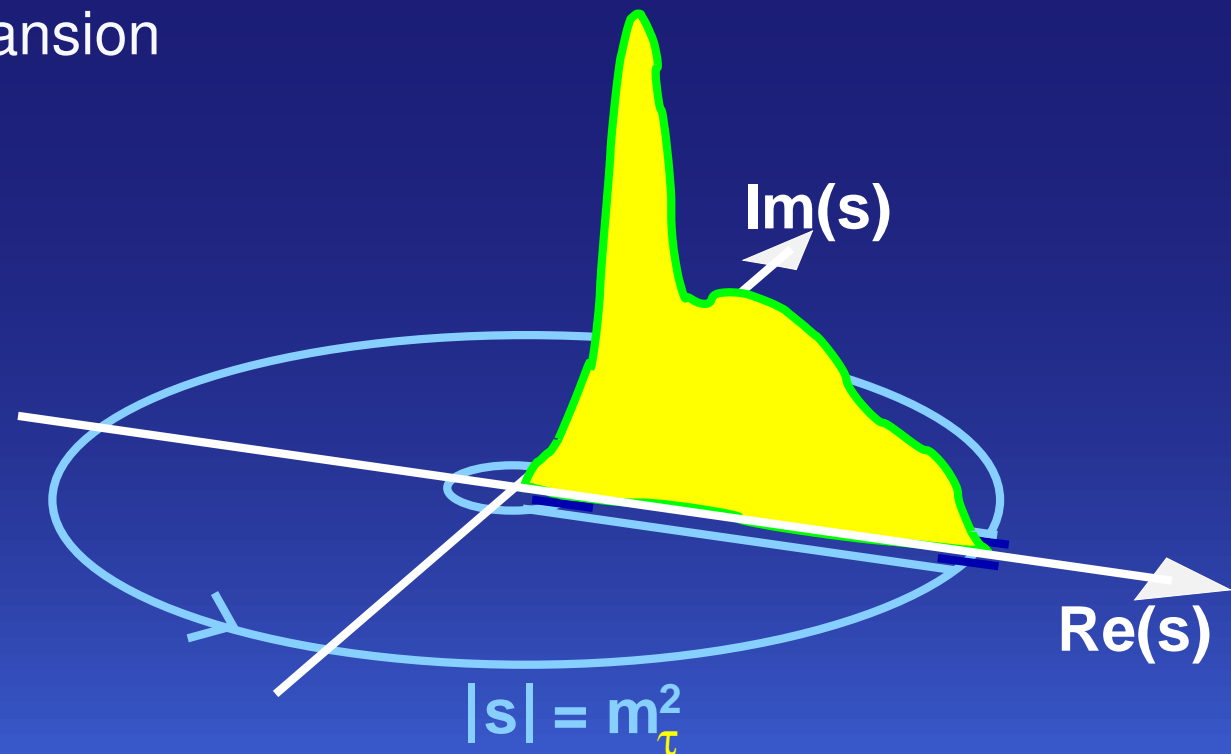# FOPT and CIPT in τ Decays

α_s Workshop   Sven Menke, MPP München   9. February 2011, MPP München

► Perturbative QCD Methods in τ Decays
- Fixed Order Perturbation Theory
- Contour Improved Perturbation Theory

► Generalized FOPT
- Modified Taylor Expansion

► Generalized RCPT

► Numerical Results

► Conclusions

arXiv:0904.1796

Im(s)

Re(s)

$|s| = m_\tau^2$

► Use the optical theorem:

- The differential decay width $\mathrm{d}\Gamma/\mathrm{d}s$ for the τ going into hadrons is proportional to the imaginary part of the vacuum polarization $\mathrm{Im}\Pi$ (also called spectral function) of the W propagator

