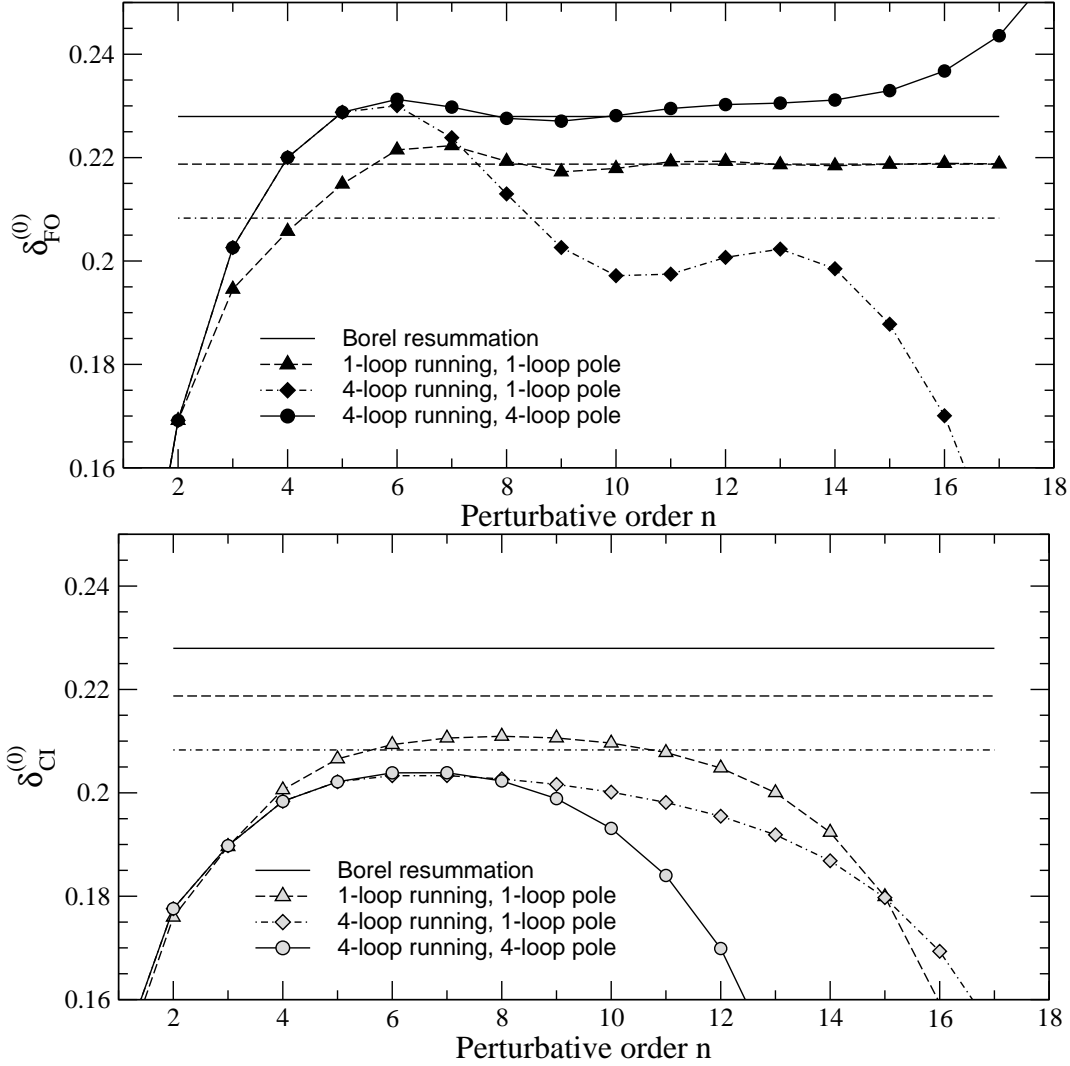


Figure 4: Results for $\delta_{\text{FO}}^{(0)}$ (full circles, diamonds, triangles) and $\delta_{\text{CI}}^{(0)}$ (grey circles, diamonds, triangles) at $\alpha_s(M_\tau) = 0.34$, for the model of eq. (4.8) at $p = 2$. The triangles correspond to a running of α_s at 1-loop, while for the diamonds a 4-loop running coupling has been employed. For the circles the pole in eq. (4.8) is modified to account for the 4-loop renormalon cut structure. The straight dashed, dash-dotted and solid lines represent the respective Borel sums.

To conclude the discussion of simple toy models, we finally investigate the ansatz (4.8) with $p = 3$. Graphically, this case is displayed in figure 5, with the same notation as used in figure 4. We see that this case very much resembles the model of figure 2, where the higher $c_{n,1}$ had been set to zero. The $p = 3$ model differs from the $p = 2$ one in two respects. First, it follows from eq. (4.6) that there is no suppression of $c_{n,1} + g_n$ relative to $c_{n,1}$ in large orders. More importantly, with $p = 3$ the divergence of the series is milder. Hence, the Adler function coefficients $c_{n,1}$ are much smaller than the running effects g_n along the complex contour, an expectation that can be verified by explicitly investigating the $c_{n,1}$ in this model. Thus, like for the first model, the truncated Adler function, CIPT provides a

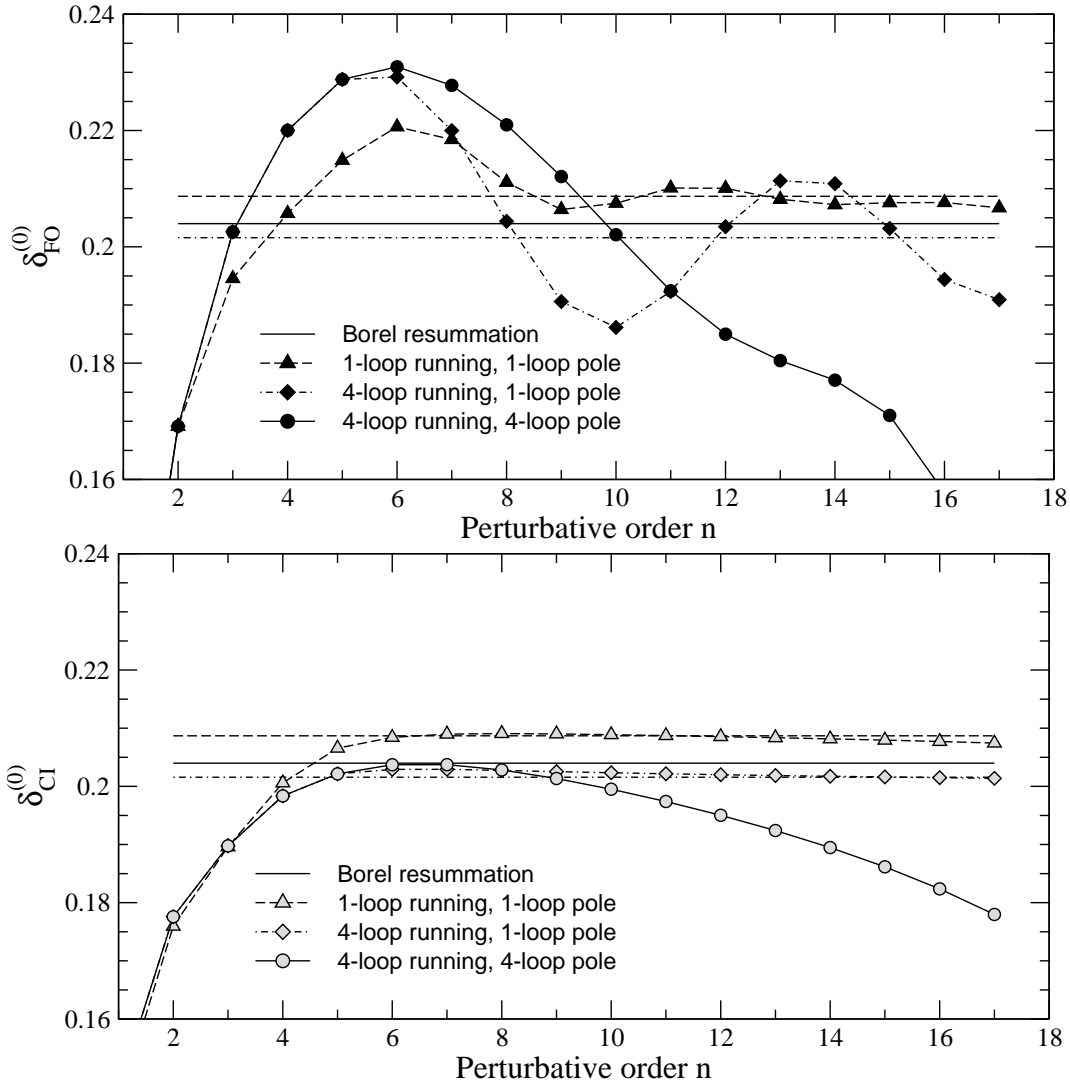


Figure 5: Results for $\delta_{FO}^{(0)}$ (full circles, diamonds, triangles) and $\delta_{CI}^{(0)}$ (grey circles, diamonds, triangles) at $\alpha_s(M_\tau) = 0.34$, for the model of eq. (4.8) at $p = 3$. The triangles correspond to a running of α_s at 1-loop, while for the diamonds a 4-loop running coupling has been employed. For the circles the pole in eq. (4.8) is modified to account for the 4-loop renormalon cut structure. The straight dashed, dash-dotted and solid lines represent the respective Borel sums.

good account of the Borel sum, and FOPT is only able to approach this value with large oscillations.

4.4 Resumé

The main conclusions from the toy examples are as follows. We find that CIPT provides the better approximation whenever running coupling effects to the series expansion of R_τ dominate over the intrinsic Adler function coefficients as should have been expected. This is the case in truncated perturbation theory and in models with weak factorial divergence, such as the $p = 3$ single-pole model. We find that FOPT provides the better approxima-

tion, whenever there are systematic cancellations between the Adler function and running coupling contributions. Such cancellations occur in the $p = 2$ single-pole model, the large- β_0 approximation, and in general for the leading IR renormalon contribution to R_τ . We also find that to correctly account for these cancellations, the running coupling effects have to be implemented at the same loop-order in the contour integral and the renormalon structure of the Adler function Borel transform. The interesting question is now which of these features is relevant to the “real world”.

The discussion of this section already allows to disfavour a large class of models for the realistic Borel-transformed Adler function that were initially considered by us. In this class fall models with renormalon poles that only have integer power, because they fail to account consistently for running coupling effects beyond 1-loop evolution. For the same reason – besides not using the available information on the known positions of the renormalon poles – also models based on Padé approximation are of limited use.

