Perturbative Input to Tau Decays

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Phys. Rev. Lett. 88 (2002) 012001
Phys. Rev. D67 (2003) 074026
Phys. Letts. B559 (2003) 245
Phys. Rev. Lett. 96 (2006) 012003
Phys. Rev. Lett. 101 (2008) 012002
Phys. Rev. Lett. 104 (2010) 132004

α_s from τ-decays

one of the most precise results for α_s

$$\frac{\Gamma(\tau \to h_{s=0}\nu)}{\Gamma(\tau \to l\overline{\nu}\nu)} = |V_{ud}|^2 S_{\mathrm{EW}} R_\tau = 3.471 \pm 0.011$$

$$R_\tau = 3\,(1 + \delta_P + \underbrace{\delta_{\mathrm{EW}}}_{\text{small}} + \underbrace{\delta_{\mathrm{NP}}}_{0.003 \pm 0.003})$$

$\delta_P = 0.1998 \pm 0.0043$ (exp)

(Davier, Höcker, Zhang, ALEPH, OPAL, CLEO)

- previous fixed order perturbation theory (FOPT):

$$\delta_P = a_s + 5.202\, a_s^2 + 26.37\, a_s^3 + ?$$

- previous contour improved perturbation theory (CIPT):

$$\delta_P = 1.364\, a_s + 2.54\, a_s^2 + 9.71\, a_s^3 + ?$$

previously:

estimates for α_s^4 (and α_s^5) terms only (FAC, PMS)

questions:

- are FAC/PMS supported by higher order calculations
- does the difference between (FOPT) and CIPT decrease upon inclusion of α_s^4? (Baikov, Chetyrkin, JK, 2002)

aim: evaluate α_s^4
$\Rightarrow$ absorptive part of 5-loop correlators

Theory:
The long march towards α_s^4

Massless Correlators: Technicalities

Correlator of two currents $\boldsymbol{j = \bar{q}\,\Gamma\,q}$ and $\boldsymbol{j^\dagger}$

$$\boldsymbol{\Pi^{jj}(q^2 = -Q^2) = i \int \mathrm{d}x e^{iqx} \langle 0 | T[\ j(x) j^\dagger(0)\ ] | 0 \rangle}$$

related to the corresponding absorptive part $R(s)$ through

$$R^{jj}(s) \approx \Im\, \Pi^{jj}(s - i\delta)$$

RG equation $(a_s \equiv \alpha_s/\pi)$

$$\Pi^{jj} = Z^{jj} + \Pi^{B}(-Q^2, \alpha_s^B)$$

$$\left(\mu^2 \frac{\partial}{\partial \mu^2} + \beta(a_s) \frac{\partial}{\partial a_s} \right) \Pi = \gamma^{jj}(a_s)$$

extremely useful for determining the absorptive part of Π^{jj}

For Π at $(L+1)$ loop

$$\frac{\partial}{\partial \log(\mu^2)}\Pi = \gamma^{jj}(a_s) - \left(\beta(a_s)\frac{\partial}{\partial a_s}\right)\Pi$$

anom.dim. at a_s^L
(L+1) loop integrals

only L-loop integrals contribute
due to the factor of $\beta(a_s)$

to find Log-dependent part of Π at (L+1)-loops one needs

- (L+1)-loop anomalous dimension γ^{jj} and L-loop Π (BUT! including its constant part)
- (L+1) loop anom.dim. reducible to combination of L-loop p-integrals

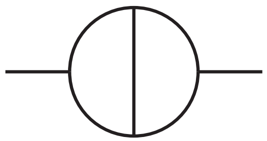
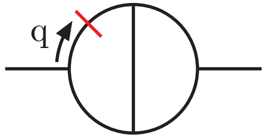
Strategy

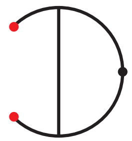
Log-dependent part of 5-loop correlator

$\hat{=}$ divergent part $(1/\epsilon)$ of 5-loop correlator

A div. part of 5-loop requires finite part of 4-loop

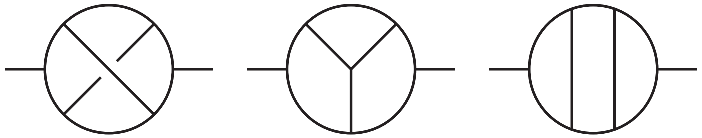
systematic, automatized algorithm (Chetyrkin)

div  $\hat{=} \int dq^2$ 

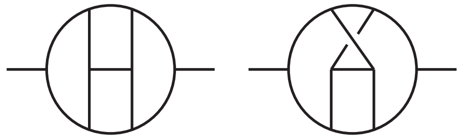
requires 

B fundamental problem:
finite part of 4-loop massless propagators
compare 3- and 4-loop calculation

