HOW DO WE COMPUTE FEYNMAN INTEGRALS?

ANTONELA MATIJAŠIĆ

IMPRS Young Scientist Workshop
Ringberg Castle

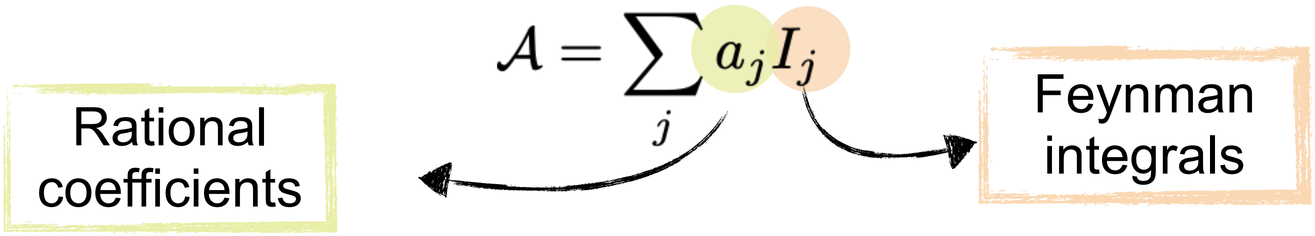
November 23th, 2023

MAX-PLANCK-INSTITUT
FÜR PHYSIK

European Research Council
Established by the European Commission

WHY DO WE WANT TO COMPUTE FEYNMAN INTEGRALS?

- Scattering amplitudes are a link between theory and experiment



$$\mathcal{A} = \sum_j a_j I_j$$

Rational coefficients

Feynman integrals

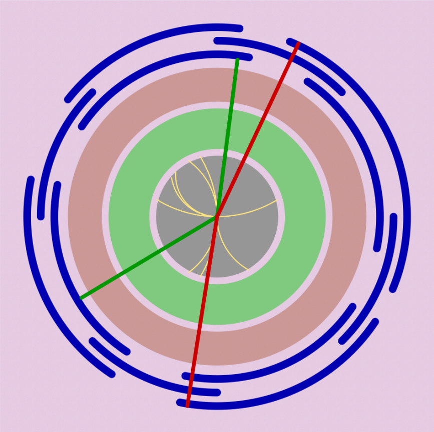
WHY DO WE WANT TO COMPUTE FEYNMAN INTEGRALS?

- Scattering amplitudes are a link between theory and experiment

$$\mathcal{A} = \sum_j a_j I_j$$

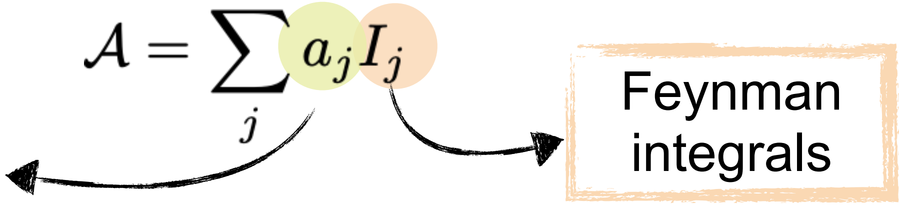
Rational coefficients

Feynman integrals



WHY DO WE WANT TO COMPUTE FEYNMAN INTEGRALS?

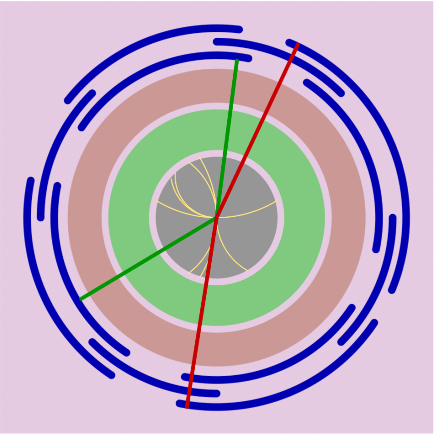
- Scattering amplitudes are a link between theory and experiment