In order to be able to build a more realistic model for $B[\widehat{D}](u)$, as a prerequisite, employing the RGE and the structure of the OPE, in the next section we shall derive the general form of the renormalon cut including β -function effects up to 4-loop level.

5. Renormalon poles at four loops

Our next aim is a physically motivated ansatz for the Borel transform of the Adler function $B[\widehat{D}](t)$, which incorporates all known exact results, and on the basis of which we will be in a position to investigate the influence of higher-order perturbative contributions, and to perform a comparison of the perturbative series in FOPT and CIPT with its Borel sum. As has been seen in the last section, if higher-order running effects are to be included in the contour integration, the renormalon pole structure should match the corresponding loop order. The derivation of the renormalon cut is based just on the structure of the OPE and the RGE, detached from any limitations of the large- β_0 approximation, up to an unknown overall constant. The expressions obtained in this section extend the analysis already presented in sections 3.2.3 and 3.3.1 of ref. [20] to one more loop order.

Let us begin with the IR renormalon poles. The central idea is that the IR renormalon ambiguity of the Borel integral arises from long-distance regions in Feynman integrals, and therefore must be consistent with the power-suppressed terms appearing in the operator product expansion [39, 40]. Comparing the energy dependence of a certain term in the OPE to the one of the complex ambiguity of the Borel integral, the renormalon singularity that gives rise to this ambiguity can be determined. A generic term in the OPE of $\widehat{D}(s)$ from an operator O_d of dimension d can be written as

$$\widehat{C}_{O_d}(a_Q) \frac{\langle \widehat{O}_d \rangle}{Q^d} = [a_Q]^{\frac{\gamma_{O_d}^{(1)}}{\beta_1}} \left[\widehat{C}_{O_d}^{(0)} + \widehat{C}_{O_d}^{(1)} a_Q + \widehat{C}_{O_d}^{(2)} a_Q^2 + \dots \right] \frac{\langle \widehat{O}_d \rangle}{Q^d}, \quad (5.1)$$

where the anomalous dimension γ_{O_d} of the operator O_d is defined by

$$-\mu \frac{d}{d\mu} O_d(\mu) \equiv \gamma_{O_d}(a_\mu) O_d(\mu) = \left[\gamma_{O_d}^{(1)} a_\mu + \gamma_{O_d}^{(2)} a_\mu^2 + \gamma_{O_d}^{(3)} a_\mu^3 + \dots \right] O_d(\mu). \quad (5.2)$$

For convenience, we have expressed eq. (5.1) in terms of the scale invariant operator \widehat{O}_d , defined by

$$\widehat{O}_d \equiv O_d(\mu) \exp \left\{ - \int \frac{\gamma_{O_d}(a_\mu)}{\beta(a_\mu)} da_\mu \right\}, \quad (5.3)$$

such that higher order coefficients of γ_{O_d} are contained in the Wilson coefficients $\widehat{C}_{O_d}^{(k)}$. Since eq. (5.1) will only be needed up to a multiplicative factor, we do not have to specify the constant of integration in (5.3), and without loss of generality, it can be assumed to be zero. Employing the RGE for a_Q , the Q -dependent part of (5.1) can be written as:

$$\begin{aligned} \frac{\widehat{C}_{O_d}(a_Q)}{Q^d} &= \text{const.} \times \widehat{C}_{O_d}(a_Q) e^{-\frac{d}{\beta_1 a_Q}} [a_Q]^{-d \frac{\beta_2}{\beta_1^2}} \exp \left\{ d \int_0^{a_Q} \left[\frac{1}{\beta(a)} - \frac{1}{\beta_1 a^2} + \frac{\beta_2}{\beta_1^2 a} \right] da \right\}, \\ &= \text{const.} \times \widehat{C}_{O_d}(a_Q) e^{-\frac{d}{\beta_1 a_Q}} [a_Q]^{-d \frac{\beta_2}{\beta_1^2}} \left[1 + b_1 a_Q + b_2 a_Q^2 + \dots \right], \end{aligned} \quad (5.4)$$

where the coefficients b_1 and b_2 are found to be:

$$b_1 = \frac{d}{\beta_1^3} (\beta_2^2 - \beta_1 \beta_3), \quad b_2 = \frac{b_1^2}{2} - \frac{d}{2\beta_1^4} (\beta_2^3 - 2\beta_1 \beta_2 \beta_3 + \beta_1^2 \beta_4). \quad (5.5)$$

To find the Borel transform that matches the Q -dependence of (5.4), we take the ansatz:

$$B[\widehat{D}_p^{\text{IR}}](u) \equiv \frac{d_p^{\text{IR}}}{(p-u)^{1+\tilde{\gamma}}} \left[1 + \tilde{b}_1(p-u) + \tilde{b}_2(p-u)^2 + \dots \right]. \quad (5.6)$$

Employing eq. (A.8), the imaginary ambiguity corresponding to the Borel integral of $B[\widehat{D}_p^{\text{IR}}](u)$ is found to be:

$$\text{Im} \left[\widehat{D}_p^{\text{IR}}(a_Q) \right] = \text{const.} \times e^{-\frac{2p}{\beta_1 a_Q}} [a_Q]^{-\tilde{\gamma}} \left[1 + \tilde{b}_1 \frac{\beta_1}{2} \tilde{\gamma} a_Q + \tilde{b}_2 \frac{\beta_1^2}{4} \tilde{\gamma}(\tilde{\gamma}-1) a_Q^2 + \dots \right]. \quad (5.7)$$

Comparing eqs. (5.4) and (5.7), one deduces:

$$p = \frac{d}{2}, \quad \tilde{\gamma} = 2p \frac{\beta_2}{\beta_1^2} - \frac{\gamma_{O_d}^{(1)}}{\beta_1}, \quad \tilde{b}_1 = \frac{2(b_1 + c_1)}{\beta_1 \tilde{\gamma}}, \quad \tilde{b}_2 = \frac{4(b_2 + b_1 c_1 + c_2)}{\beta_1^2 \tilde{\gamma}(\tilde{\gamma}-1)}, \quad (5.8)$$

where $c_1 \equiv \widehat{C}_{O_d}^{(1)}/\widehat{C}_{O_d}^{(0)}$ and $c_2 \equiv \widehat{C}_{O_d}^{(2)}/\widehat{C}_{O_d}^{(0)}$. Taylor expanding the ansatz (5.6) in u and performing the Borel integral term by term yields the perturbative series:

$$\begin{aligned} \widehat{D}_p^{\text{IR}}(a_Q) &= \frac{\pi d_p^{\text{IR}}}{p^{1+\tilde{\gamma}} \Gamma(1+\tilde{\gamma})} \sum_{n=0}^{\infty} \Gamma(n+1+\tilde{\gamma}) \left(\frac{\beta_1}{2p} \right)^n a_Q^{n+1} \\ &\times \left[1 + \frac{2p(b_1 + c_1)}{\beta_1(n+\tilde{\gamma})} + \left(\frac{2p}{\beta_1} \right)^2 \frac{(b_2 + b_1 c_1 + c_2)}{(n+\tilde{\gamma})(n+\tilde{\gamma}-1)} + \mathcal{O} \left(\frac{1}{n^3} \right) \right]. \end{aligned} \quad (5.9)$$

Eq. (5.9) extends the corresponding eq. (3.51) of ref. [20] to include terms of order $1/n^2$ in the large-order behaviour of the perturbative series.⁷

⁷Note a missing sign in the global factor containing β_0 in eq. (3.51) of [20], which should read $(-2\beta_0/d)^n$.

The corresponding expression for a general UV renormalon pole when higher orders in the running are included can be deduced by formally considering negative coupling a_Q and a Borel integral that ranges from zero to minus infinity [41]. Then one finds poles leading to ambiguities on the negative real axis, and the complex ambiguities can again be identified with the RGE properties of some higher-dimensional operators. The result is that the constants in the parametrisation of the general UV singularity,

$$B[\widehat{D}_p^{\text{UV}}](u) \equiv \frac{d_p^{\text{UV}}}{(p+u)^{1+\bar{\gamma}}} \left[1 + \bar{b}_1(p+u) + \bar{b}_2(p+u)^2 \right], \quad (5.10)$$

can be obtained from the corresponding parameters of the IR renormalon pole (5.6) with the replacement $p \rightarrow -p$, leading to:

$$\bar{\gamma} = -2p \frac{\beta_2}{\beta_1^2} + \frac{\gamma_{O_d}^{(1)}}{\beta_1}, \quad \bar{b}_1 = -\tilde{b}_1[p \rightarrow -p], \quad \bar{b}_2 = \tilde{b}_2[p \rightarrow -p]. \quad (5.11)$$

The perturbative expansion corresponding to eq. (5.10) takes the form:⁸

$$\begin{aligned} \widehat{D}_p^{\text{UV}}(a_Q) &= \frac{\pi d_p^{\text{UV}}}{p^{1+\bar{\gamma}} \Gamma(1+\bar{\gamma})} \sum_{n=0}^{\infty} \Gamma(n+1+\bar{\gamma}) \left(-\frac{\beta_1}{2p} \right)^n a_Q^{n+1} \\ &\times \left[1 + \bar{b}_1 \frac{p\bar{\gamma}}{(n+\bar{\gamma})} + \bar{b}_2 \frac{p^2\bar{\gamma}(\bar{\gamma}-1)}{(n+\bar{\gamma})(n+\bar{\gamma}-1)} + \mathcal{O}\left(\frac{1}{n^3}\right) \right]. \end{aligned} \quad (5.12)$$

In the next section, the general forms of the IR and UV renormalon singularities (5.6) and (5.10) will be employed to construct a physically motivated model for the Borel transform of the Adler function $B[\widehat{D}](u)$. In order to describe the leading IR renormalon at $u=2$ as well as possible, in this case we include the known Wilson coefficient function and anomalous dimension of the gluon condensate $\langle aG^2 \rangle$ [42], which results in [43]

$$c_1[\langle aG^2 \rangle] = \frac{C_A}{2} - \frac{C_F}{4} - \frac{\beta_2}{\beta_1}, \quad (5.13)$$

while all other c 's appearing in the equations above will be set to zero.

6. A physical model for the Adler function series

To clarify whether FOPT or CIPT results in a better approximation to the τ hadronic width, we need to construct a physically motivated model for the Adler function series beyond the order $n=4$, up to which it is known exactly. Since the four lowest coefficients are available, it is reasonable to attempt to merge the low-order series with the expected large-order behaviour. Furthermore, it is convenient to generate the series from its Borel transform.⁹ A physical model should account for the following features:

⁸There is a sign mistake in the term proportional to $1/n$ in the corresponding eq. (3.48) of [20].