- this means it is enough to calculate the inclusive vacuum polarization of the W instead of every single final state of the hadronic τ decays
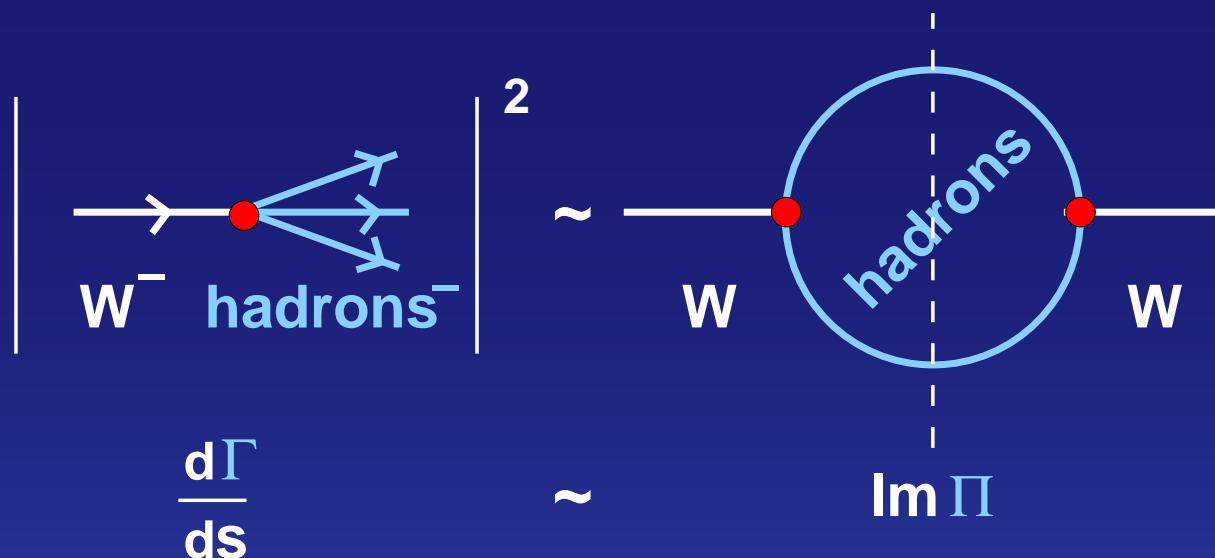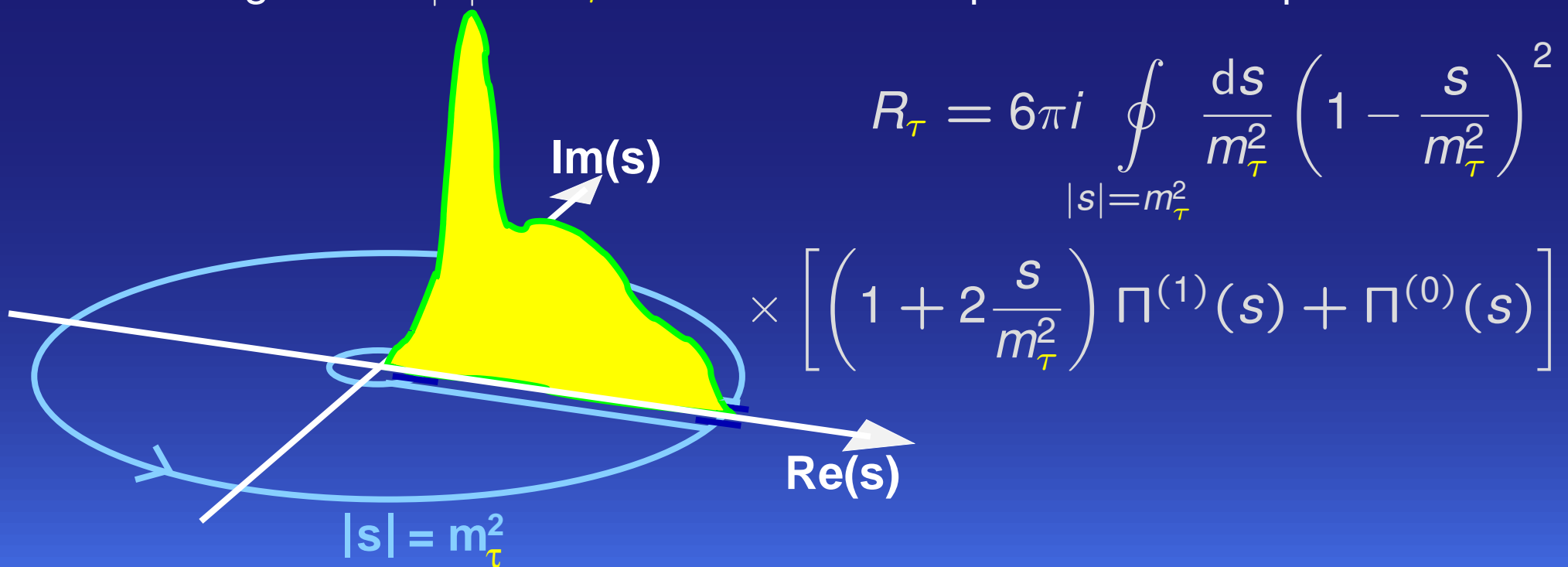

$$R_\tau = 12\pi \int\limits_0^{m_\tau^2} \frac{\mathrm{d}s}{m_\tau^2} \left(1 - \frac{s}{m_\tau^2}\right)^2 \left[\left(1 + 2\frac{s}{m_\tau^2}\right)\mathrm{Im}\Pi^{(1)}(s) + \mathrm{Im}\Pi^{(0)}(s)\right]$$

► Use the Cauchy theorem:

- since Π is analytic (except for the real positive s-axis where it might have poles) in the entire complex s-plane

- and due to the identity $\text{Im}\Pi(s_+ + i\epsilon) = \dfrac{1}{2i}(\Pi(s_+ + i\epsilon) - \Pi(s_+ - i\epsilon))$

- the integral along the real positive s-axis can be expressed as a circular integral at $|s| = m_\tau^2$

- this means all QCD calculations are done at a well defined rather high scale $|s| = m_\tau^2$ over the vacuum polarization amplitude

$$R_\tau = 6\pi i \oint_{|s|=m_\tau^2} \frac{ds}{m_\tau^2}\left(1 - \frac{s}{m_\tau^2}\right)^2$$

$$\times \left[\left(1 + 2\frac{s}{m_\tau^2}\right)\Pi^{(1)}(s) + \Pi^{(0)}(s)\right]$$

Im(s)

Re(s)

|s| = m$_\tau^2$

▶ All the dynamics of the vacuum polarization amplitude is in its logarithmic derivative, the so-called Adler-function: $D(s) = -s\dfrac{\mathrm{d}\Pi(s)}{\mathrm{d}s}$

▶ the Adler-function can be written as polynomial in the strong coupling constant α_s: $D(s) = \dfrac{1}{4\pi^2}\displaystyle\sum_{n=0}^{\infty} K_n\left(\dfrac{\alpha_s(-s)}{\pi}\right)^n$, where the K_n are known up to $n = 4$ and K_5 has been estimated:

$$
\begin{aligned}
K_0 &= K_1 = 1 \\
K_2 &= \frac{299}{24} - 9\zeta(3) = 1.63982\ldots \\
K_3 &= \frac{58057}{288} - \frac{779}{4}\zeta(3) + \frac{75}{2}\zeta(5) = 6.37101\ldots \\
K_4 &= \frac{78631453}{20736} + \frac{4185}{8}\zeta(3)^2 - \frac{1704247}{432}\zeta(3) \\
&\quad + \frac{34165}{96}\zeta(5) - \frac{1995}{16}\zeta(7) = 49.0757\ldots \\
K_5 &\simeq 275, \quad \text{I'll use } 400 \pm 400
\end{aligned}
$$

▶ convert the contour integral over $\Pi(s)$ via partial integration in a contour integral over $D(s)$

$$\oint_{|s|=m_\tau^2} ds\, g(s)\, \Pi(s) \;=\; \oint_{|s|=m_\tau^2} \frac{ds}{s} \left[G(s) - G(m_\tau^2) \right] (-s) \frac{d}{ds} \Pi(s)$$

$$=\; \oint_{|s|=m_\tau^2} \frac{ds}{s} \left[G(s) - G(m_\tau^2) \right] D(s)$$

- where the polynomial $g(s) = 6\pi i \dfrac{1}{m_\tau^2} \left(1 - \dfrac{s}{m_\tau^2}\right)^2 \left(1 + 2\dfrac{s}{m_\tau^2}\right)$

