

# Electroweak Radiative Corrections to Semileptonic $\tau$ Decays

Jens Erler

*Instituto de Física, Universidad Nacional Autónoma de México, 04510 México D.F., MÉXICO*

e-mail: erler@fisica.unam.mx

(November 2002)

I present an update on the electroweak radiative correction factor to semileptonic  $\tau$  decays, including a next-to-leading order resummation of large logarithms. My result differs both qualitatively and quantitatively from the one recently obtained by Davier *et al.*. As two consequences, (i) the discrepancy between the predictions for the muon  $g-2$  based on  $\tau$  decay data and  $e^+e^-$  annihilation data increases, and (ii) the  $g-2$  prediction based on  $\tau$  decay data appears to be consistent (within about one standard deviation) with the experimental result from BNL.

PACS numbers: 11.10.Hi, 12.15.Lk, 13.35.Dx, 13.40.Em.

The largest theoretical uncertainty in the Standard Model prediction of the anomalous magnetic moment of the muon,  $a_\mu = (g_\mu - 2)/2$ , arises from the hadronic two-loop vacuum polarization contribution,  $\Delta a_\mu^{\text{had},(2)}$ . This contribution is two orders of magnitude larger than the ultimate experimental error anticipated by the Muon  $g-2$  Collaboration at BNL [2], so it needs to be controlled at the 1% level or better. While non-perturbative QCD effects prevent a first principles calculation,  $\Delta a_\mu^{\text{had},(2)}$  can be rigorously obtained experimentally from a dispersion relation which relates it to an integral over  $e^+e^-$  annihilation cross sections. Using the conserved vector current (CVC) hypothesis one can obtain additional information by studying the invariant mass distribution of  $\tau$  decay hadronic final states. This necessitates a careful assessment of CVC breaking effects, which was done in a recent article by Davier *et al.* [1]. In this note, I present an update of the short distance electroweak radiative corrections to  $\tau$  decays, representing a particular CVC breaking effect. This update is motivated by two mistakes in one of the formulas of Ref. [1]. Numerically, the corresponding shifts are modest, but not negligible, and have the same sign.

The leading electroweak radiative corrections to  $\tau$  decays are enhanced by a large logarithm [3,4],

$$S_{\text{EW}} = 1 + \frac{3\alpha}{4\pi}(1 + 2\bar{Q}) \ln \frac{M_Z^2}{m_\tau^2} = 1.01878 \quad (1)$$

where  $M_Z = 91.1876(2)$  GeV [5] is the  $Z$  boson mass, and  $\alpha = \alpha(m_\tau) = 1/133.50(2)$  [6] is the QED coupling at the  $\tau$  lepton mass,  $m_\tau = 1776.99(3)$  MeV [7], evaluated in the  $\overline{\text{MS}}$  renormalization scheme.  $\bar{Q}$  is the hypercharge of the weak doublet produced in the final state. Therefore,  $\bar{Q} = 1/6$  for semileptonic decays,  $\tau^- \rightarrow \nu_\tau \bar{u}d(s)$ . Since

$\bar{Q} = -1/2$  for leptons, there are no large logarithms for leptonic  $\tau$  decays.

The remaining (not logarithmically enhanced) corrections at  $\mathcal{O}(\alpha)$  have been obtained in Ref. [8] (final state fermion masses are neglected throughout). In the notation of Eq. (17) of Ref. [1] they are,

$$S_{\text{EW}}^{\text{sub,had}} = 1 + \frac{\alpha(m_\tau)}{\pi} \left( \frac{85}{24} - \frac{\pi^2}{2} \right), \quad (2)$$

$$S_{\text{EW}}^{\text{sub,lep}} = 1 + \frac{\alpha(m_\tau)}{\pi} \left( \frac{25}{8} - \frac{\pi^2}{2} \right), \quad (3)$$

for semileptonic and leptonic decays, respectively. In Ref. [1], however,  $S_{\text{EW}}^{\text{sub,had}}$  was erroneously identified with the ratio,

$$\frac{S_{\text{EW}}^{\text{sub,had}}}{S_{\text{EW}}^{\text{sub,lep}}} = 1 + \frac{5}{12} \frac{\alpha(m_\tau)}{\pi} = 1.00099. \quad (4)$$

This amounts to a double counting of the correction  $S_{\text{EW}}^{\text{sub,lep}} - 1 = -0.00432$ : the hadronic spectral functions are normalized relative to the leptonic branching ratio (see Eq. (10) of Ref. [1]) so that the ratio (4) must be included, but it is incorrect to perform an additional division by  $S_{\text{EW}}^{\text{sub,lep}}$ .