⁹A previous attempt to resum the perturbative series for R_τ , based on Borel transforms of the Adler function, has been made in ref. [44].

- It should reproduce the exactly known $c_{n,1}$, $n \leq 4$.
- The very large order behaviour is governed by a sign-alternating UV renormalon divergence. Since the low-order series shows no sign of a sizeable alternating component, it should be sufficient to include the leading singularity at $u = -1$. On the other hand, since the intermediate orders are governed by IR renormalons, at least two IR renormalon singularities, at $u = 2$ and $u = 3$, should be included to merge the large-order behaviour with the low-order exact coefficients.
- Since four-loop running is employed in the contour integral that relates the Adler function to R_τ , the renormalon singularities in the Borel transform cannot be simple poles, but should be extended consistently to cuts. This is particularly important for $u = 2$. We use the equations from section 5, incorporating $c_1[\langle aG^2 \rangle]$ into the description of the $u = 2$ cut. For the UV renormalon singularity at $u = -1$, we use $\bar{\gamma} = 1 - 2\beta_2/\beta_1^2$ in eq. (5.11), since it is known to be a double pole in the large- β_0 approximation.¹⁰

We are thus led to the ansatz:

$$B[\widehat{D}](u) = B[\widehat{D}_1^{\text{UV}}](u) + B[\widehat{D}_2^{\text{IR}}](u) + B[\widehat{D}_3^{\text{IR}}](u) + d_0^{\text{PO}} + d_1^{\text{PO}}u, \quad (6.1)$$

where the first three terms use (5.6) and (5.10) as building blocks for the three leading renormalon singularities.¹¹ The model (6.1) depends on five parameters, the three residues of the renormalon poles, d_1^{UV} , d_2^{IR} and d_3^{IR} , as well as the two polynomial parameters d_0^{PO} and d_1^{PO} . To fix these parameters, we match the perturbative expansion of our model to the known coefficients $c_{n,1}$ of eqs. (2.13), (2.15) and (3.10). First, the residues are fixed using the coefficients $c_{3,1}$, $c_{4,1}$ and $c_{5,1}$, and then the polynomial coefficients are adjusted to also reproduce the first two orders $c_{1,1}$ and $c_{2,1}$. The motivation for also making use of the fifth datum, namely the estimate for the coefficient $c_{5,1} = 283$, as well as including the polynomial terms, is that we wish to avoid fixing the residue of a renormalon term by using orders as low as $n = 2$. However, as we discuss below, our result is surprisingly independent of this extra assumption.

Following the outlined procedure, the parameters of the model (6.1) are found to be:

$$d_1^{\text{UV}} = -1.56 \cdot 10^{-2}, \quad d_2^{\text{IR}} = 3.16, \quad d_3^{\text{IR}} = -13.5, \quad (6.2)$$

¹⁰The additional “1” arises from the anomalous dimension term $\gamma_{O_d}^{(1)}/\beta_1$ in eq. (5.11) related to a four-quark operator [38].

¹¹A similar approach to the heavy quark mass has been put forward in refs. [45, 46].

$c_{6,1}$	$c_{7,1}$	$c_{8,1}$	$c_{9,1}$	$c_{10,1}$	$c_{11,1}$	$c_{12,1}$
3275	$1.88 \cdot 10^4$	$3.88 \cdot 10^5$	$9.19 \cdot 10^5$	$8.37 \cdot 10^7$	$-5.19 \cdot 10^8$	$3.38 \cdot 10^{10}$

Table 2: Predictions for the Adler-function coefficients $c_{6,1}$ to $c_{12,1}$ from our model (6.1) for $B[R](u)$, employing the estimate for the coefficient $c_{5,1} = 283$ of eq. (3.10).

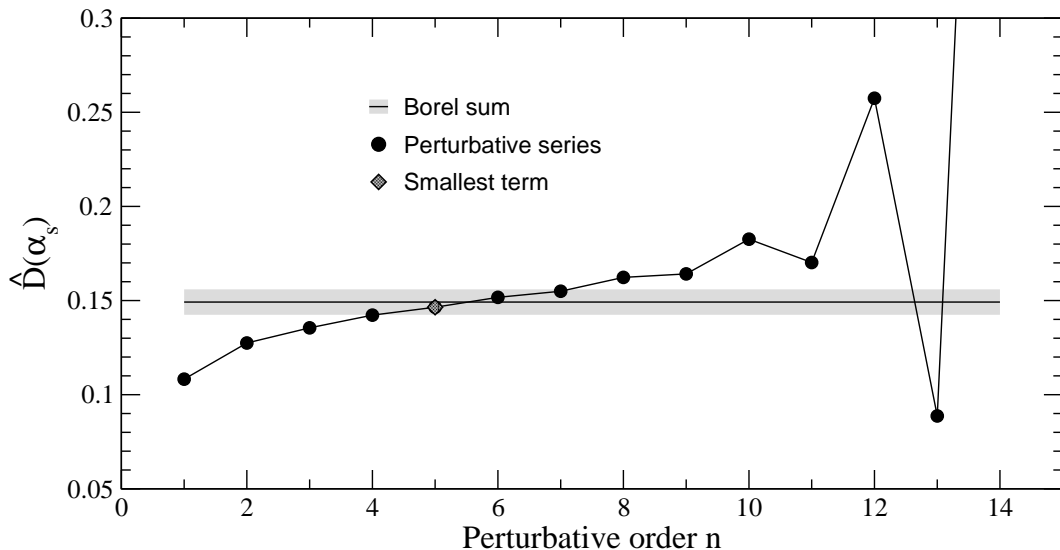


Figure 6: Results for $\hat{D}(M_\tau^2)$ (full circles) at $\alpha_s(M_\tau) = 0.34$, employing the higher-order coefficients $c_{n,1}$ of table 2 obtained from our model (6.1), as a function of the order n up to which the terms in the perturbative series have been summed. The straight line represents the result for the Borel sum of the series, and the shaded band provides an error estimate inferred from the complex ambiguity.

$$d_0^{\text{PO}} = 0.781, \quad d_1^{\text{PO}} = 7.66 \cdot 10^{-3}.$$

The fact that the parameter d_1^{PO} turns out to be so small implies that the coefficient $c_{2,1}$ is already reasonably well described by the renormalon pole contribution, although it was not used to fix the residua. Another implication of this observation will be discussed below. With the Borel transform thus determined, we are now in the position to investigate the higher-order coefficients. The perturbative series of the Adler function up to $c_{12,1}$ is presented in table 2. We observe that the first negative coefficient is found at the 11th order, after which the series retains its sign-alternating behaviour. The successive approximations to the (reduced) Adler function $\hat{D}(M_\tau^2)$ defined in (4.4) are displayed in figure 6, which shows that the model is well-behaved: the series goes through a number of small terms at order $n = 4$ to 7 , the minimal term being reached at $n = 5$, such that the truncated series agrees nicely with its Borel sum. The sign-alternating UV renormalon divergence takes over around $n = 10$.

To gain further insight into the contribution of a certain renormalon singularity to the coefficients $c_{n,1}$, in table 3 the relative contributions (in %) for the two IR and the UV

	$c_{3,1}$	$c_{4,1}$	$c_{5,1}$	$c_{6,1}$	$c_{7,1}$	$c_{8,1}$	$c_{9,1}$	$c_{10,1}$	$c_{11,1}$	$c_{12,1}$
IR ₂	82.4	100.4	135.9	97.5	155.9	76.3	359.9	48.3	-103.6	22.9
IR ₃	28.7	-10.0	-20.2	-13.3	-17.3	-6.5	-23.2	-2.3	3.6	-0.6
UV ₁	-11.2	9.7	-15.6	15.8	-38.6	30.3	-236.7	54.0	200.0	77.7

Table 3: Relative contributions (in %) of the different IR and UV renormalon poles to the Adler-function coefficients $c_{3,1}$ to $c_{12,1}$.

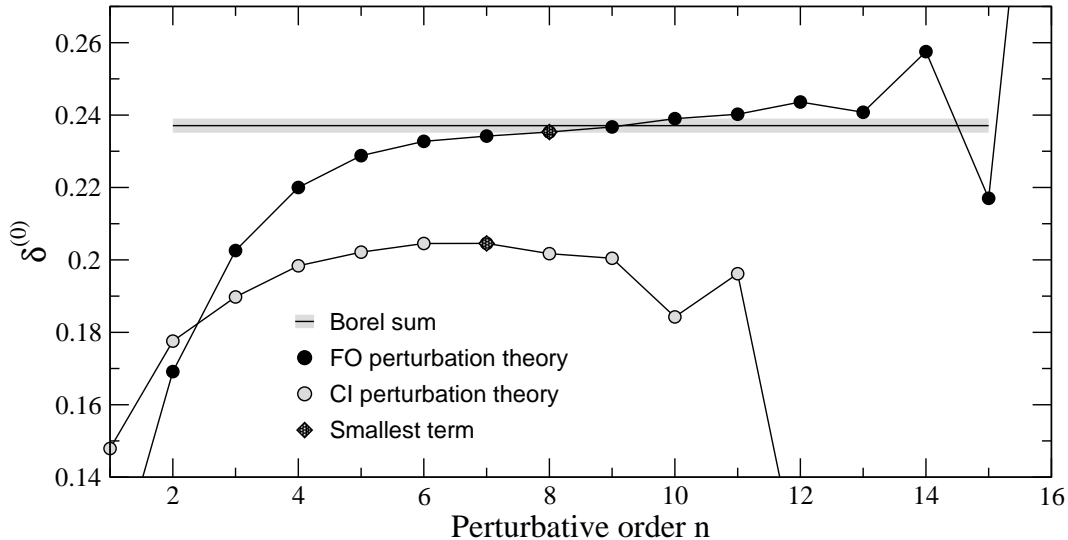


Figure 7: Results for $\delta_{\text{FO}}^{(0)}$ (full circles) and $\delta_{\text{CI}}^{(0)}$ (grey circles) at $\alpha_s(M_\tau) = 0.34$, employing the higher-order coefficients $c_{n,1}$ of table 2 obtained from our model (6.1), as a function of the order n up to which the terms in the perturbative series have been summed. The straight line represents the result for the Borel sum of the series, and the shaded band provides an error estimate inferred from the complex ambiguity.

pole are displayed. As can be seen from table 3, already for the third order there is a reasonable dominance of the leading IR renormalon at $u = 2$, and a sizeable but not too large contribution of the second one at $u = 3$. For the next few orders the leading IR pole becomes even more dominant as it should. Then, after a region of cancellations between the leading IR and UV poles, the leading UV renormalon, which dictates the large order behaviour, takes over. Nevertheless, already at the fifth order, the last we have included in fixing our parameters, with 16% there is a noticeable contribution from the UV renormalon, so that we have sensitivity to this term in the construction of the model.