- has the antiderivative $G(s) = 3\pi i \left(2\dfrac{s}{m_\tau^2} - 2\dfrac{s^3}{m_\tau^6} + \dfrac{s^4}{m_\tau^8}\right)$

- note that $g(s)$ describes the $J=1$ part only since for massless quarks the scalar and pseudo-scalar parts vanish

$$R_\tau = 3 \sum_{n=0}^{5} \frac{K_n}{2\pi i} \oint_{|s|=m_\tau^2} \frac{ds}{s} \left(1 - 2\frac{s}{m_\tau^2} + 2\frac{s^3}{m_\tau^6} - \frac{s^4}{m_\tau^8}\right) \left(\frac{\alpha_s(-s)}{\pi}\right)^n$$

▶ the perturbative descriptions of hadronic τ decays start all with the same integral given on the previous slide

▶ they differ in the way the integral is calculated

▶ most interesting is the treatment of $\alpha_s(-s)$ on the circle $|s| = m_\tau^2$

▶ QCD does not tell us how large $\alpha_s(\mu^2)$ at a given scale μ is, but QCD does tell us what $\alpha_s(\mu_1^2)$ at some scale μ_1 is if we know it at some other scale μ_0

▶ this prediction is made with the so-called β-function:

$\beta(a_s) = \dfrac{\mathrm{d}a_s}{\mathrm{d}\ln s} = -\beta_0 a_s^2 - \beta_1 a_s^3 - \beta_2 a_s^4 - \beta_3 a_s^5 + O(a_s^6)$, with $a_s = \dfrac{\alpha_s(\mu^2)}{4\pi}$, and where the β_n are known up to $n = 3$:

$$\beta_0 = 11 - \frac{2}{3}n_f, \qquad \beta_1 = 102 - \frac{38}{3}n_f, \qquad \beta_2 = \frac{2857}{2} - \frac{5033}{18}n_f + \frac{325}{54}n_f^2,$$

$$\beta_3 = \frac{149753}{6} + 3564\zeta(3) - \left(\frac{1078361}{162} + \frac{6508}{27}\zeta(3)\right)n_f + \left(\frac{50065}{162} + \frac{6472}{81}\zeta(3)\right)n_f^2 + \frac{1093}{729}n_f^3.$$

► The Adler-function shown used K_n for a fixed choice of renormalization scale $-\mu^2 = m_\tau^2$

► Physics should not depend on this choice

► The truncation of the perturbative series introduces a residual dependency: $\mu^2 \dfrac{\mathrm{d}}{\mathrm{d}\mu^2} D(s, \mu^2) = 0 + O\left(\alpha_s^5(\mu^2)\right)$

► This can be solved for each order with the β-function ($\eta = \ln(-s/\mu^2)$):

$$K_0(\eta) = K_1(\eta) = K_0 = K_1 = 1$$

$$K_2(\eta) = K_2 - \frac{\beta_0}{4}\eta$$

$$K_3(\eta) = K_3 - \left(\frac{\beta_1}{16} + \frac{\beta_0}{2}K_2\right)\eta + \frac{\beta_0^2}{16}\eta^2$$

$$K_4(\eta) = K_4 - \left(\frac{\beta_2}{64} + \frac{\beta_1}{8}K_2 + \frac{3\beta_0}{4}K_3\right)\eta + \left(\frac{5\beta_1\beta_0}{128} + \frac{3\beta_0^2}{16}K_2\right)\eta^2 - \frac{\beta_0^3}{64}\eta^3$$

$$K_5(\eta) = K_5 - \left(\frac{\beta_3}{256} + \frac{\beta_2}{32}K_2 + \frac{3\beta_1}{16}K_3 + \beta_0 K_4\right)\eta +$$

$$\left(\frac{3\beta_0^2}{8}K_3 + \frac{7\beta_1\beta_0}{64}K_2 + \frac{3\beta_2\beta_0}{256} + \frac{3\beta_1^2}{512}\right)\eta^2 + \left(\frac{\beta_0^3}{16}K_2 + \frac{13\beta_1\beta_0^2}{768}\right)\eta^3 + \frac{\beta_0^4}{256}\eta^4$$

▶ **Fixed Order Perturbation Theory**

- make Taylor-expansion of $\alpha_s(-s)$ on the circle $|s| = m_\tau^2$ around $\alpha_s(m_\tau^2)$

- insert Taylor-expansion in the integral which becomes solvable in all orders

- order the result in powers of $\alpha_s(m_\tau^2)$

- keep only the terms up to a fixed order in $\alpha_s(m_\tau^2)$

$$
\begin{aligned}
R_\tau &= 3\left(1 + \frac{\alpha_s(m_\tau^2)}{\pi} + 5.2023\ldots\frac{\alpha_s^2(m_\tau^2)}{\pi^2} + 26.3659\ldots\frac{\alpha_s^3(m_\tau^2)}{\pi^3}\right. \\
&\quad + 127.0786\ldots\frac{\alpha_s^4(m_\tau^2)}{\pi^4} \\
&\quad \left. + (K_5 + 307.7825\ldots)\frac{\alpha_s^5(m_\tau^2)}{\pi^5} + O(\alpha_s^6)\right)
\end{aligned}
$$

► Contour Improved Perturbation Theory

- evolve $\alpha_{\mathrm{s}}(-s)$ in small steps numerically using the β-function on the circle $|s| = m_\tau^2$

- insert the numerical values in the integral and solve it numerically too

- the result contains integrals over terms of the form
$$s^k \left[\left(\frac{\alpha_{\mathrm{s}}(m_\tau^2)}{\pi} \right)^l \ln^m \frac{-s}{m_\tau^2} \right]^n, \text{ with } k = 0, 1, 3, 4; n = 1, \ldots, 5 \text{ (up to } K_5$$
for $D(s)$); $l = 2, \ldots, 5$ (up to β_3 for the evolution of α_{s}); $m = 1, \ldots, l - 1$

- the terms with total power in α_{s} larger than the chosen order in FOPT are creating the first difference between CIPT and FOPT

- for up to the 5$^{\text{th}}$ order the neglected terms in FOPT are those with $l \times n > 5$