Since numerically,

$$\frac{\alpha_s}{\pi} \ln \frac{M_Z^2}{m_\tau^2} \sim \mathcal{O}(1), \quad (5)$$

short distance QCD effects are of similar size as the  $\mathcal{O}(\alpha)$  corrections discussed in the previous paragraph. They have been computed in Ref. [9] and modify Eq. (1),

$$S_{\text{EW}} = 1 + \frac{3\alpha}{4\pi} \ln \frac{M_Z^2}{m_\tau^2} \left[ (1 + 2\bar{Q}) - 2\bar{Q} \frac{\alpha_s}{\pi} \right]. \quad (6)$$

The short distance QCD correction corresponding to the term proportional to the strong coupling constant,  $\alpha_s$ , has been approximated [9] in order to obtain an analytic result. I have checked that this approximation reproduces the exact  $\mathcal{O}(\alpha\alpha_s \ln M_Z^2)$  result within about 1%. Since two scales enter Eq. (6), it is clear that a next-to-leading order renormalization group analysis is in order.

Resummation of the leading order logarithms in Eq. (1) is done using the renormalization group equation (RGE) [10],

$$\left[ \mu^2 \frac{\partial}{\partial \mu^2} + \beta_0^{(1)} \frac{\alpha^2}{\pi} \frac{\partial}{\partial \alpha} - \frac{\alpha}{\pi} \right] S(\mu_0, \mu) = 0, \quad (7)$$

where  $\beta_0^{(1)}$  is the lowest order QED  $\beta$ -function coefficient. This RGE is subject to the initial condition,  $S(\mu_0, \mu_0) = 1$ , and its solution is given by,

$$S(\mu_0, \mu) = \left[ 1 - \frac{\alpha(\mu_0)}{\pi} \beta_0^{(1)} \ln \frac{\mu^2}{\mu_0^2} \right]^{-\frac{1}{\beta_0^{(1)}}}. \quad (8)$$

Applied to the case at hand, this is often written as,

$$\begin{aligned} S(m_\tau, M_Z) &= \left[ \frac{\alpha(m_b)}{\alpha(m_\tau)} \right]^{\frac{9}{19}} \left[ \frac{\alpha(M_W)}{\alpha(m_b)} \right]^{\frac{9}{20}} \left[ \frac{\alpha(M_Z)}{\alpha(M_W)} \right]^{\frac{36}{17}} \\ &= 1.01937, \end{aligned} \quad (9)$$

where the solution to the one-loop RGE of QED,

$$\mu^2 \frac{d}{d\mu^2} \alpha(\mu) = \beta_0^{(1)} \frac{\alpha^2(\mu)}{\pi}, \quad (10)$$

has been employed. It should be stressed, that consistency with the RGE demands one-loop evolution of  $\alpha(\mu)$  *within* each of the factors in Eq. (9). On the other hand, the values used *across* the various factors, may be related to each other either by one-loop evolution or including higher order running effects, since the difference is of higher order in the RGE (7). The increase of  $S(m_\tau, M_Z)$  in Eq. (9) relative to Eq. (1) due to the summation of  $\mathcal{O}(\alpha^n \ln^n M_Z^2)$  effects is about 3% of the non-resummed correction.

I will now extend the RGE analysis of the previous paragraph to properly sum up all logarithms of  $\mathcal{O}(\alpha_s^n \ln^n M_Z^2)$ . Eq. (7) is to be replaced by,

$$\left[ \mu^2 \frac{d}{d\mu^2} - \frac{\alpha}{\pi} \left( 1 - \frac{\alpha_s}{4\pi} \right) \right] S(\mu_0, \mu) = \quad (11)$$

$$\left[ \mu^2 \frac{\partial}{\partial \mu^2} + \beta_0^{(1)} \frac{\alpha^2}{\pi} \frac{\partial}{\partial \alpha} - \beta_0^{(3)} \frac{\alpha_s^2}{\pi} \frac{\partial}{\partial \alpha_s} - \frac{\alpha}{\pi} \left( 1 - \frac{\alpha_s}{4\pi} \right) \right] S = 0,$$

where  $\beta_0^{(3)}$  is the lowest order QCD  $\beta$ -function coefficient. With the definitions,

$$\begin{aligned} \eta_\tau &= \frac{\alpha_s(m_\tau)}{4\pi} \left[ 1 + \frac{75}{76} \frac{\alpha_s(m_\tau)}{\alpha(m_\tau)} \right]^{-1}, \\ \eta_b &= \frac{\alpha_s(m_b)}{4\pi} \left[ 1 + \frac{69}{80} \frac{\alpha_s(m_b)}{\alpha(m_b)} \right]^{-1}, \\ \eta_W &= \frac{\alpha_s(M_W)}{4\pi} \left[ 1 + \frac{69}{17} \frac{\alpha_s(M_W)}{\alpha(M_W)} \right]^{-1}, \end{aligned} \quad (12)$$