Let us now move to the implications of our model (6.1) for the τ hadronic width, that is for $\delta^{(0)}$ in FOPT and CIPT. A graphical representation of the series behaviours is displayed in figure 7. Like in section 4, the full circles denote our result for $\delta_{\text{FO}}^{(0)}$ and the grey circles the one for $\delta_{\text{CI}}^{(0)}$, as a function of the order n up to which the perturbative series has been summed. The straight line corresponds to the principal value Borel sum of the series, and, like in figure 3 for the large- β_0 approximation, the shaded band provides an error estimate based on the imaginary part divided by π . The order at which the FO and CI series have their smallest terms is indicated by the grey diamonds. For CIPT this happens at the 7th and for FOPT at the 8th order. The essential conclusion is that the qualitative behaviour of the realistic series is determined by the features that were discussed in section 4 in the context of the $p = 2$ single-pole model. The cancellations between the Adler function coefficients $c_{n,1}$ and the contributions from the contour integral, g_n , soften the divergence of the R_τ series relative to the Adler function series shown in figure 6, allowing the FO series to approach the “true” result around its minimal term, after which the large-order asymptotic behaviour takes over. On the contrary, CIPT misses this cancellation and

always stays much below the “true” result. If CIPT were used to determine α_s from the measured value of R_τ , i.e. $\delta^{(0)}$, a too large value of α_s would be extracted to compensate for the deficit. We emphasise that the clear preference for FOPT holds even if we discard all higher-order terms and use only the series up to $n = 4$, as can be seen from figure 7, provided the gross features of our ansatz are correct.¹²

Numerically, our central result for the Borel sum is found to be

$$\delta_{\text{BS}}^{(0)} = 0.2371 \pm 0.0060 i. \quad (6.3)$$

Due to the suppression of the leading IR renormalon divergence by the contour integration, only one third of the imaginary part arises from the first IR pole, and two thirds from the second. This is consistent with the power-suppressed terms in the OPE, where for R_τ the $1/M_\tau^6$ terms dominate over the $1/M_\tau^4$ terms for the same reasons. The ambiguity that our procedure assigns to the “true” value is also of order of the power corrections as discussed in section 7. At the order of its minimal term, the FOPT result reads $\delta_{\text{FO}}^{(0)} = 0.2353$ with a minimal term of 0.0011. Thus, the difference of the Borel sum and the supposedly best result for $\delta_{\text{FO}}^{(0)}$ is about 1.6 times the minimal term, and the imaginary ambiguity divided by π precisely is able to account for this difference. Thus, our ansatz passes the requirements for a sensible asymptotic series and OPE expansion. In the remainder of this section, we shall make further checks as to the robustness of the prediction for $\delta^{(0)}$ from our model for $B[\widehat{D}](u)$, and we estimate the uncertainty of our value for $\delta_{\text{BS}}^{(0)}$ given in (6.3).

Our first test consists in dropping $c_{5,1}$ as an input, and instead using $c_{2,1}$ to determine the residua of the renormalon poles. In this case we set $d_1^{\text{PO}} = 0$, since we have only four data to fix the parameters. The visual appearance of the corresponding plot for $\delta^{(0)}$ is indistinguishable from figure 7, so we do not show it again. Also numerically, with $\delta_{\text{BS}}^{(0)} = 0.2370 \pm 0.0061 i$, we have practically the same result as in our main model. However, now we are in a position to predict the coefficient $c_{5,1}$, with the surprising result $c_{5,1} = 280$, almost the same as our naive estimate (3.10) that was used as an input before. This outcome was already anticipated by the fact that for our main model (6.1), d_1^{PO} (which we now set to 0) turned out to be very small. However, when one breaks down $c_{2,1}$ into the contributions from the different renormalon terms, one finds that the IR pole at $u = 3$ is twice as important as the leading IR pole, so the agreement with the previous ansatz might be fortuitous. For this reason, we prefer to base our discussion on an estimate of $c_{5,1}$, despite the obvious drawback of having to rely on an additional assumption.

Another modification to corroborate our assumptions and the findings for the main model is to add a third IR renormalon pole at $u = 4$, again only adding one constant d_0^{PO} , and fitting the parameters to the first five perturbative coefficients. Hence, like in the last model, the coefficient $c_{2,1}$ influences the residua of the renormalon poles. The numerical outcome is $\delta_{\text{BS}}^{(0)} = 0.2377 \pm 0.0065 i$, once more only a slight shift compared to (6.3). The small change in the complex ambiguity only originates from the first two IR poles with almost no contribution from the third. Also graphically, no difference would be visible in comparison to figure 7. When inspecting the separate pole contributions, the third IR pole

¹²Further insights into the origin of the difference between $\delta_{\text{CI}}^{(0)}$ and $\delta_{\text{BS}}^{(0)}$ can be found in appendix B.

mainly contributes to the first and second coefficient. For $c_{3,1}$, we are left with a small 5% contribution, and beyond the third order, the IR pole at $u = 4$ is completely negligible. Thus, our assumptions on the relevant singularities appear to be justified.

Two further sources of uncertainty in our result (6.3) arise from the dependence on the estimate of the coefficient $c_{5,1}$, which is not exactly known, and from neglecting higher order running effects beyond β_4 , where the β -function coefficients are not known. In view of the fact that $c_{5,1}$ turned out extremely stable when it was predicted above from our model, to estimate the corresponding uncertainty, we think it is a safe choice to vary $c_{5,1}$ by $\pm 50\%$. This results in variations of $(\pm 0.0042, \pm 0.0030)$ in the real and imaginary part of $\delta_{\text{BS}}^{(0)}$, respectively, with no qualitative change in the behaviours of the FOPT and CIPT series relative to figure 7.¹³ To get an idea about the importance of higher order running effects, we can set β_4 , \bar{b}_2 and \tilde{b}_2 to zero, and check how much our result changes. This results in a $(-0.0021, -0.0011)$ variation of $\delta_{\text{BS}}^{(0)}$. Adding both variations in quadrature, we arrive at our final result for $\delta_{\text{BS}}^{(0)}$ including uncertainties,

$$\delta_{\text{BS}}^{(0)} = (0.2371 \pm 0.0047) \pm (0.0060 \pm 0.0032) i, \quad (6.4)$$

which will be used in the next section for our determination of α_s from the hadronic τ decay width. This should be compared to the perturbative corrections from eqs. (3.8) and (3.9), which give 0.2200 (0.2288) in FOPT and 0.1984 (0.2021) in CIPT when the series is truncated at $n = 4$ ($n = 5$).

Note that there is no uncertainty due to scale- or scheme-dependence in the usual sense, since we are considering a series to all orders that is formally scale- and scheme-independent. A question that can be asked, however, is whether our ansatz for the Borel transform would still be meaningful and lead to the same results regarding the validity of FOPT and CIPT, if the input data $c_{n,1}$, $n \leq 5$, were given in another scheme for α_s than the $\overline{\text{MS}}$ scheme or at another scale. This is not obvious, but it is not clear what conclusions should be drawn from this. An arbitrary scheme change can produce arbitrary irregularities in the input data, making any attempt to merge low with high orders meaningless. Similarly, a scheme change defined by the relation $1/\alpha_s = 1/\alpha_s^{\overline{\text{MS}}} + \beta_1 C/(2\pi)$ would produce an additional factor $\exp(-Cu)$ in the new Borel transform, which changes the renormalon residues and therefore the balance between contributions from the different leading singularities. With no guidance at hand, we adopt the point of view that the $\overline{\text{MS}}$ scheme has proven useful and stable in so many applications of perturbative QCD that there is little motivation to consider significant departures. Nevertheless, to acquire an estimate of the model dependence of our approach, in the next section we also fit our model parameters to the Adler function coefficients which correspond to the expansion in the strong coupling evaluated at a scale $\alpha_s(\xi M_\tau)$, and determine the impact of this variation on the α_s determination. With such variations, the strength of the leading UV renormalon is modified as compared to the lowest IR renormalon pole. For $\xi < 1$ the

¹³This remains true for larger variations of $c_{5,1}$. However, for $c_{5,1} = 0$ or negative, large cancellations between the two IR renormalon poles are required in our model to produce such small $c_{5,1}$ and the assumption to fit the low orders with only a few renormalon singularities becomes questionable.

leading UV renormalon becomes stronger, leading to an earlier onset of the asymptotic regime, while for $\xi > 1$ it is suppressed, which imposes limitations on sensible values of ξ .

7. Determination of α_s

The starting point for a determination of α_s from hadronic τ decays is eq. (1.2) for the decay rate of the τ lepton into light u and d quarks. The general strategy for our extraction of α_s below will be as follows. We concentrate on the sum of the vector and axialvector channels, $R_{\tau,V+A}$, as in this case some of the higher-dimensional operator contributions cancel. A more elaborate analysis of the separate channels along the lines of refs. [7, 8, 9] employing moments of the τ decay spectral function is left for the future. We expect significantly larger perturbative corrections and ambiguities for the moments, along with the enhanced condensate contributions, which makes the α_s analysis more complicated. We base our analysis on FOPT, and FOPT together with the ansatz for higher-order terms of section 6, but do not use CIPT, despite the fact that the CI perturbation series appears to be better behaved in low orders. The reason is that the analysis in the previous sections shows that the better stability and smaller scale dependence of CIPT reflects the plateau of the CIPT curve in figure 7, but the plateau value of the perturbative correction is far from the true value. In such a situation, theoretical error estimates based on scale dependence provide a very large underestimate of the true uncertainty.

The first step of the α_s analysis consists in estimating the values of the power corrections $\delta_{ud,V+A}^{(D)}$ in eq. (1.2), which arise from higher-dimensional operators in the framework of the OPE. The power corrections will be calculated in FOPT.¹⁴ Given these estimates and experimental data, we calculate a phenomenological value of $\delta^{(0)}$ using eq. (1.2). We then determine the value of $\alpha_s(M_\tau)$ by requiring that the theoretical value $\delta_{\text{theo}}^{(0)}$ in FOPT, and in our model of section 6 matches the phenomenological value $\delta_{\text{phen}}^{(0)}$. Errors are estimated by varying all parameters within their uncertainties.