- from the 50 logarithmic terms 39 are neglected in FOPT

▶ let's have a closer look to the α_s expansion used in FOPT

▶ from the β function the following equation for $\alpha_s(-s)$ as function of $\alpha_s(s_0)$ follows:

$$
\begin{aligned}
\frac{\alpha_s(-s)}{\pi} \;=\; & \frac{\alpha_s(s_0)}{\pi} - \frac{1}{4}\beta_0 \ln\frac{-s}{s_0}\left(\frac{\alpha_s(s_0)}{\pi}\right)^2 + \\[4pt]
& \frac{1}{16}\left(\beta_0^2\ln^2\frac{-s}{s_0} - \beta_1\ln\frac{-s}{s_0}\right)\left(\frac{\alpha_s(s_0)}{\pi}\right)^3 - \\[4pt]
& \frac{1}{128}\left(2\,\beta_0^3\ln^3\frac{-s}{s_0} - 5\,\beta_0\beta_1\ln^2\frac{-s}{s_0} + 2\,\beta_2\ln\frac{-s}{s_0}\right)\left(\frac{\alpha_s(s_0)}{\pi}\right)^4 + \\[4pt]
& \frac{1}{1536}\left(6\,\beta_0^4\ln^4\frac{-s}{s_0} - 26\,\beta_0^2\beta_1\ln^3\frac{-s}{s_0} + \right. \\[4pt]
& \left. 9\left(\beta_1^2 + 2\,\beta_0\beta_2\right)\ln^2\frac{-s}{s_0} - 6\,\beta_3\ln\frac{-s}{s_0}\right)\left(\frac{\alpha_s(s_0)}{\pi}\right)^5 + O(\alpha_s(s_0)^6)
\end{aligned}
$$

► we can compare the numerically solved α_s with the Taylor expansion from the prev. slide on the circle $s = m_\tau^2$.

► note that we even keep all known orders (up to α_s^5) for the best

► while in FOPT the term for α_s^5 enters with just first order (i.e. the constant value $\alpha_s(s) = \alpha_s(s_0)$)

► the usage of the Taylor expansion over a wide distance from s to s_0 creates the second difference between FOPT and CIPT

► plots (real part: left, imaginary part: right) show Taylor-expanded $\alpha_s(s = m_\tau^2 \exp(i\varphi))$ (blue) and numerical result (black) for various orders
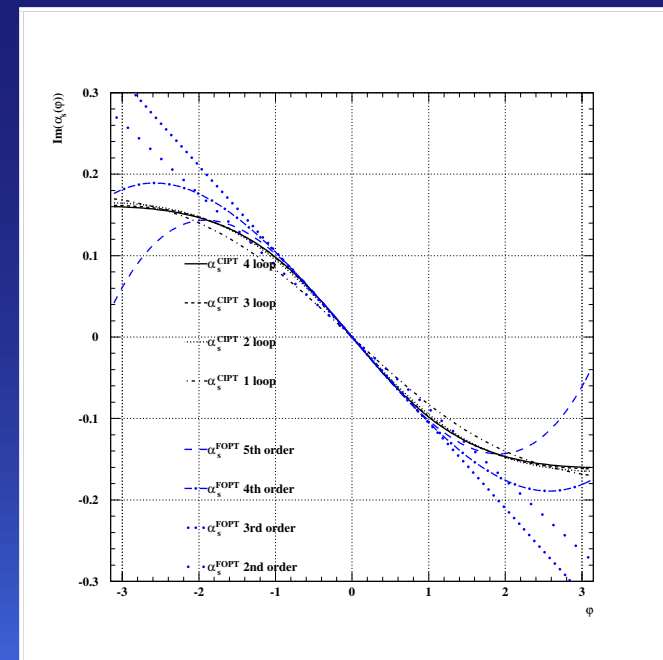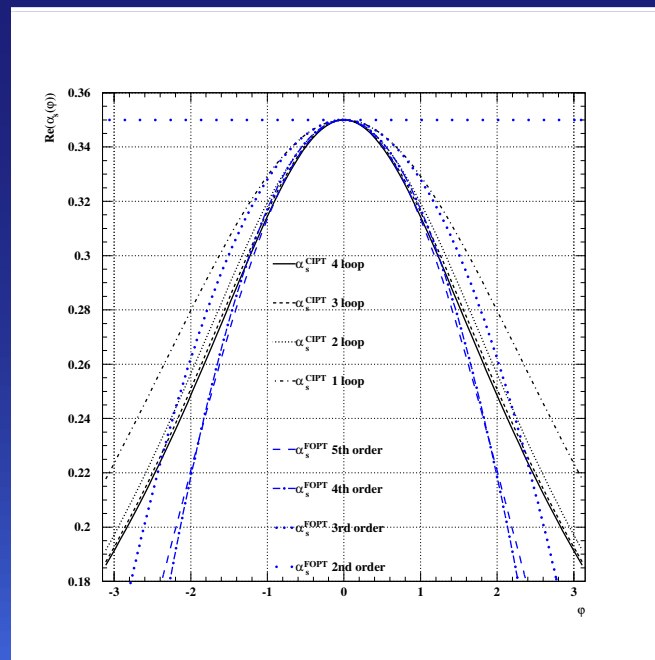
▶ another way of showing the differences between FOPT and CIPT is to apply the two methods to the evolution of α_s from the τ to the Z^0