Eq. (11) is solved by,

$$\begin{aligned} S(m_\tau, M_Z) &= \left[ \frac{\alpha(m_b)}{\alpha(m_\tau)} \right]^{\frac{9}{19}(1-\eta_\tau)} \left[ \frac{\alpha_s(m_b)}{\alpha_s(m_\tau)} \right]^{\frac{9}{19}\eta_\tau} \\ &\quad \left[ \frac{\alpha(M_W)}{\alpha(m_b)} \right]^{\frac{9}{20}(1-\eta_b)} \left[ \frac{\alpha_s(M_W)}{\alpha_s(m_b)} \right]^{\frac{9}{20}\eta_b} \\ &\quad \left[ \frac{\alpha(M_Z)}{\alpha(M_W)} \right]^{\frac{36}{17}(1-\eta_W)} \left[ \frac{\alpha_s(M_Z)}{\alpha_s(M_W)} \right]^{\frac{36}{17}\eta_W} \\ &= 1.01907 \pm 0.00001, \end{aligned} \quad (13)$$

where I used the solutions to the one-loop RGE of QCD,

$$\mu^2 \frac{d}{d\mu^2} \alpha_s(\mu) = -\beta_0^{(3)} \frac{\alpha_s^2(\mu)}{\pi}, \quad (14)$$

and of QED<sup>1</sup>. The shift,  $-0.00030$ , between Eqs. (9) and (13) is somewhat larger than the shift,  $-0.00022$ , obtained in Ref. [4], which is in part due to the summation, but mainly due to the inputs. The uncertainty in Eq. (13) is from the current uncertainty in  $\alpha_s = 0.120 \pm 0.002$ , while other parametric uncertainties are miniscule<sup>2</sup>. Neglecting two-loop  $\mathcal{O}(\alpha^{n+1} \ln^n M_Z^2)$  effects, Eq. (13) simplifies,

$$\begin{aligned} S(m_\tau, M_Z) &= \left[ \frac{\alpha(m_b)}{\alpha(m_\tau)} \right]^{\frac{9}{19}} \left[ \frac{\alpha(M_W)}{\alpha(m_b)} \right]^{\frac{9}{20}} \left[ \frac{\alpha(M_Z)}{\alpha(M_W)} \right]^{\frac{36}{17}} \\ &\quad \left[ \frac{\alpha_s(m_b)}{\alpha_s(m_\tau)} \right]^{\frac{3}{25} \frac{\alpha(m_\tau)}{\pi}} \left[ \frac{\alpha_s(M_Z)}{\alpha_s(m_b)} \right]^{\frac{3}{23} \frac{\alpha(m_b)}{\pi}}, \end{aligned} \quad (15)$$

which differs by only  $\approx 3 \times 10^{-6}$ . Neglecting further the numerically similar three-loop  $\mathcal{O}(\alpha_s \alpha^n \ln^n M_Z^2)$  effects, one can expand the second line in Eq. (15) to linear order in  $\alpha$ . If one then rewrites the resulting expression in terms of QCD scale parameters,  $\Lambda_{\text{QCD}}$ , one encounters the double-logarithmic form originally obtained 20 years ago [9].

To summarize, next-to-leading order effects *reduce* the leading order summation by about 50%, *i.e.*, they are numerically of the same order. Both effects are in turn numerically of order  $\alpha$ , so they must be included for a complete  $\mathcal{O}(\alpha)$  evaluation. Unknown higher orders are suppressed by at least a factor of  $\alpha_s/\pi$  relative to any of the effects mentioned before. Thus, the uncertainty due to higher order effects is of order  $\mathcal{O}(\alpha\alpha_s) \sim 0.0003$ .

Following Ref. [1] one can write,

$$S_{\text{EW}} \equiv S(m_\tau, M_Z) \frac{S_{\text{EW}}^{\text{sub, had}}}{S_{\text{EW}}^{\text{sub, lep}}} = 1.0201 \pm 0.0003, \quad (16)$$

but there  $S_{\text{EW}} = 1.0267 \pm 0.0027$  is quoted instead. Almost 2/3 of the difference is due to the error pointed out after Eq. (4), and about 5% due to neglecting next-to-leading order contributions to  $S(m_\tau, M_Z)$ . Another 15% difference is likely due to applying Eq. (9) incorrectly (as discussed above). The remaining 15% can perhaps be

<sup>1</sup>QCD corrections to Eq. (10) are suppressed by an additional factor  $\alpha_s/\pi$ . Their inclusion gives rise to the summation of  $\mathcal{O}(\alpha_s \alpha^n \ln^n M_Z^2)$  effects, but the integration cannot be performed analytically. Numerically this summation affects the result at the  $10^{-5}$  level which can safely be neglected.