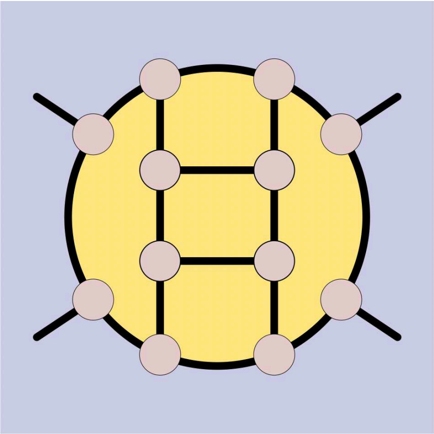


$$\mathcal{A} = \sum_j a_j I_j$$

Rational coefficients

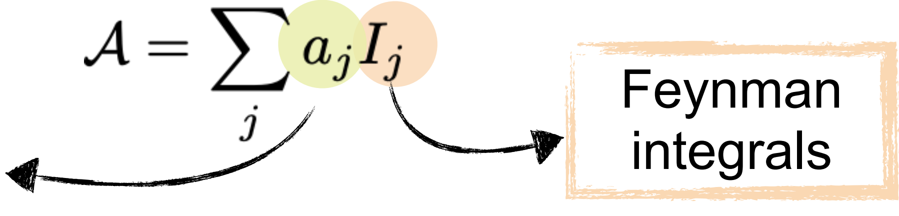
Feynman integrals





WHY DO WE WANT TO COMPUTE FEYNMAN INTEGRALS?

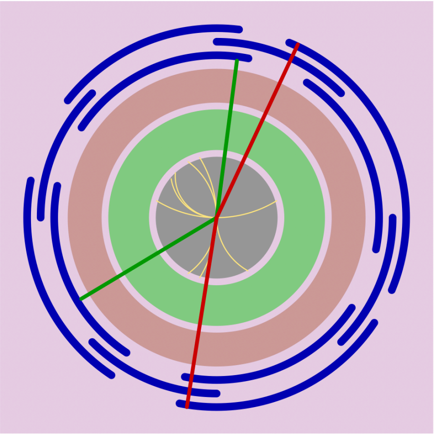
- Scattering amplitudes are a link between theory and experiment

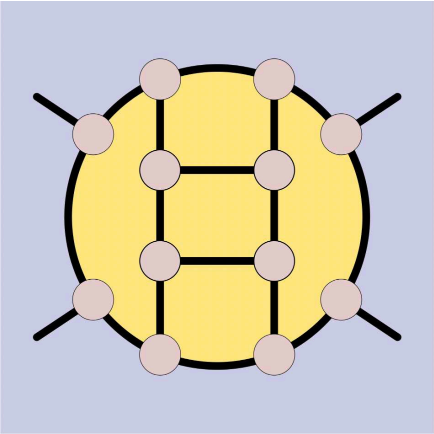


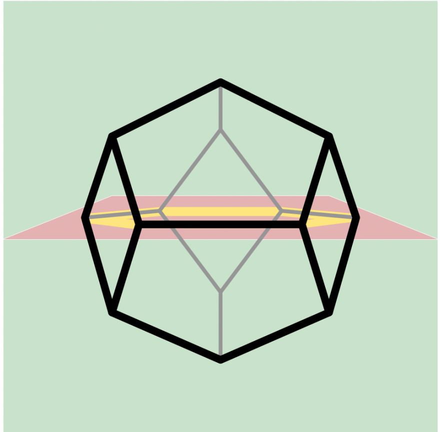
$$\mathcal{A} = \sum_j a_j I_j$$

Rational coefficients

Feynman integrals



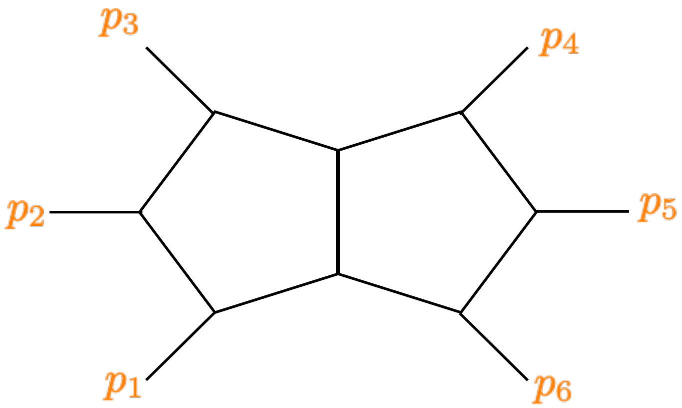




WHAT ARE FEYNMAN INTEGRALS?