▶ for CIPT 100 small logarithmic steps are used

▶ for FOPT just 2 steps (at flavor thresholds) are used

▶ beyond 2 loop the CIPT results become almost identical

▶ but even at 5th order the deviation for FOPT is 30%

▶ again the main difference comes not from higher orders but from the large step width
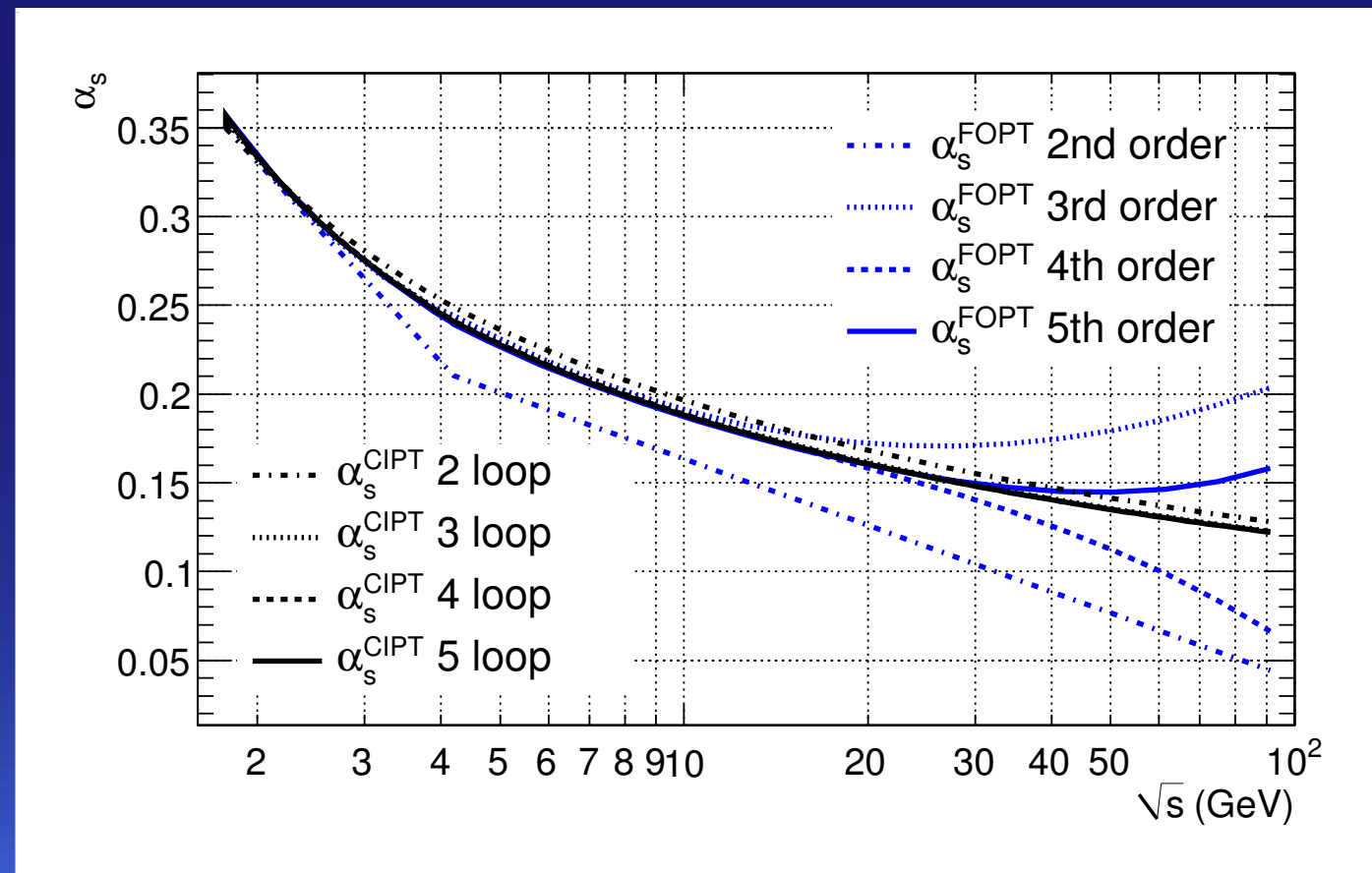
Generalized FOPT

► How can the difference between FOPT and CIPT be absorbed into an uncertainty?

- The Taylor expansion of α_s on the circle $|s| = m_\tau^2$ in FOPT is used from $\alpha_s(m_\tau^2)$ to $\alpha_s(-m_\tau^2 - i\epsilon)$ and from $\alpha_s(m_\tau^2)$ to $\alpha_s(-m_\tau^2 + i\epsilon)$
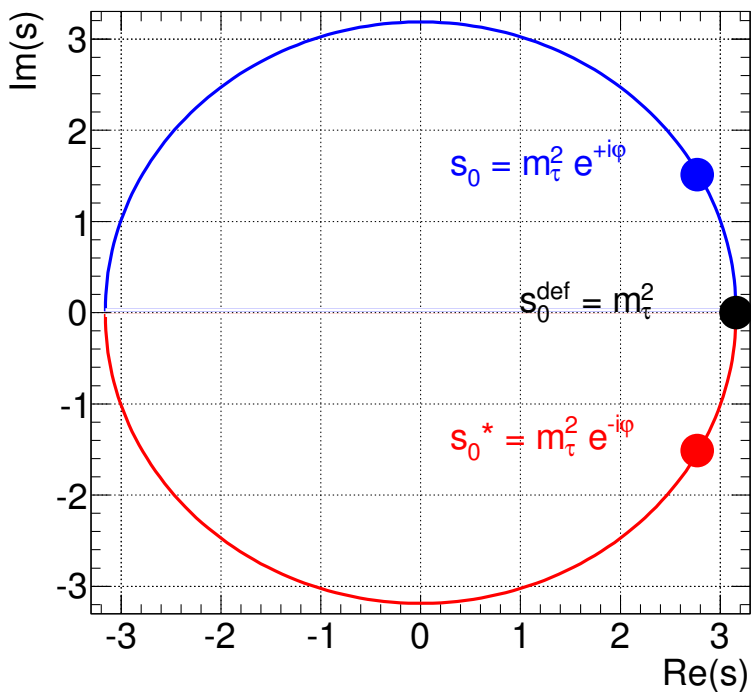
- The maximum angular distance on the circle is $180°$ on both semi-cricles
- Splitting each semi-circle in 2 parts and developing α_s around the splitting point ensures that again the angular distance is at most $180°$
- The splitting points should be symmetric on both semi-cricles to keep the symmetry $\alpha_s(s^*) = \alpha_s(s)^*$ which is important to keep integrals real
- If more and more equidistant splitting points would be added the result would converge towards CIPT

► Moving the splitting point on the semi-circles from $\phi = 0°$ (the usual FOPT) to $\phi = 180°$ (anti FOPT) shows the impact of this arbitrary choice