3 topologies without insertions



11 topologies without insertion



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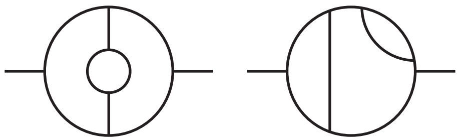
14 topologies with+without insertions



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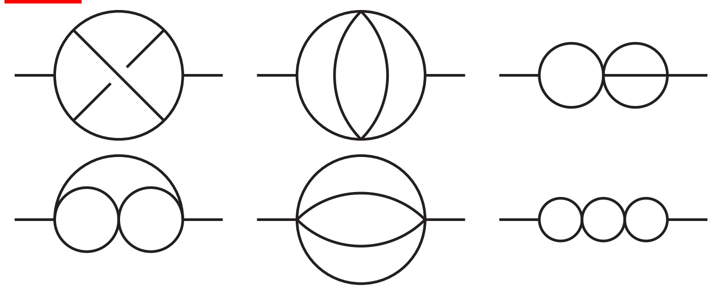
∼150 topologies with+without insertions

. . .

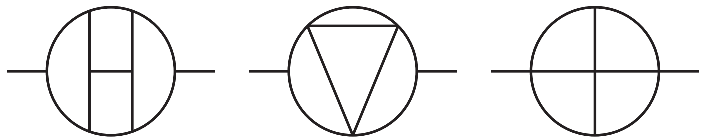
reduction to master integrals: MINCER

6 master integrals



reduction to master integrals ???

28 master integrals



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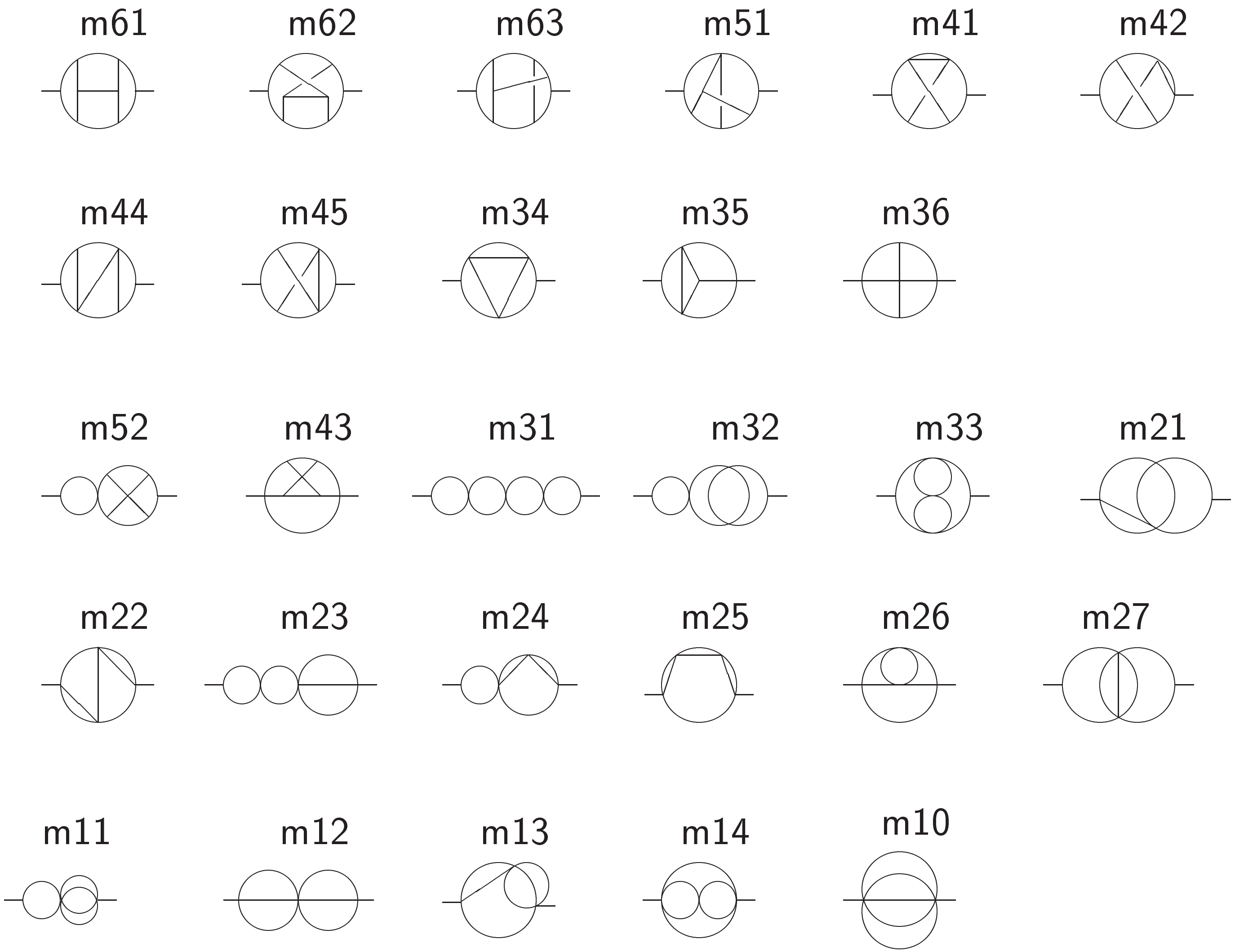
All relevant Master Integrals solved (2004)