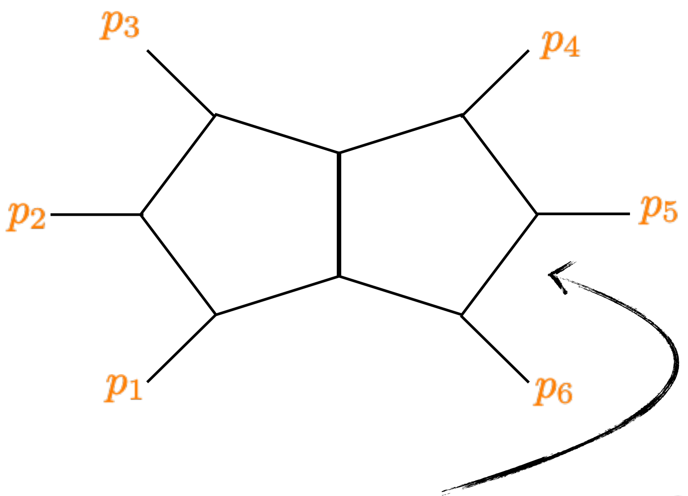
WHAT ARE FEYNMAN INTEGRALS?

Loop Feynman diagram



WHAT ARE FEYNMAN INTEGRALS?

Loop Feynman diagram + Feynman rules

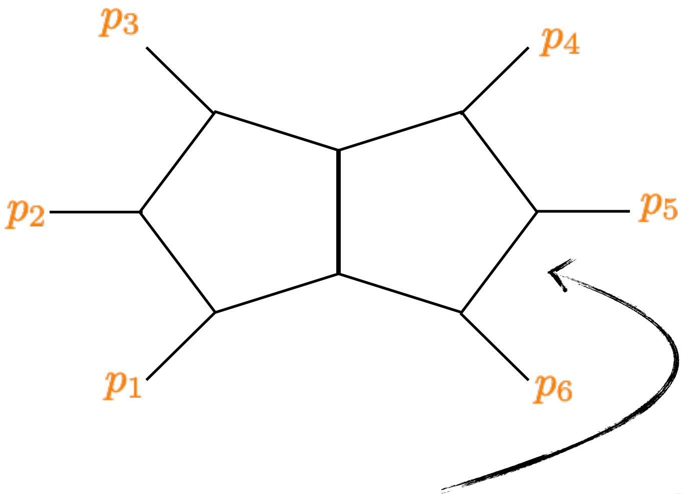


$$D_{12} = -(l_2 + p_1 + p_2 + p_3 + p_4 + p_5)^2$$

- Internal lines get propagators $1/D_i$
- Use momentum conservation at each vertex
- Integrate over loop momenta l_i

WHAT ARE FEYNMAN INTEGRALS?

Loop Feynman diagram + Feynman rules = Feynman integrals



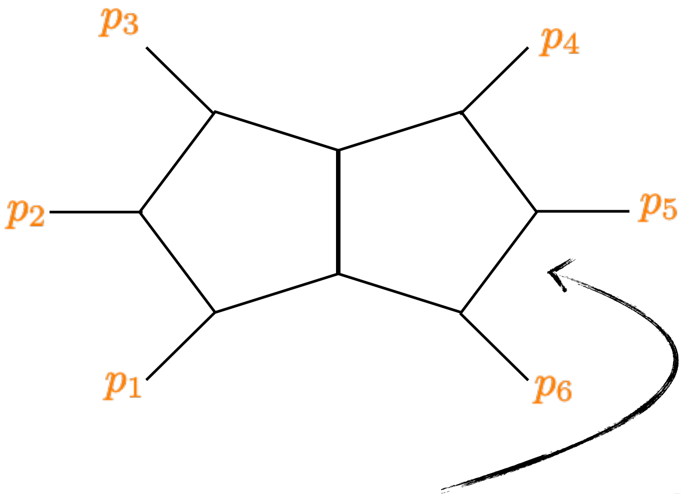
$$D_{12} = -(l_2 + p_1 + p_2 + p_3 + p_4 + p_5)^2$$