<sup>2</sup>What enters Eqs. (9) and (13) is the  $\overline{\text{MS}}$   $b$ -quark definition, which is free of renormalon ambiguities and therefore much better known (see Ref. [11] for a recent sub-percent determination) than the  $b$ -quark pole mass. Higher order matching corrections are also smaller if one uses the  $\overline{\text{MS}}$  mass definition.

traced to use of the on-shell definition of  $\alpha$  in place of the  $\overline{\text{MS}}$  definition as used in the present work. Note, that the derivation of Eq. (9) assumes a *mass-independent* renormalization scheme (such as the  $\overline{\text{MS}}$  scheme), in which at each fermion threshold the  $\beta$ -function coefficients change by a finite amount: this is the origin of the product form of Eq. (9). Thus, the solution (9) cannot be applied to *mass-dependent* schemes, such as the on-shell renormalization scheme. It is emphasized again, that the numerical difference to Ref. [1] should not be viewed as a scheme-dependance and thus as an estimate of uncalculated higher order corrections (which are much smaller as discussed above). On the contrary, one should expect that a self-consistent treatment within the on-shell scheme will reproduce the result of the present work.

As far as the  $\tau$ -based analysis of Ref. [1] is concerned, about 77% of the data is affected by  $S_{\text{EW}}$ . Since  $S_{\text{EW}}$  obtained in this paper differs by about 0.65% from the one in Ref. [1], one expects a 0.5% shift in the extracted  $\Delta a_{\mu}^{\text{had},(2)}$ . Including an update of the CKM matrix element  $|V_{ud}|$  entering the analysis (the value,  $|V_{ud}| = 0.9752 \pm 0.0007$ , is replaced by,  $|V_{ud}| = 0.97485 \pm 0.00046$ , from the fit result of Ref. [7]), this amounts to about one half of the current experimental uncertainty of 0.8 parts per billion [2] for the muon magnetic moment. The  $\tau$ -based Standard Model prediction would then be consistent with the measurement [2] within about one standard deviation. The discrepancy to the  $e^+e^-$  based analysis of Ref. [1] would correspondingly be larger. Furthermore, the smaller errors in Eq. (16) and in  $|V_{ud}|$  compared to Ref. [1] should lead to a slight reduction of the overall uncertainty of the  $\tau$ -based result. As a final remark, the recent determination [11] of  $\alpha_s$  from the  $\tau$  lifetime, when updated with the present next-to-leading order analysis, increases  $\alpha_s(M_Z)$  by less than 0.0001.

#### Acknowledgements:

It is a pleasure to thank Eric Braaten and Michael Ramsey-Musolf for their comments.

Electroweak Working Group and SLD Heavy Flavor and Electroweak Groups: D. Abbaneo *et al.*, *A Combination of Preliminary Electroweak Measurements and Constraints on the Standard Model*, e-print [hep-ex/0112021](#).

- [6] J. Erler, *Phys. Rev.* **D59**, 054008 (1999).
- [7] Particle Data Group: K. Hagiwara *et al.*, *Phys. Rev.* **D66**, 010001 (2002).
- [8] E. Braaten and C.S. Li, *Phys. Rev.* **D42**, 3888 (1990).
- [9] A. Sirlin, *Nucl. Phys.* **B196**, 83 (1982).
- [10] W.J. Marciano and A. Sirlin, *Phys. Rev. Lett.* **56**, 22 (1986).
- [11] J. Erler and M. Luo, *Precision Determination of Heavy Quark Masses and the Strong Coupling Constant*, e-print [hep-ph/0207114](#).

- 
- [1] M. Davier, S. Eidelman, A. Höcker and Z. Zhang, *Confronting Spectral Functions from  $e^+e^-$  Annihilation and  $\tau$  Decays: Consequences for the Muon Magnetic Moment*, e-print [hep-ph/0208177](#).
  - [2] Muon g-2 Collaboration: G.W. Bennett *et al.*, *Phys. Rev. Lett.* **89**, 101804 (2002).
  - [3] A. Sirlin, *Rev. Mod. Phys.* **50**, 573 (1978).
  - [4] W.J. Marciano and A. Sirlin, *Phys. Rev. Lett.* **61**, 1815 (1988).
  - [5] ALEPH, DELPHI, L3, and OPAL Collaborations, LEP