(method: “glue and cut” (Chetyrkin, Tkachov))



MINCER: 3-loop (Larin, Tkachov, Vermaseren)

recursion relations based on integration by parts identities!

reduction algorithm and program constructed "manually" for 14 topologies.

4-loop:

more complicated identities

$\sim$ 150 topologies . . .

straightforward generalization of MINCER difficult

$\Rightarrow$ fully automatized construction of program; new concept?

C

Baikov: recursion relations can be solved "mechanically" in the limit of large dimension d:

consider amplitude f:

$f(\text{topology, power of prop, } d)$
$= \sum_{\alpha=\text{masters}} C^{(\alpha)}(\text{topology, power of prop, } d) \star f^{(\alpha)}(d)$

$f^{(\alpha)}$: 28 masters, analytically solved

$C^{(\alpha)}$: rational function $\frac{P^n(d)}{Q^m(d)}$, to be calculated;
$m + n \approx 60$ corresponds to ~ 60 coefficients

expand $C^{(\alpha)}$:

$C^{(\alpha)} = \sum_k c_k^{(\alpha)}(\text{topology, power of prop})(1/d)^k + \ldots$

$c_k^{(\alpha)}$ in terms of Gass-integrals $\Rightarrow$ sufficiently many terms $c_k^{(\alpha)}$
$\Rightarrow C^{(\alpha)}$

additional information on structure of $P^n(d)$, $Q^m(d)$ may lead to drastic reduction of hardware requirements:

originally ~ 60 numbers

additional information on structure of $Q^m(d)$ and using already calculated integrals

$\Rightarrow \; (m+n)_{\text{eff}} \approx 20$

evaluation of $c_k^{(\alpha)}$:

handling of polynomials of 9 variables of degree 2 k

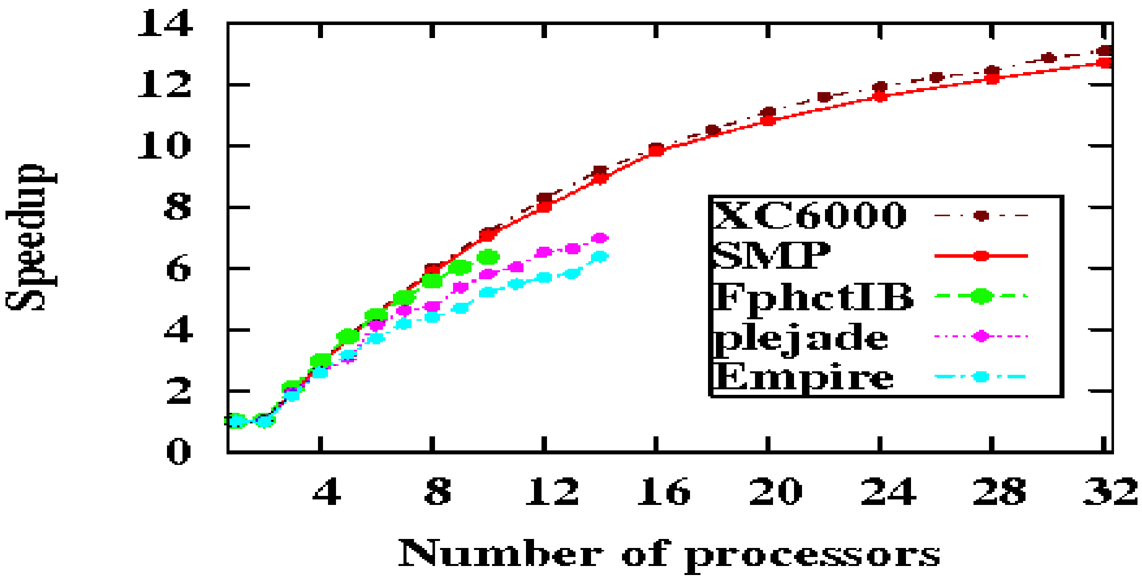
$\frac{(9+2k)!}{9!\,(2k)!}$ terms $\quad 2\,k = 40 \;\Rightarrow\; 2 \cdot 10^9$ terms

(200 GB storage, 1 TB for operation))

months of runtime

Computing

- 32+8 node SGI (SMP architecture)
- HP XC 4000 "Supercomputer"
- PARFORM
  (Tentyukov, Vermaseren, Fliegner,Retey . . .)