Generalized FOPT

► The FOPT result around the new development points of
$\alpha_s(m_\tau^2 \exp(\pm i\varphi_0))/\pi = a \pm ib$ looks like this:

$$
\begin{aligned}
\delta_{\text{pert}} = \quad & a + 0.8488\, b + (-4.5\, \varphi_0 + 0.8082)\, a\, b + \\
& (1.9099\, \varphi_0 + 5.2023)\,(a^2 - b^2) + \\
& (-5.0625\, \varphi_0^2 + 5.2138\, \varphi_0 + 26.366)\,(a^3 - 3\, a\, b^2) + \\
& (4.2972\, \varphi_0^2 + 27.410\, \varphi_0 + 12.356)\,(b^3 - 3\, a^2\, b) + \\
& (-9.6687\, \varphi_0^3 - 101.51\, \varphi_0^2 - 71.629\, \varphi_0 + 127.08)\,(a^4 - 6\, a^2\, b^2 + b^4) + \\
& (45.563\, \varphi_0^3 - 100.94\, \varphi_0^2 - 918.59\, \varphi_0 - 521.11)\,(a^3\, b - a\, b^3) + \\
& (25.629\, \varphi_0^4 - 92.897\, \varphi_0^3 - 1220.5\, \varphi_0^2 - 1272.5\, \varphi_0 + \\
& \quad K_5 + 307.78)\,(a^5 - 10\, a^3\, b^2 + 5\, a\, b^4) + \\
& (21.755\, \varphi_0^4 + 324.78\, \varphi_0^3 + 271.83\, \varphi_0^2 - 1612.0\, \varphi_0 + \\
& \quad 0.8488\, K_5 - 1413.5)\,(b^5 + 5\, a^4\, b - 10\, a^2\, b^3),
\end{aligned}
$$

► δ_{pert} remains real for all choices of α_s and φ_0

► the usual FOPT result is retained for $\varphi_0 = 0$ and $b = 0$

► re-inserting the Taylor expanded α_s and Taylor expanding again also gives the usual FOPT result

► the plot shows δ_{pert} for the generalized FOPT (blue) vs. CIPT (black) as function of φ_0

► the 3 lines for each are different choices of K_5 (0, 400, 800)

► CIPT remains constant for all choices of φ_0

► variation for FOPT due to φ_0 is 4 times larger than the variation due to K_5

Generalized RCPT
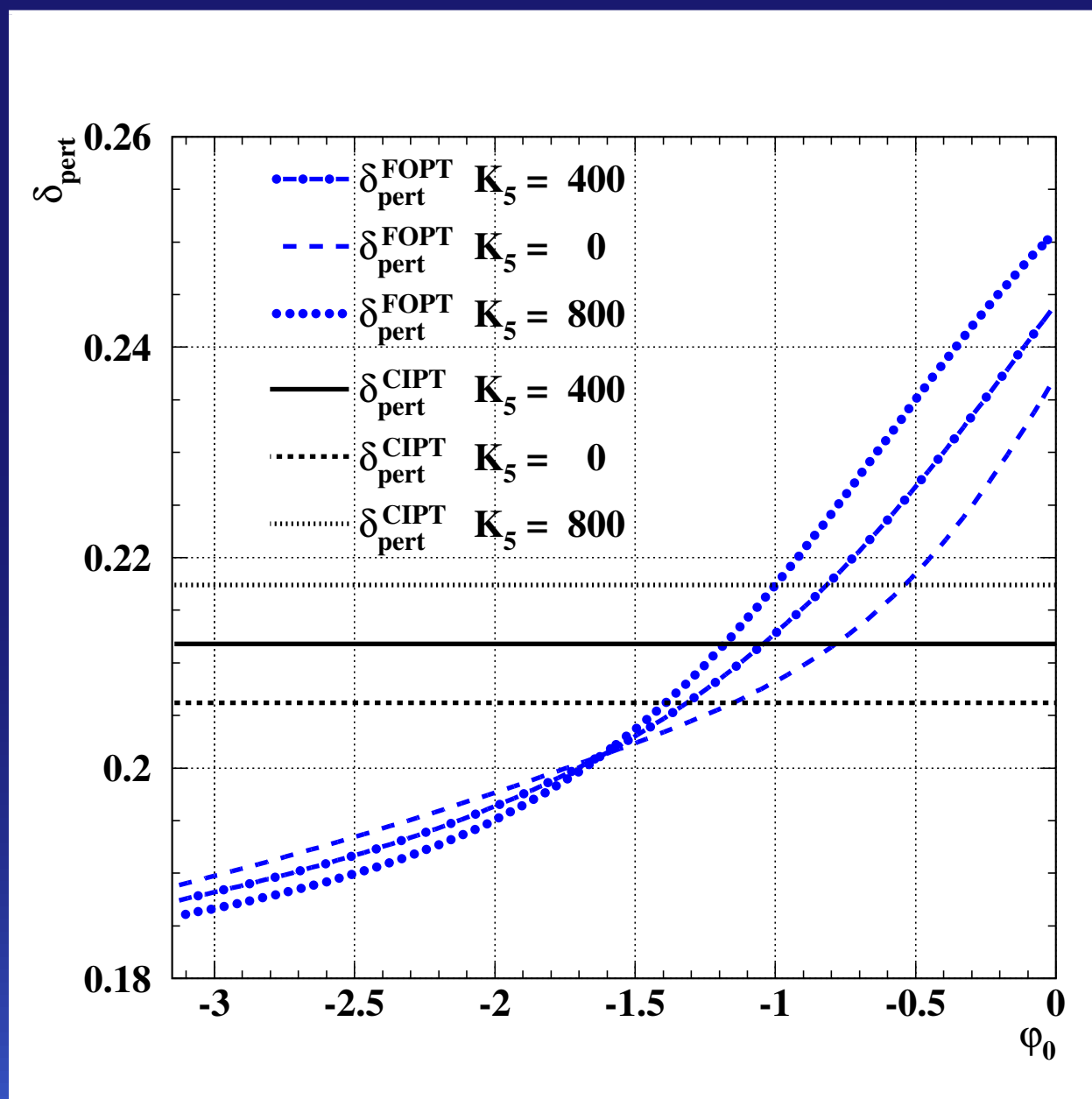
► For Renormalon Cchain Perturbation Theory a similar generalization approach can be used