7.1 Estimate of power corrections

Before going through the estimation of the power corrections, let us remark that these will only be considered for the transversal part of the correlator $\Pi_{ud}^{V+A,(1+0)}(s)$. The longitudinal contribution $\delta_{ud,S+P}$ to $R_{\tau,V+A}$, arising from $\Pi_{ud}^{V+A,(0)}(s)$ which is related to scalar and pseudoscalar correlation functions, will be included later according to a phenomenological approach which was already employed in the determination of $|V_{us}|$ from R_τ [47, 48]. This approach has much smaller uncertainties than making use of the corresponding QCD expressions for the scalar/pseudoscalar light-quark correlators since their perturbative expansions converge substantially slower than those for the vector/axialvector correlation functions. With foresight, we shall use $\alpha_s(M_\tau) = 0.3156 \pm 0.006$ in the numerical estimate of the power correction below.

¹⁴In principle, for quantities other than the purely perturbative corrections to the Adler function, the preference of FOPT over CIPT has to be investigated anew. However, the power corrections are already a small correction, and for our application the difference in treating the perturbative expansion of the Wilson coefficients is irrelevant.

The lowest-dimensional power corrections to $R_{\tau,V+A}$ are of dimension-2 and only arise from terms proportional to quark masses squared. Detailed expressions for these contributions were given in ref. [22, 49] up to order α_s^2 , and the next corrections at $\mathcal{O}(\alpha_s^3)$ were first presented in ref. [50]. From these expressions we calculate the corresponding $\delta_{m^2,V+A}^{(2)}$ in FOPT. Contributions to this quantity arise from two sources: one is proportional to $(m_u^2 + m_d^2)$ and the other to m_s^2 . The second is due to internal strange-quark loops and only starts at $\mathcal{O}(\alpha_s^2)$. While the perturbative series for the $(m_u^2 + m_d^2)$ term displays a reasonable convergence, the m_s^2 one is very badly behaved, such that in total the $\mathcal{O}(\alpha_s^3)$ term is about a factor of three times the $\mathcal{O}(\alpha_s^2)$ term. To estimate the uncertainty in $\delta_{m^2,V+A}^{(2)}$, we average the results either including or omitting the $\mathcal{O}(\alpha_s^3)$ term, and take the spread as the error. For the quark masses we use $m_u(M_\tau) = 2.8 \pm 0.5$ MeV, $m_d(M_\tau) = 5.0 \pm 0.6$ MeV and $m_s(M_\tau) = 97 \pm 9$ MeV, which derive from ref. [51]. Also quadratically including the parametric uncertainties, which albeit play a minor role, yields

$$\delta_{m^2,V+A}^{(2)} = (3.1 \pm 8.6) \cdot 10^{-5}. \quad (7.1)$$

Although this contribution has a very large uncertainty, it will turn out to be immaterial for the α_s determination.

Dimension-4 contributions arise from three possible sources: the gluon condensate $\langle aG^2 \rangle$, the quark condensate $\langle \bar{q}q \rangle$, and m_q^4 corrections. (See e.g. ref. [22] and references therein for explicit expressions.) The quartic mass corrections are tiny, and thus we shall drop them. Furthermore, being suppressed by $1/s^2$ because of the weight function in R_τ , the contour integral is only non-vanishing if there are additional logarithms $\ln(-s)$, which first appear at order α_s^2 .¹⁵ The explicit expressions for the two contributions including the known terms are then found to be

$$\delta_{\langle G^2 \rangle, V+A}^{(4)} = \frac{11\pi^2}{4} a(M_\tau)^2 \frac{\langle aG^2 \rangle}{M_\tau^4}, \quad (7.2)$$

$$\delta_{\langle \bar{q}q \rangle, V+A}^{(4)} = 54\pi^2 \frac{(m_u + m_d)\langle \bar{q}q \rangle}{M_\tau^4} \left[a(M_\tau)^2 + \left(\frac{517}{36} - \frac{8}{3}\zeta_3 + \left(\frac{5}{6} - \frac{4}{3}\zeta_3 \right) \kappa R_s \right) a^3 \right], \quad (7.3)$$

where $\kappa \equiv \langle \bar{s}s \rangle / \langle \bar{q}q \rangle$, $R_s \equiv 2m_s / (m_u + m_d)$, and we have assumed isospin symmetry for the up and down quark condensates. Historically, the standard value for the gluon condensate is $\langle aG^2 \rangle = 0.012 \text{ GeV}^4$ [52], and not much progress has been made since then. Therefore, it will be employed as our central value. As we only know the leading term for $\delta_{\langle G^2 \rangle, V+A}^{(4)}$, to be conservative, we assign a 100% uncertainty. For $\delta_{\langle \bar{q}q \rangle, V+A}^{(4)}$, the required quark condensate can be calculated from the GMOR relation [53, 54], with the result $\langle \bar{q}q \rangle(M_\tau) = -(272 \pm 15 \text{ MeV})^3$. With 40%, the next-to-leading order α_s^3 correction is large, but still has perturbative character. Thus, we include this term and take its size as an estimate for the missing higher orders. Further employing $\kappa = 0.8 \pm 0.3$ [54], and the light quark masses from above, we obtain

$$\delta_{\langle G^2 \rangle, V+A}^{(4)} = (3.3 \pm 3.3) \cdot 10^{-4}, \quad \delta_{\langle \bar{q}q \rangle, V+A}^{(4)} = (-4.9 \pm 6.2) \cdot 10^{-5}. \quad (7.4)$$

¹⁵This is yet another manifestation of the suppression of the leading IR renormalon at $u = 2$.

It may be remarked that the upper range of $\delta_{\langle G^2 \rangle, V+A}^{(4)}$ just corresponds to the complex ambiguity of the $u = 2$ renormalon pole divided by π in our all-order ansatz. It is gratifying to observe that they are of the same order of magnitude, giving support to the size of this term. Nevertheless, also the dimension-4 contributions only play a minor role in the determination of α_s .

At dimension six, $\delta_{ud, V+A}^{(D)}$ receives contributions from the three-gluon condensate $\langle g^3 G^3 \rangle$, 4-quark operators, and lower-dimensional operators times appropriate powers of quark masses. The coefficient function of $\langle g^3 G^3 \rangle$ vanishes at the leading order and is therefore suppressed. Also the operators which get multiplied by quark masses only give very small contributions. Thus we neglect these two types of contributions and concentrate on the 4-quark condensates. At leading order, one is confronted with three four-quark operators, while at higher orders many more are generated [3, 55, 56]. As it appears impossible to determine all required condensates from phenomenology, the so-called vacuum-saturation approximation (VSA) had been proposed [52]. This reduces all 4-quark condensates to squares of the quark condensate. With this simplifying assumption, the leading dimension-6 contribution takes the form

$$\delta_{\langle \bar{q}q\bar{q}q \rangle, V+A}^{(6)} = -\frac{512}{27} \pi^3 \alpha_s \frac{\rho \langle \bar{q}q \rangle^2}{M_\tau^6} = (-4.8 \pm 2.9) \cdot 10^{-3}, \quad (7.5)$$

where we assume $\rho = 2 \pm 1$ for the numerical estimate. Our reasoning for this is as follows: VSA is incompatible with the scale and scheme dependence of the 4-quark operators [56, 57]. Therefore, it does not make sense to include the next-to-leading order corrections. Conventionally, one introduces the parameter ρ , which comprises the violation of the VSA. A rough idea about the size of the parameter ρ can be gleaned from phenomenological fits of the dimension-6 contributions to Π_{ud}^V , Π_{ud}^A , Π_{ud}^{V+A} , as well as Π_{ud}^{V-A} [7, 8, 9, 58] (and references therein). Comparing the phenomenological fits with the VSA, values for ρ from one to about three are found, which motivates the chosen range. At any rate, $\delta_{\langle \bar{q}q\bar{q}q \rangle, V+A}^{(6)}$ constitutes the dominant power correction.

We also include a crude estimate for the still higher-dimensional operator corrections. Not much is known about these contributions, apart from the fact that they should be suppressed as compared to $\delta_{\langle \bar{q}q\bar{q}q \rangle, V+A}^{(6)}$, since they carry at least two more powers of $1/M_\tau$. Inspecting the fits of refs. [7, 8, 9] where also a dimension-8 contribution was included, one infers that roughly it could be of order 10^{-3} . Therefore, we have added

$$\delta_{ud, V+A}^{(8)} = (0 \pm 1) \cdot 10^{-3} \quad (7.6)$$

in our total estimate of the uncertainty for power corrections. There are potentially also (short-distance) instanton contributions to the τ hadronic width. The leading contribution of this type is a rapidly increasing and uncertain function of $\alpha_s(M_\tau)$. For $\alpha_s(M_\tau) = 0.32$ it has been estimated to contribute $2 \cdot 10^{-3}$ to $\delta_{ud, V+A}^{(D)}$ [59]. Nonetheless, we do not include a further power correction uncertainty for this term.

To complete our summary of power corrections to R_τ , we still have to compute the longitudinal contributions which arise from scalar and pseudoscalar correlators. Because the perturbative series for these correlators do not converge very well, here we shall follow

the approach of refs. [47, 48]. The contribution from the scalar correlator is suppressed by a factor $(m_u - m_d)^2$, and it can be altogether neglected. The main idea then is to replace the QCD expressions for the pseudoscalar correlator by a phenomenological representation. The dominant contribution to the pseudoscalar spectral function stems from the well known pion pole, giving

$$\delta_{ud,S+P}^\pi = -16\pi^2 \frac{f_\pi^2 M_\pi^2}{M_\tau^4} \left(1 - \frac{M_\pi^2}{M_\tau^2}\right)^2, \quad (7.7)$$

plus small corrections from higher-excited pionic resonances. Repeating the analysis of section 3 of ref. [47] and updating the input parameters, we find

$$\delta_{ud,S+P} = (-2.64 \pm 0.05) \cdot 10^{-3}. \quad (7.8)$$

The uncertainty in (7.8) has been estimated by setting the higher-resonance contribution to zero, which clearly demonstrates that the pion pole is dominant. Collecting all contributions, and adding the errors in quadrature, we arrive at our total estimate of all power corrections:

$$\delta_{\text{PC}} = (-7.1 \pm 3.1) \cdot 10^{-3}. \quad (7.9)$$

Our value (7.9) is consistent with the most recent fit to the τ spectral functions performed in ref. [9].

As a matter of principle, the OPE of the correlation functions in the complex s plane, even when integrated over a suitable energy interval [25], could be afflicted with so-called “duality violations” [60]. In our case, these would come from the contour integral close to the physical region, and even though suppressed by a double zero, could lead to additional contributions or uncertainties. Within a model, these contributions were recently investigated for hadronic τ decays [61], and in ref. [9] the model was fitted to the τ spectral functions, with the finding that possible additional contributions are below the 10^{-3} level. In view of these results, we shall omit possible duality violating terms, but in future analyses which perform a simultaneous fit of higher order OPE contributions, in analogy to [9], it might be worthwhile to include them.