Reliability?

No independent evaluation!

Master integrals:

numerical evaluation (agreement up to $10^{-4} - 10^{-5}$)

Algebra:

Crewther relation:

$$D_{\text{NS}} \quad \star \quad C_{\text{Bjp}} \quad \approx \quad \left(1 + \frac{\beta(a_s)}{a_s} \quad K_{NS}(a_s)\right)$$

$$\Uparrow \qquad\qquad \Uparrow \qquad\qquad\qquad\qquad \Uparrow$$

Adler fcn Bjorken SR (evaluated in α_s^4) $a_s\,K_1^{NS} + a_s^2\,K_2^{NS} + a_s^3\,K_3^{NS}$

color structure $\Rightarrow$ 6 nontrivial constraints (fulfilled!)

Results

consider $D(Q^2) \equiv -12\pi^2 Q^2 \frac{d}{dQ^2}\Pi = \int_0^\infty ds \frac{Q^2}{(Q^2+s)^2} R(s)$

(Adler function, μ independent)

$$D(q^2) = 1 + a_s + a_s^2\,(-0.1153\,n_f + 1.968) + a_s^3\,(0.08621\,n_f^2 - 4.216\,n_f + 18.24) + a_s^4\,(-0.010\,n_f^3 + 1.88\,n_f^2 - 34.4\,n_f + 135.8)$$

Available in analytical form and for generic gauge group

relation to FAC/PMS

n_f	$d_4^{\mathrm{FAC/PMS}}$	d_4^{exact}	$r_4^{\mathrm{FAC/PMS}}$	r_4^{exact}
3	27 ± 16	49.08	-129 ± 16	-106.88
4	8 ± 28	27.39	112 ± 30	-92.90
5	-8 ± 44	9.21	97 ± 44	-79.98

impact on α_s from Z-decays

$$R(s) = D(s) - \pi^2 \beta_0^2 \left\{ \frac{d_1}{3} a_s^3 + \left(d_2 + \frac{5\beta_1}{6\beta_0} d_1 \right) a_s^4 \right\}$$

$$\Rightarrow \delta\alpha_s(M_Z) = 0.0005$$

$$\alpha_s(M_Z)^{\text{NNNLO}} = 0.1190 \pm 0.0026$$

impact on α_s from τ-decays

$\delta_P = 0.1998 \pm 0.043$ (exp) scale: $\mu^2/M_\tau^2 = 0.4 - 2$ (theor)

	$\alpha_s^{FO}(M_\tau)$	$\alpha_s^{CI}(M_\tau)$
no α_s^4	$0.337 \pm 0.004 \pm 0.03$	$0.354 \pm 0.006 \pm 0.02$
$d_4 = 25$	$0.325 \pm 0.004 \pm 0.02$	$0.347 \pm 0.006 \pm 0.009$
$d_4 = 49.08$	$0.322 \pm 0.004 \pm 0.02$	$0.342 \pm 0.005 \pm 0.01$

$\Rightarrow$ use mean value between FOPT and CIPT $\Leftarrow$ difference beweeen FOPT andf CIPT unchanged, as anticipated in 2002

$$\alpha_s(M_\tau) = 0.332 \pm 0.005_{\rm exp} \pm 0.015_{\rm theo}$$

RUNDEC:

four-loop running (β_0, β_1, β_2, β_3)
four-loop matching at quark thresholds
($m_c(m_c) = 1.286(13)$ GeV, $m_b(m_b) = 4.164(25)$ GeV)

$$\begin{aligned} \alpha_s(M_Z) &= 0.1202 \pm 0.0006_{\rm exp} \pm 0.0018_{\rm theo} \pm 0.0003_{\rm evol} \\ &= 0.1202 \pm 0.0019 \end{aligned}$$

consistent with α_s from Z

Summary

- Adler function, $R(s)$, R_τ available to $\mathcal{O}(\alpha_s^4)$
- First and only N^{3}LO results

$$\alpha_s(M_z) = \begin{cases} 0.1190 \pm 0.0026 & \text{from } Z \\ 0.1202 \pm 0.0019 & \text{from } \tau \end{cases}$$

- α_s^4 terms move Z and τ closer together

combined

$$\alpha_s(M_Z) = 0.1198 \pm 0.0015$$