- Internal lines get propagators $1/D_i$
- Use momentum conservation at each vertex
- Integrate over loop momenta l_i

$$I^{(d_0)}(\vec{v}; d_0) = e^{2\epsilon\gamma_E} \int \frac{d^{d_0-2\epsilon} l_1 d^{d_0-2\epsilon} l_2}{i\pi^{(d_0-2\epsilon)}} \frac{1}{D_1 \ldots D_{13}}$$

WHAT ARE FEYNMAN INTEGRALS?

Loop Feynman diagram + Feynman rules = Feynman integrals



$D_{12} = -(l_2 + p_1 + p_2 + p_3 + p_4 + p_5)^2$

- Internal lines get propagators $1/D_i$
- Use momentum conservation at each vertex
- Integrate over loop momenta l_i

$$I^{(d_0)}(\vec{v}; d_0) = e^{2\epsilon\gamma_E} \int \frac{d^{d_0-2\epsilon} l_1 d^{d_0-2\epsilon} l_2}{i\pi^{(d_0-2\epsilon)}} \frac{1}{D_1 \dots D_{13}}$$

- Mandelstam invariants

$$\vec{v} = \{s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{61}, s_{123}, s_{234}, s_{345}\} \quad s_{ij} = (p_i + p_j)^2, \qquad s_{ijk} = (p_i + p_j + p_k)^2$$

WHAT APPROACHES CAN WE USE?

- The ultimate goal is to compare the result with the experiment ⇒ we need numbers

WHAT APPROACHES CAN WE USE?

- The ultimate goal is to compare the result with the experiment ⇒ we need numbers

Numerical computation

- Sector decomposition [pySecDec]
- Auxiliary mass flow [AMFlow]
- Monte Carlo methods [feyntrop]

WHAT APPROACHES CAN WE USE?

- The ultimate goal is to compare the result with the experiment ⇒ we need numbers

Numerical computation

- Sector decomposition [pySecDec]
- Auxiliary mass flow [AMFlow]
- Monte Carlo methods [feyntrop]

Analytic computation

- Direct integration
- Bootstrap methods
- Differential equation method

WHAT APPROACHES CAN WE USE?

- The ultimate goal is to compare the result with the experiment ⇒ we need numbers