7.2 α_s analysis

Employing the value $R_{\tau,V+A} = 3.479 \pm 0.011$, which results from eq. (1.1) in conjunction with $R_{\tau,S} = 0.1615 \pm 0.0040$ [9], as well as $|V_{ud}| = 0.97418 \pm 0.00026$ [62], from eq. (1.2) the phenomenological value for $\delta^{(0)}$ can be derived after accounting for the electroweak corrections and subtracting the power correction given in eq. (7.9):

$$\delta_{\text{phen}}^{(0)} = 0.2042 \pm 0.0038_{\text{exp}} \pm 0.0033_{\text{PC}} = 0.2042 \pm 0.0050. \quad (7.10)$$

By far the dominant experimental uncertainty is due to $R_{\tau,V+A}$. In the second error, we have also included the ones from S_{EW} and δ'_{EW} which should be considered theoretical (though they are not power corrections). The final step in the extraction of $\alpha_s(M_\tau)$ now consists in finding the values of α_s for which the phenomenological value $\delta_{\text{phen}}^{(0)}$ matches the theoretical prediction, either from the finite order series, or from our model to all orders.

Let us begin by analysing the impact of using FOPT when the perturbative expansion is employed up to and including the fifth order. Besides the exactly known coefficients, we make use of our value (3.10), $c_{5,1} = 283$, and to estimate the corresponding uncertainty, we have either removed or doubled the fifth order term. A further theoretical error is added to account for the residual renormalisation scale dependence of the fixed order series. We estimate this by expressing the FO series in terms of $a(\mu^2)$ rather than $a(M_\tau^2)$ and by varying μ between 1 GeV and 2.5 GeV. Going through this procedure, and keeping the errors separate to clearly see the importance of the various contributing uncertainties, we obtain:

$$\begin{aligned}\alpha_s(M_\tau) &= 0.3203 \pm 0.0032_{\text{exp}} \pm 0.0028_{\text{PC}} \pm 0.0027_{c_{5,1}} \pm_{-0.0052}^{+0.0105} (\text{scale}) \\ &= 0.320 \pm 0.003_{\text{exp}} \pm_{-0.006}^{+0.011} (\text{th}) = 0.320 \pm_{-0.007}^{+0.012}.\end{aligned}\tag{7.11}$$

Comparing our result (7.11) with other determinations of $\alpha_s(M_\tau)$ from hadronic τ decays [9, 18] that include the exact fourth order term, we observe that our value is smaller by 0.012 [18] and 0.024 [9], up to twice the estimated error. This should not come as a surprise since the analysis of ref. [9] relied on the use of CIPT, which increases $\alpha_s(M_\tau)$, see figure 7, while in ref. [18] an average of FOPT and CIPT is performed. The error in eq. (7.11) is dominated by the scale error, which is significantly larger than in CIPT.¹⁶ However, as argued above, we find that the small scale-dependence in CIPT is misleading when interpreted as a measure of the theoretical uncertainty.

We evolve $\alpha_s(M_\tau)$ to the Z -boson mass scale, using the β -function known to four loops [63, 64], and the matching coefficients at flavour thresholds up to order α_s^3 [65].¹⁷ The central values of the flavour thresholds $\mu_c^* = 3.729$ GeV and $\mu_b^* = 10.558$ GeV are taken to correspond to the physical thresholds of open D and B meson production. In the matching coefficients also the charm and bottom quark masses are required, which we assume to be $m_c(m_c) = 1.28 \pm 0.05$ GeV and $m_b(m_b) = 4.20 \pm 0.05$ GeV [67] in the $\overline{\text{MS}}$ -scheme. The uncertainties resulting from the evolution are estimated as follows: like in section 6, we compare to consistent 3-loop running, i.e. setting $\beta_4 = 0$ and removing the $\mathcal{O}(\alpha_s^3)$ coefficient in the matching relation, and we vary the quark masses as well as the flavour thresholds in the ranges $\mu_c^*/2 < \mu_c < 2\mu_c^*$ and $\mu_b^*/2 < \mu_b < 2\mu_b^*$. Adding all uncertainties in quadrature, we arrive at:

$$\begin{aligned}\alpha_s(M_Z) &= 0.1185 \pm 0.0004_{\text{exp}} \pm_{-0.0008}^{+0.0013} (\text{th}) \pm 0.0002_{\text{evol}} \\ &= 0.1185 \pm_{-0.0009}^{+0.0014}.\end{aligned}\tag{7.12}$$

Again, this value is lower than previous results from hadronic τ decays, but it is in perfect agreement with the global averages of $\alpha_s(M_Z)$ [1, 2].

¹⁶The scale variation has a minimum at the scale $\mu \approx 1.22$ GeV, which might therefore be considered as an “optimal” scale. It is amusing to note that evaluating α_s at this scale, we obtain $\alpha_s(M_\tau) = 0.3151$, very close to the improved result (7.13).

¹⁷The evolution has been performed independently with the Mathematica package RunDec [66] and a private routine coded by one of us (MJ), finding complete agreement.

To further improve our determination of α_s , we now include the higher order perturbative terms according to our model of section 6. To this end, we need to find the value of α_s such the value of the Borel sum $\delta_{\text{BS}}^{(0)}$ matches the phenomenological value (7.10).¹⁸ Again keeping the contributing uncertainties separate, we find:

$$\begin{aligned}\alpha_s(M_\tau) &= 0.3156 \pm 0.0030_{\text{exp}} \pm 0.0026_{\text{PC}} \pm 0.0025_{c_{5,1}} \pm 0.0011_{\beta_4=0} \stackrel{+0.0034}{-0.0029} (\text{scale}) \\ &= 0.3156 \pm 0.0030_{\text{exp}} \pm 0.0051_{\text{th}} = 0.3156 \pm 0.0059.\end{aligned}\tag{7.13}$$

Besides the uncertainties already discussed previously, in eq. (7.13) we have also included a scale/model uncertainty according to the reasoning put forward at the end of section 6. As the scale variation, we chose $0.5 < \xi^2 < 1.5$, where $\mu \equiv \xi M_\tau$. (The corresponding scale interval is $1.26 \text{ GeV} < \mu < 2.18 \text{ GeV}$.) The employed range in ξ is dictated by the observation that for lower scales the contribution of the leading UV renormalon is enhanced as compared to the leading IR renormalon pole, while for larger scales it is suppressed. At the lower end for ξ , the contribution of the leading UV renormalon to the coefficient $c_{3,1}$ is of a similar size than the leading IR renormalon, while at the upper end, we are only left with a mere 3% contribution of the first UV renormalon to the coefficient $c_{5,1}$, practically losing the sensitivity to this contribution. Therefore, beyond the used range for ξ , our model ceases to make sense. For given ξ , we determine $\alpha_s(\xi M_\tau)$ such that the value (7.10) is obtained and then evolve back to the scale M_τ . In quoting the final uncertainty of $\alpha_s(M_\tau)$, in (7.13) we have used the larger scale variation in order to have a symmetric final error. The dependence of the successive perturbative approximations to R_τ on the choice of ξ is shown in figure 8, upper panel. The lower panel shows the corresponding approximations in CIPT, which are again seen to lie below the “true” result for any reasonable value of ξ .

Evolving the result (7.13) to the Z -boson mass scale, we arrive at our final value for $\alpha_s(M_Z)$:

$$\begin{aligned}\alpha_s(M_Z) &= 0.11795 \pm 0.00038_{\text{exp}} \pm 0.00063_{\text{th}} \pm 0.00020_{\text{evol}} \\ &= 0.11795 \pm 0.00076.\end{aligned}\tag{7.14}$$

This result is slightly smaller than eq. (7.12) based on the FOPT up to the fifth order as could be anticipated from figure 7, where one observes that $\delta^{(0)}$ at $\mathcal{O}(\alpha_s^5)$ is smaller than the full Borel sum. We consider eq. (7.14) as our best estimate of the strong coupling in the $\overline{\text{MS}}$ scheme from hadronic τ decays.

8. Conclusions

Hadronic τ decays provide an especially clean environment for the study of QCD effects and in particular the determination of QCD parameters. Of prime interest in this respect is the QCD coupling α_s . Still, due to the relatively low scale M_τ , an adequate control over the perturbative series should be achieved for such applications. Besides explicitly

¹⁸We do not include a separate error from the ambiguity of $\delta_{\text{BS}}^{(0)}$ (at $\alpha_s(M_\tau) = 0.3156$: $\delta_{\text{BS}}^{(0)} = 0.2042 \pm 0.0029 i$), since it is subsumed in the error of δ_{PC} , which is several times larger.