- usually the RCPT results are corrected up to the order used in FOPT with the known exact FOPT terms, while higher order terms are resummed to all orders in the large β_0 approximation
  ► $\delta_{\text{pert}}^{\text{RCPT}} =$
  $\delta_{\text{renormalon}} - \delta_{\text{large-}\beta_0}^{\text{FOPT}} + \delta_{\text{pert}}^{\text{FOPT}}$

- in this equation the same arbitrary development point $\varphi_0 = 0$ is usually used for $-\delta_{\text{large-}\beta_0}^{\text{FOPT}} + \delta_{\text{pert}}^{\text{FOPT}}$

- therefore the same generalization idea to allow for other values of φ_0 can be applied

- in addition I use 2-loop instead of 0-loop matching to go from the $\overline{\text{MS}}$-scheme to the V scheme

- the plot shows the comparison of generalized RCPT and CIPT like for generalized FOPT and CIPT on the slide before
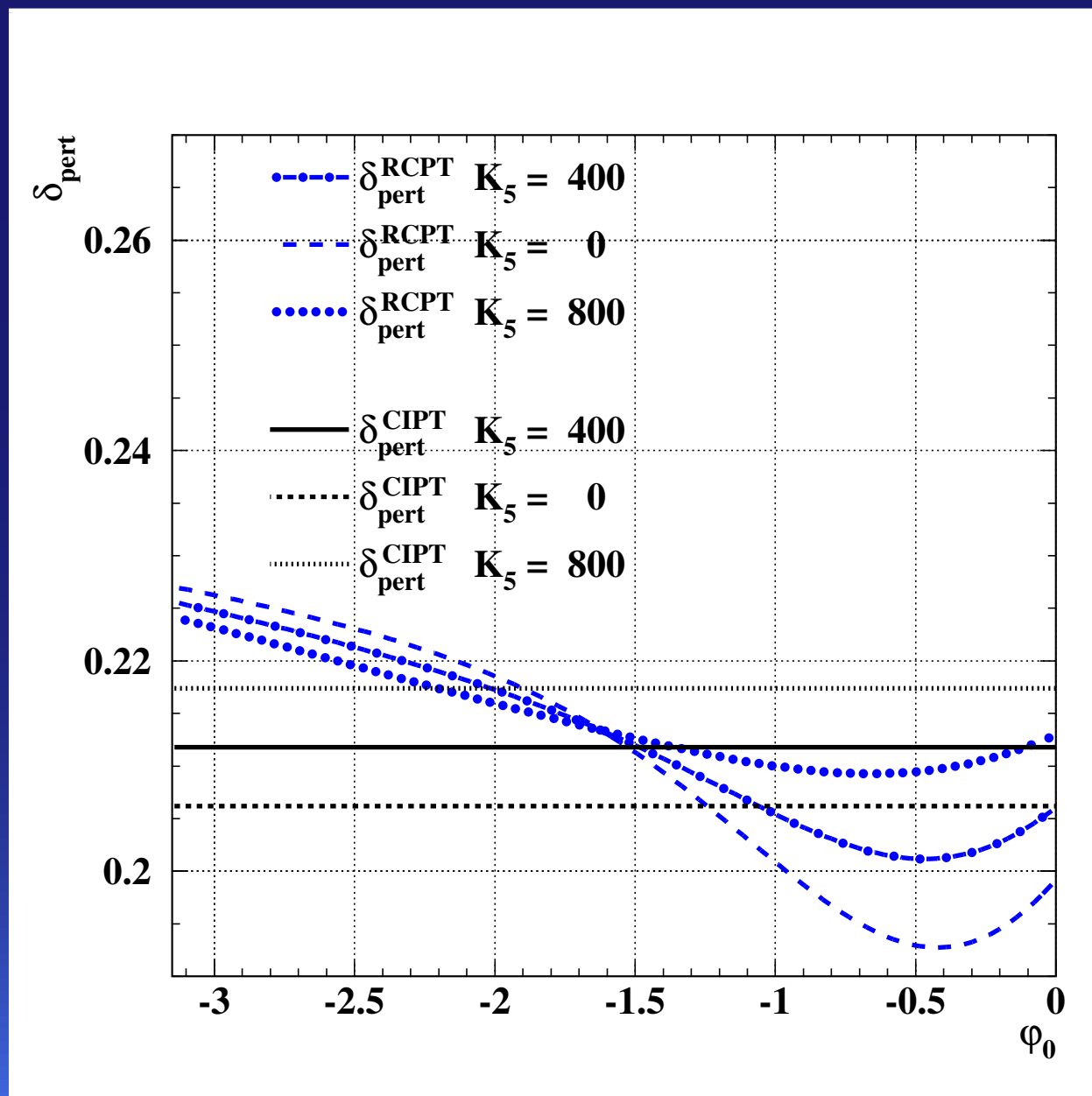
Numerical Results

► For numerical comparisons I'll use the δ_{pert} from M. Davier et al. Eur.Phys.J.C56, 305 (2008), arXiv:0803.0979:

► $\delta_{\text{pert}} = 0.2042 \pm 0.0038_{\text{exp}} \pm 0.0033_{\text{non-pert}}$

$$\alpha_{\text{s}}^{\text{CIPT}}(m_\tau^2) = 0.3406 \pm 0.0047_{\text{exp}} \pm 0.0041_{\text{non-pert}} \pm 0.0066_{K_5}$$

$$\alpha_{\text{s}}^{\text{FOPT}}(m_\tau^2) = 0.3535 \pm 0.0061_{\text{exp}} \pm 0.0053_{\text{non-pert}} \pm 0.0208_{\varphi_0}{}_{-0.0001}^{+0.0005}{}_{K_5}$$

$$\alpha_{\text{s}}^{\text{RCPT}}(m_\tau^2) = 0.3440 \pm 0.0030_{\text{exp}} \pm 0.0026_{\text{non-pert}} \pm 0.0061_{\varphi_0} \pm 0.0019_{K_5}$$