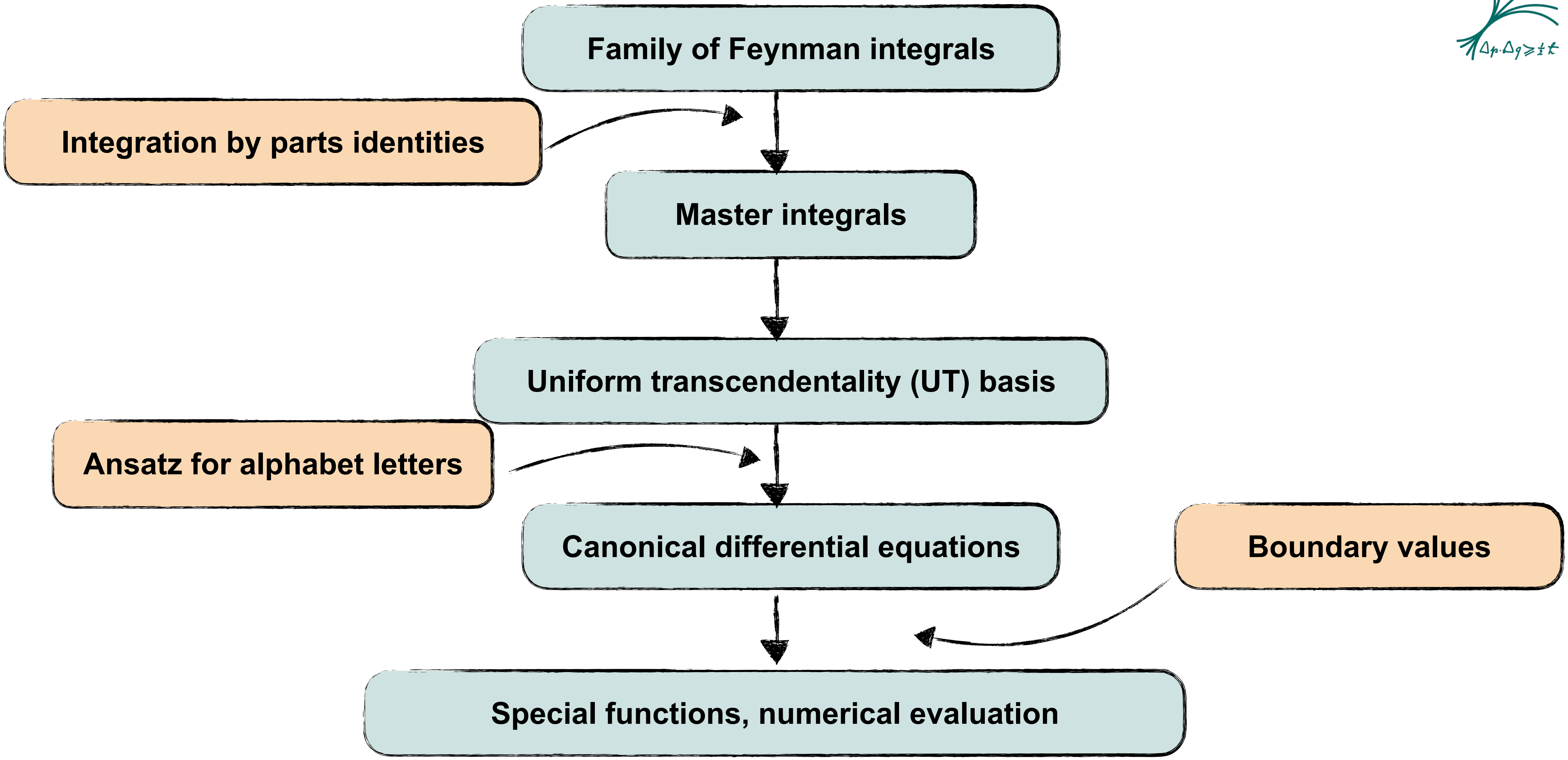
Numerical computation

- Sector decomposition [pySecDec]
- Auxiliary mass flow [AMFlow]
- Monte Carlo methods [feyntrop]