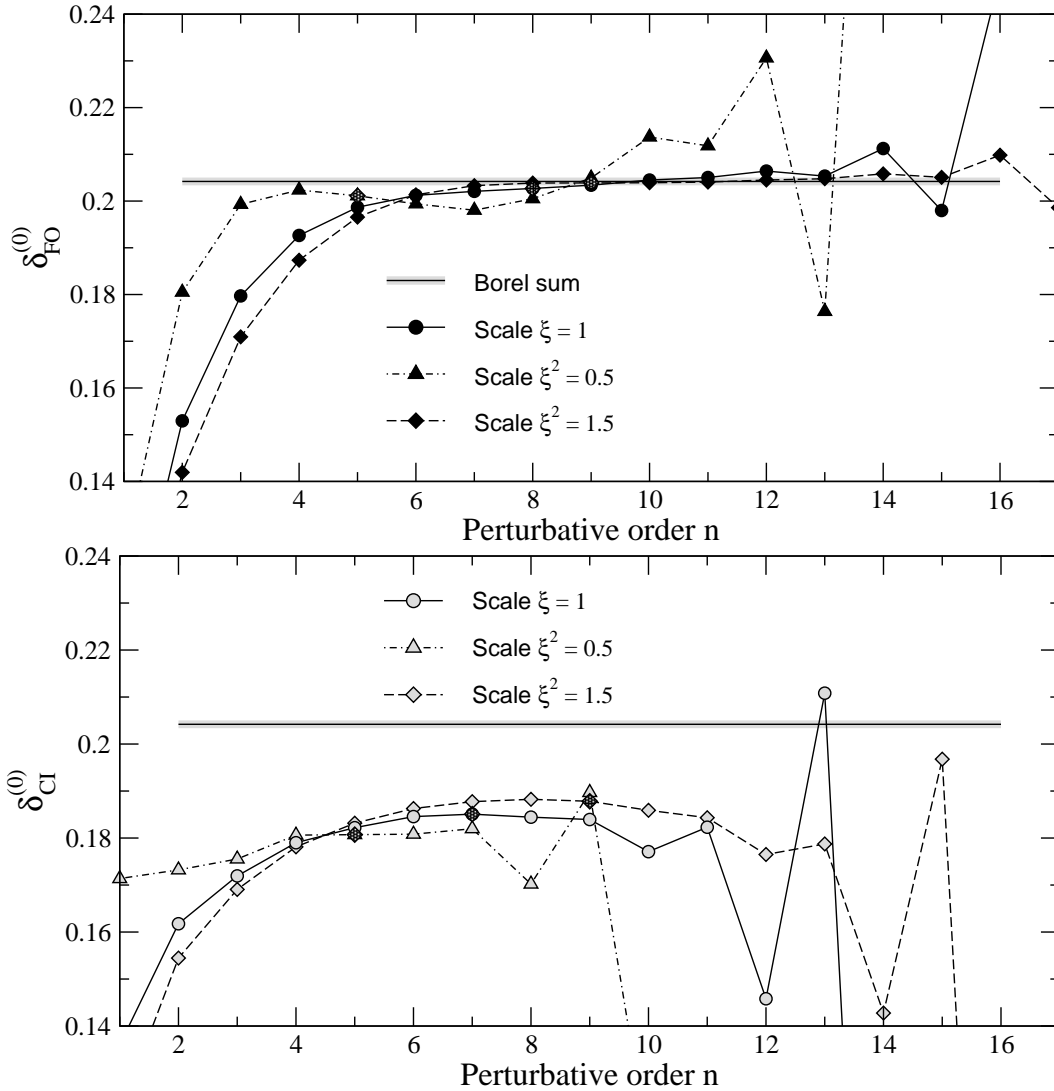


Figure 8: Results for $\delta_{\text{FO}}^{(0)}$ (full circles, diamonds, triangles) and $\delta_{\text{CI}}^{(0)}$ (grey circles, diamonds, triangles) for the model of eq. (6.1) as a function of the order n up to which the terms in the perturbative series have been summed. The parameters of the model are fit to the first five coefficients of the Adler function expanded in $\alpha_s(\xi\sqrt{s})$ as described in section 6. The value of $\alpha_s(\xi M_\tau)$ is determined by requiring that the Borel sum equals $\delta_{\text{phen}}^{(0)} = 0.2042$ in each case.

computing terms in the perturbative expansion, progress in this direction can be attained by inspecting the renormalon divergence structure that the perturbative series should have on general grounds.

A long-standing question in the interpretation of the QCD correction to R_τ is a numerical discrepancy between two ways of performing the renormalisation group improvement, namely CIPT and FOPT. While CIPT resums running effects of α_s along the complex integration contour which are known to be large, FOPT performs a consistent expansion in α_s at each loop order. The CIPT sum is significantly below the FOPT sum requiring a larger strong coupling to reproduce the experimental R_τ . Resolving this discrepancy

has become a major issue for improving the accuracy of the α_s determination from R_τ , in particular since recently the fourth order coefficient of the series expansion has been computed [18], and further terms cannot be expected to be available any time soon.

CIPT has long been considered as the method of choice due to its apparent better convergence and smaller renormalisation scale dependence of the truncated series as compared to FOPT. However, the argument that FOPT should be discarded, since the expansion of the running coupling along the contour produces a series of finite radius of convergence that is avoided in CIPT, does not take into account that both the CIPT and FOPT series have zero radius of convergence anyway due to factorial divergence in higher orders. This leads to interesting cancellations between the Adler function coefficients and running coupling effects [17]. Having a sufficiently large number of exact Adler function coefficients at our disposal, and without prospects of further improvements by exact computations, makes it timely and possible to attempt to merge the exactly known terms with what is known on general grounds about large-order behaviour, and to reinvestigate the conceptual issues of the FOPT/CIPT comparison.

We investigated several toy models to find out under which circumstances FOPT or CIPT provide a better approximation to the “true” result, which we may define as the Borel sum of the series, since power corrections to R_τ are small. Two extreme cases can be singled out. If the Adler function coefficients are set to zero beyond a certain order, CIPT becomes exact, while FOPT performs large oscillations around the exact result [19]. On the other hand, if the Adler function series is dominated by an infrared renormalon at $u = 2$, large cancellations occur in the R_τ series, which are only manifest in FOPT. In this case, FOPT accounts well for the Borel sum, while CIPT stays systematically below, despite an apparent better convergence and smaller scale dependence!

Taking these lessons we constructed a realistic ansatz for the entire Borel transform of the Adler function such that the exactly known coefficients plus an estimate of the fifth order coefficient are exactly reproduced, while including knowledge about the leading three renormalon singularities. Given the regularity of the exactly known low-order terms, the weight of the three contributions was fit to the four- to six-loop terms. With our central model of eq. (6.1) we were able to achieve a very coherent picture of the known and higher order coefficients. Our main conclusion from a study of the realistic model is that, given the particular features of the R_τ series, FOPT provides the better approximation, in general and at fourth order, than CIPT and is to be preferred on these grounds. In this respect the full QCD case resembles the large- β_0 approximation discussed in ref. [17].

We believe this conclusion to be valid in QCD beyond particular models, since it is based on generic properties of the R_τ perturbation series – although it must be admitted that statements about the accuracy of FOPT and CIPT, and the validity of perturbation theory in higher orders in general, are always open to some amount of speculation. The main point is that conventional wisdom that favours CIPT is based on the first three orders in perturbation theory when the running coupling effects in g_n dominate over the Adler function coefficients $c_{n,1}$. This dominance weakens as n grows, as can be seen from the known exact coefficients, and the situation is expected to reverse beyond $n = 5$. The fact that the $c_{n,1}$ ultimately diverge, cannot be ignored. This as well as the crucial $1/n^2$

cancellation in the sum $c_{n,1} + g_n$ for the leading non-sign-alternating component of the R_τ perturbation series are model-independent consequences of QCD.

Making use of these results, two determinations of the strong coupling α_s from $R_{\tau,V+A}$ were presented in section 7. First, we performed a conventional FOPT analysis based on the fourth-order result plus an estimate of the fifth order, whose size is taken as an uncertainty. The resulting value for $\alpha_s(M_\tau)$ can be found in eq. (7.11). Evolving to M_Z , we obtain

$$\alpha_s(M_Z) = 0.1185^{+0.0014}_{-0.0009} \quad (\text{FOPT}), \quad (8.1)$$

where the error is dominated by residual renormalisation scale dependence; see eq. (7.12). This value is lower than those presented in refs. [9, 18] mainly because we propose to not use CIPT as the result of our study. Our preferred and best result is obtained by incorporating available structural information on the large-order behaviour as embodied in the model of eq. (6.1). The resulting value for $\alpha_s(M_\tau)$ can be found in eq. (7.13). Evolving it to M_Z , the main result of our paper reads:

$$\begin{aligned} \alpha_s(M_Z) &= 0.11795 \pm 0.00038_{\text{exp}} \pm 0.00063_{\text{th}} \pm 0.00020_{\text{evol}} \\ &= 0.11795 \pm 0.00076. \end{aligned} \quad (8.2)$$

Our approach towards the investigation of the higher-order behaviour of perturbative series employed in this work can certainly also be applied to other quantities of interest, like the scalar correlation function, mass squared corrections to R_τ , or moments of the spectral functions. It should be interesting to see what can be said about the issue of renormalisation group resummation when contour integrations are involved in these cases. We shall return to this question in the near future. Finally, our results can be useful in other places where the QCD Adler function plays a role.

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Appendix A: Borel integral for renormalon poles

In this appendix, we provide a few useful expressions for the Borel integral of generic renormalon poles, and the corresponding complex ambiguity in the case of IR renormalons.

Let us begin with a general UV-renormalon pole which we assume to have the form:

$$B[R_p^{\text{UV}}](u) \equiv \frac{d_p^{\text{UV}}}{(p+u)^\gamma}, \quad (\text{A.1})$$

where $u = \beta_0 t$ with $\beta_0 \equiv \beta_1/(2\pi)$, $p \in \mathbb{N}$ and $\gamma \in \mathbb{R}^+$. Then, the corresponding Borel integral is given by

$$R_p^{\text{UV}}(\alpha) = d_p^{\text{UV}} \int_0^\infty \frac{e^{-t/\alpha}}{(p+\beta_0 t)^\gamma} dt, \quad (\text{A.2})$$

which after the substitution $t = \alpha z - p/\beta_0$ can be expressed in terms of the incomplete Γ -function:

$$R_p^{\text{UV}}(\alpha) = d_p^{\text{UV}} \frac{\alpha}{(\beta_0 \alpha)^\gamma} e^{p/(\beta_0 \alpha)} \int_{p/(\beta_0 \alpha)}^\infty z^{-\gamma} e^{-z} dz = d_p^{\text{UV}} \frac{\alpha}{(\beta_0 \alpha)^\gamma} e^{p/(\beta_0 \alpha)} \Gamma\left(1-\gamma, \frac{p}{\beta_0 \alpha}\right). \quad (\text{A.3})$$

Expanding the UV-renormalon pole ansatz (A.1), and performing the Borel integration term by term, yields the corresponding perturbative expansion:

$$R_p^{\text{UV}}(\alpha) = \frac{d_p^{\text{UV}}}{p^\gamma \Gamma(\gamma)} \sum_{n=0}^\infty \Gamma(n+\gamma) \left(-\frac{\beta_0}{p}\right)^n \alpha^{n+1}. \quad (\text{A.4})$$