► All three results agree within errors due to the variation of φ_0

► Averaging the RCPT and CIPT result only (since FOPT has a much larger uncertainty) yields:

$$\alpha_{\text{s}}(m_\tau^2) = 0.3423 \pm 0.005_{\text{exp}} \pm 0.007_{\triangle K_5} \pm 0.004_{\text{non-pert}}{}_{-0.001}^{+0.005}{}_{\mu}$$

► the last error is due to the variation of the renormalization scale $0.4 \leq \mu^2/m_\tau^2 \leq 1.6$

Numerical Results

► evolution to the Z^0 gives finally:

$$\alpha_{\mathrm{s}}(m_{Z^0}^2) = 0.1213 \pm 0.0006_{\mathrm{exp}} \pm 0.0008_{\triangle K_5} \pm 0.0004_{\mathrm{non\text{-}pert}}{}^{+0.0005}_{-0.0001}{}_{\mu} \pm 0.0002_{\mathrm{ev}}$$

► where the last error is the evolution uncertainty due to the variation of the thresholds $m_q < m_{\mathrm{thresh}} < 2m_q$ and the quark masses itself within their respective errors

Conclusions

▶ The differences between FOPT and CIPT can be mainly attributed to two reasons
- truncation of the Taylor series of α_s
- usage of the Taylor series over a long distance

▶ Uncertainties due to this can be made visible by allowing alternative development points of α_s on the complex circle $s_0 = m_\tau^2 \exp(i\varphi_0)$
- these uncertainties are large:
  ▶ about 4 times larger than uncertainties due to neglected higher orders
- the usual chosen point $s_0 = m_\tau^2$ sits at an extreme end making α_s from FOPT appearing smaller than from CIPT
- averaging over all possible choices gives a result much closer to CIPT – but still with larger error

▶ My proposal is to test FOPT type results always with the generalization procedure presented here in order to estimate the uncertainty and central value

► in some comments I rececived to the paper it was suggested to allow $\varphi_0 > 0$ as well

► of course this is possible but one has to keep in mind that for all $\varphi_0 > 0$ the angular distance in the Taylor expansion for α_s is larger than $180°$ und thus larger than for default FOPT

► the plot shows the full range for possible φ_0
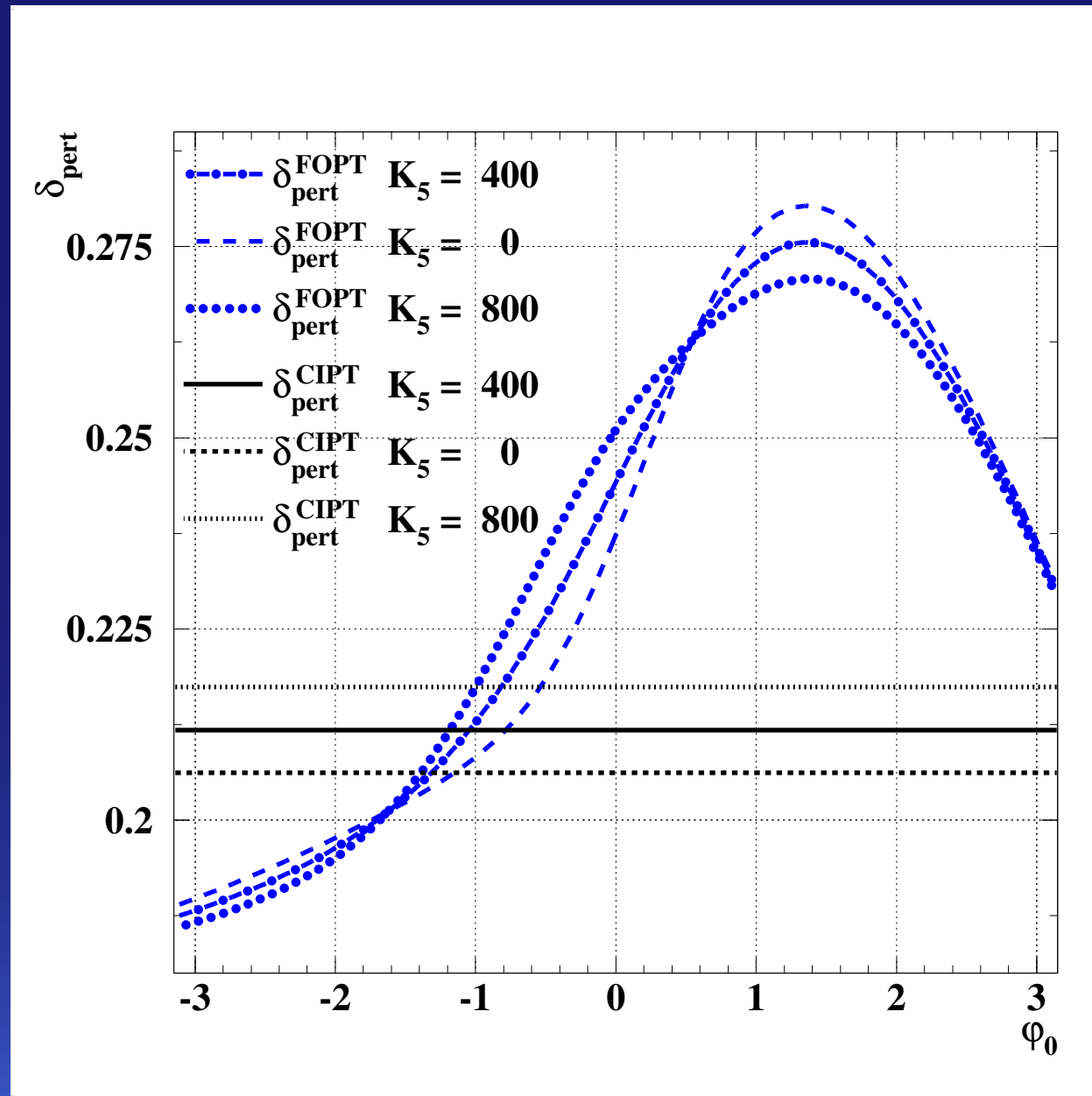
► to illustrate the effect of more and more development points on the circle here is an example with 4 such points

► the points are arranged equidistantly on the semicircle and the angle φ_0 is the angle relative to the quadrant

► the variation of δ_{pert} with φ_0 is already half of the value for 2 development points