For the general IR-renormalon pole, let us assume the generic form:

$$B[R_p^{\text{IR}}](u) \equiv \frac{d_p^{\text{IR}}}{(p-u)^\gamma}, \quad (\text{A.5})$$

where again $u = \beta_0 t$, $p \in \mathbb{N}$ and $\gamma \in \mathbb{R}^+$. The corresponding Borel integral is given by

$$R_p^{\text{IR}}(\alpha) = d_p^{\text{IR}} \int_0^\infty \frac{e^{-t/\alpha}}{(p-\beta_0 t)^\gamma} dt, \quad (\text{A.6})$$

and can be expressed in terms of the exponential integral function $E_\gamma(z)$. However, because of the singularity at $t = p/\beta_0$ and the branch cut for $t > p/\beta_0$, we still have to specify the defining integration path. We shall integrate on the real axis up to $t = p/\beta_0 - \varepsilon$, then on a semicircle of radius ε in a clockwise, or anti-clockwise direction, and finally again along the real axis from $t = p/\beta_0 + \varepsilon$ up to infinity, either above or below the cut, taking the limit $\varepsilon \rightarrow 0$ in the end. Performing all integrals, this leads to the expression:

$$R_p^{\text{IR}}(\alpha) = d_p^{\text{IR}} \frac{\alpha}{(\beta_0 \alpha)^\gamma} e^{-p/(\beta_0 \alpha)} \left\{ -\left(\frac{p}{\beta_0 \alpha}\right)^{1-\gamma} E_\gamma\left(-\frac{p}{\beta_0 \alpha}\right) + \left[(-1)^{\pm\gamma} - (-1)^{\text{sig}(\text{Im}[\alpha])\gamma}\right] \Gamma(1-\gamma) \right\}. \quad (\text{A.7})$$

In the above equation, $(-1)^z$ should be interpreted as $\exp(i\pi z)$, and the function $\text{sig}(z)$ represents the sign of z , with the additional definition $\text{sig}(0) \equiv 1$. The imaginary ambiguity of $R_p^{\text{IR}}(\alpha)$ can be readily computed from eq. (A.7), and for $\alpha > 0$ turns out to be:

$$\text{Im} [R_p^{\text{IR}}(\alpha)] = \pm \frac{d_p^{\text{IR}}}{\beta_0^\gamma} \sin(\pi\gamma) \Gamma(1-\gamma) \alpha^{1-\gamma} e^{-p/(\beta_0 \alpha)}. \quad (\text{A.8})$$

We can also straightforwardly obtain the perturbative expansion of $R_p^{\text{IR}}(\alpha)$, which reads:

$$R_p^{\text{IR}}(\alpha) = \frac{d_p^{\text{IR}}}{p^\gamma \Gamma(\gamma)} \sum_{n=0}^{\infty} \Gamma(n + \gamma) \left(\frac{\beta_0}{p}\right)^n \alpha^{n+1}. \quad (\text{A.9})$$

These results can be used to construct analytic expressions for our models, which can all be decomposed into polynomials and a sum of poles of the above form.

Appendix B: Adler function in the complex s -plane

Further understanding of the origin for the difference of $\delta^{(0)}$ as calculated within FOPT or CIPT can be gained by inspecting the reduced Adler function $\widehat{D}(s)$ on the circle $s = M_\tau^2 e^{i\varphi}$ in the complex s -plane. From the general perturbative expansion (2.10) it is clear that the ambiguity in the choice of the RG resummation already exists in this case. Analogous to the discussion of section 3 for R_τ , FOPT is defined by employing the constant scale choice $\mu^2 = M_\tau^2$, while CIPT corresponds to a resummation of the series on each point of the circle separately through the variable choice $\mu^2 = -s = -M_\tau^2 e^{i\varphi}$. The perturbative expansions for \widehat{D} as a function of the angle φ in the two cases take the form:

$$\widehat{D}_{\text{FO}}(\varphi) = \sum_{n=1}^{\infty} a(M_\tau^2)^n \sum_{k=1}^n k c_{n,k} [i(\varphi - \pi)]^{k-1}, \quad (\text{B.1})$$

$$\widehat{D}_{\text{CI}}(\varphi) = \sum_{n=1}^{\infty} c_{n,1} a(-M_\tau^2 e^{i\varphi})^n. \quad (\text{B.2})$$

In figure 9, we display a graphical account of our numerical results for the real parts $\text{Re}[\widehat{D}_{\text{FO}}(\varphi)]$ and $\text{Re}[\widehat{D}_{\text{CI}}(\varphi)]$ in the range $0 < \varphi < 2\pi$ and for $\alpha_s(M_\tau) = 0.3156$. We have drawn four different lines corresponding to a truncation of the perturbative series at the 4th (dotted line), 5th (dashed-double-dotted line), 6th (dashed-dotted line) and 7th (dashed line) order. In addition, the solid line corresponds to a resummation of the perturbative series according to the Borel sum of the physical model for the Adler function introduced in section 6. The required perturbative coefficients $c_{6,1}$ and $c_{7,1}$ for this model are given in table 2. The reason for showing the summation of the series up to $n = 7$ lies in the fact that the smallest summand of both series is found in this range.

Various observations can be made on the basis of figure 9. First of all, as can also be seen from eqs. (B.1) and (B.2), at the euclidian point $\varphi = \pi$ on the negative real s -axis, \widehat{D}_{FO} and \widehat{D}_{CI} are identical. The closest approach of the partial sums to the Borel sum for $\varphi = \pi$ is reached at the 6th order (compare figure 6). Close to the minkowskian region (the positive real s -axis), that is φ near zero or 2π , FOPT converges rather slowly, and only at the 7th order, the series approaches the shape displayed by the Borel sum. The slow convergence of FOPT close to the minkowskian region has already been noted in ref. [9], and was taken as an argument that CIPT should be preferable. Indeed, as is obvious from the lower plot of figure 9, CIPT converges very nicely for all angles and mostly even better for φ not near π than at the euclidian point. However, the resulting sums do not at all

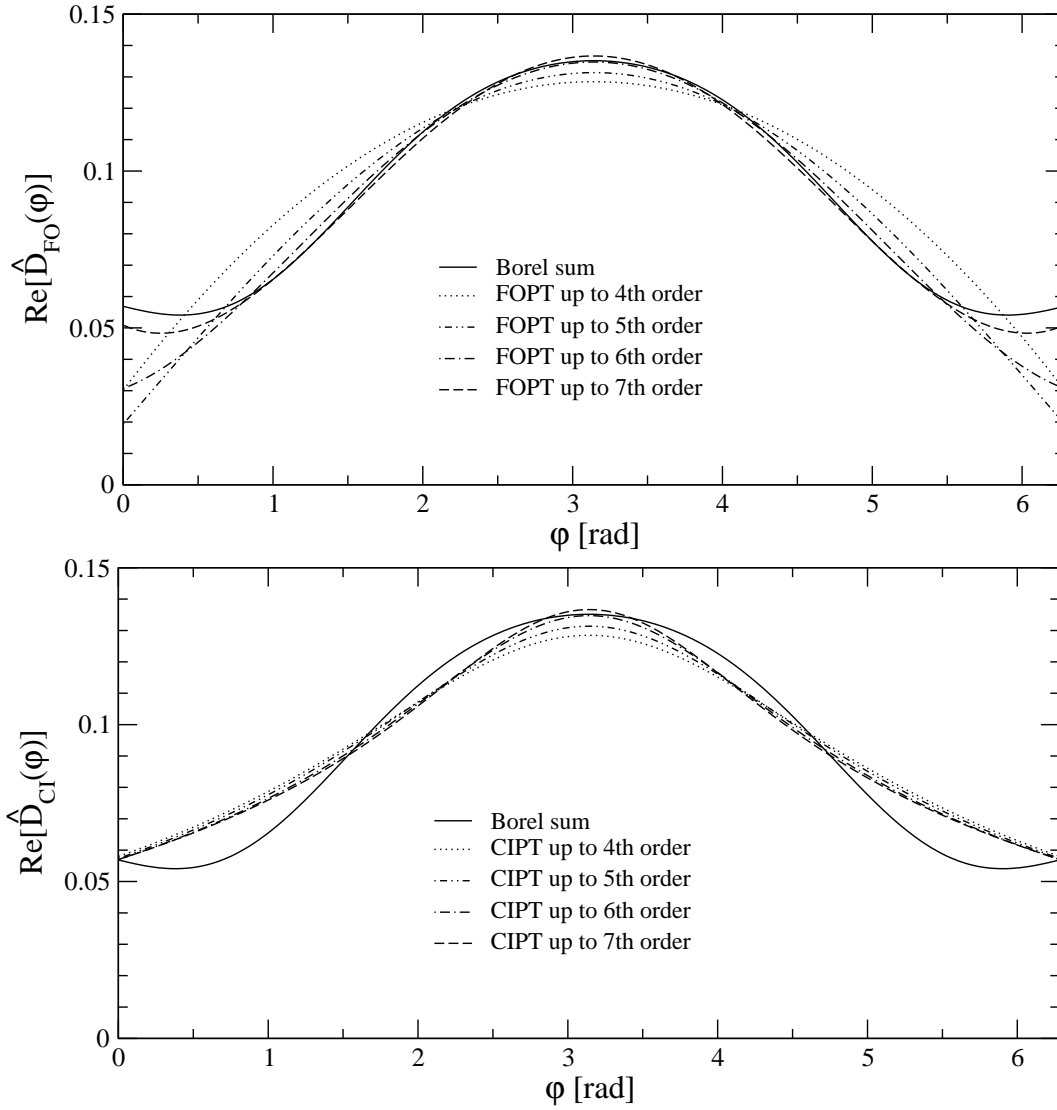


Figure 9: The real parts $\text{Re}[\hat{D}_{\text{FO}}(\varphi)]$ and $\text{Re}[\hat{D}_{\text{CI}}(\varphi)]$ of eqs. (B.1) and (B.2) for a summation of the perturbative series up to four different orders, namely the 4th (dotted line), 5th (dashed-double-dotted line), 6th (dashed-dotted line) and 7th (dashed line) order. The input for the QCD coupling was chosen to be $\alpha_s(M_\tau) = 0.3156$. Also displayed as the solid line is the Borel sum $\text{Re}[\hat{D}_{\text{BS}}(\varphi)]$ according to our physical model presented in section 6.

approach the shape of the Borel sum. Thus, except at a few isolated points, CIPT for the Adler function in the complex plane exhibits a similar problem as for R_τ , shown in figure 7, that the series exhibits apparent convergence and stability, but the resultant value is much further from the “true result” than any theoretical error estimate would suggest. For the Adler function CIPT lies below the Borel sum in the central region and above elsewhere, while for R_τ the weight function $(1-x)^3(1+x)$ with $x = e^{i\varphi}$ in the integration over the angle φ implies that $\delta_{\text{CI}}^{(0)}$ is below the Borel sum $\delta_{\text{BS}}^{(0)}$ as found in section 6. On the other hand, FOPT turns out to be much closer to the Borel sum, since the problematic

region close to the minkowskian axis is strongly suppressed in R_τ by the triple zero of the kinematical weight function at $s = M_\tau^2$. The problems with CIPT for the Adler function in the complex plane arise essentially for the same reason as for R_τ . In FOPT at each order we observe sizeable cancellations between the independent coefficients $c_{n,1}$ and the running effects which are missed by CIPT, although here, due to the angular dependence, the situation is less transparent.

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