Analytic computation

- Direct integration
- Bootstrap methods
- Differential equation method

Method of choice



Family of Feynman integrals

Integration by parts identities

Master integrals

Uniform transcendentality (UT) basis

Ansatz for alphabet letters

Canonical differential equations

Boundary values

Special functions, numerical evaluation

Family of Feynman integrals

Master integrals

Uniform transcendentality (UT) basis

Canonical differential equations

Special functions, numerical evaluation

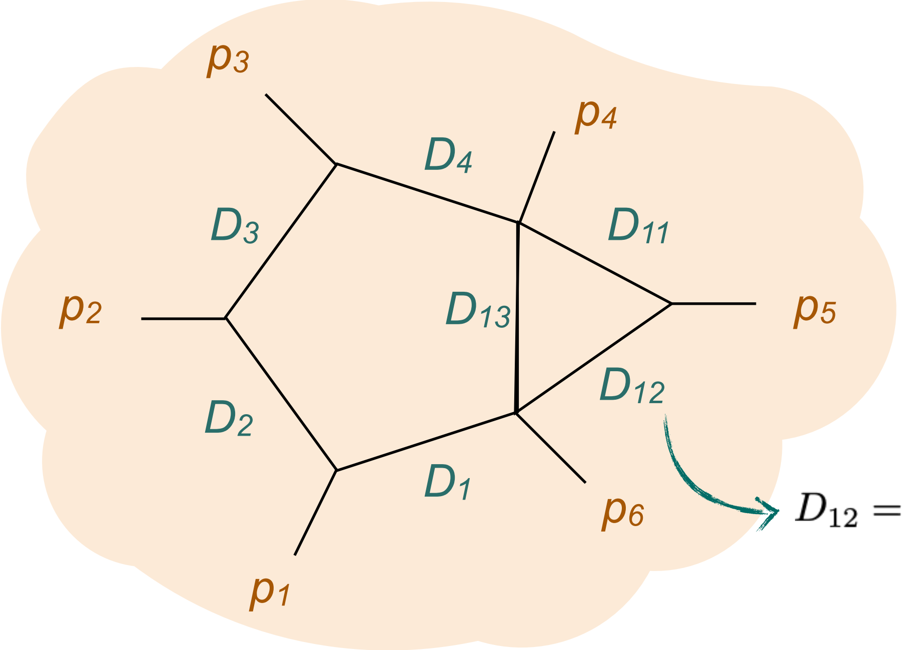
Family of Feynman integrals

Master integrals

Uniform transcendentality (UT) basis

Canonical differential equations

Special functions, numerical evaluation



$$D_{12} = -(l_2 + p_1 + p_2 + p_3 + p_4 + p_5)^2$$

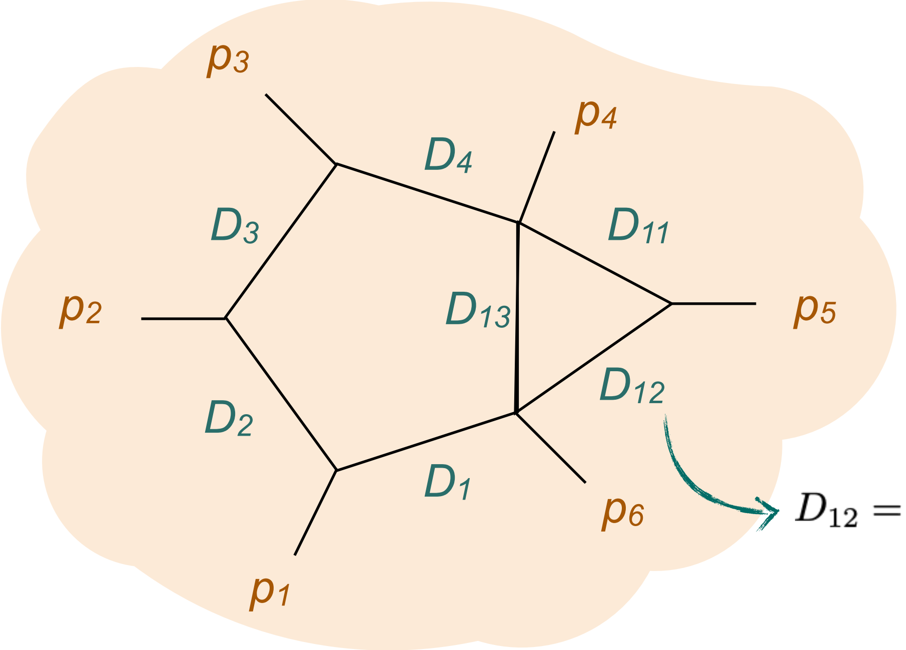
Family of Feynman integrals

Master integrals

Uniform transcendentality (UT) basis

Canonical differential equations

Special functions, numerical evaluation



$$D_{12} = -(l_2 + p_1 + p_2 + p_3 + p_4 + p_5)^2$$

$$I^{(d_0)}(a_1, \ldots, a_{13}) = e^{2\epsilon\gamma_E} \int \frac{d^{d_0-2\epsilon} l_1 d^{d_0-2\epsilon} l_2}{i\pi^{(d_0-2\epsilon)}} \frac{1}{D_1^{a_1} \ldots D_{13}^{a_{13}}}$$

$$a_1, \ldots, a_{13} \in \mathbb{Z} \qquad p_i^2 = 0, \quad i = 1, \ldots, 6$$

$$\sum_{i=1}^{6} p_i = 0 \qquad\qquad p_i \in \mathbb{R}^{D_{\text{ext}}}$$

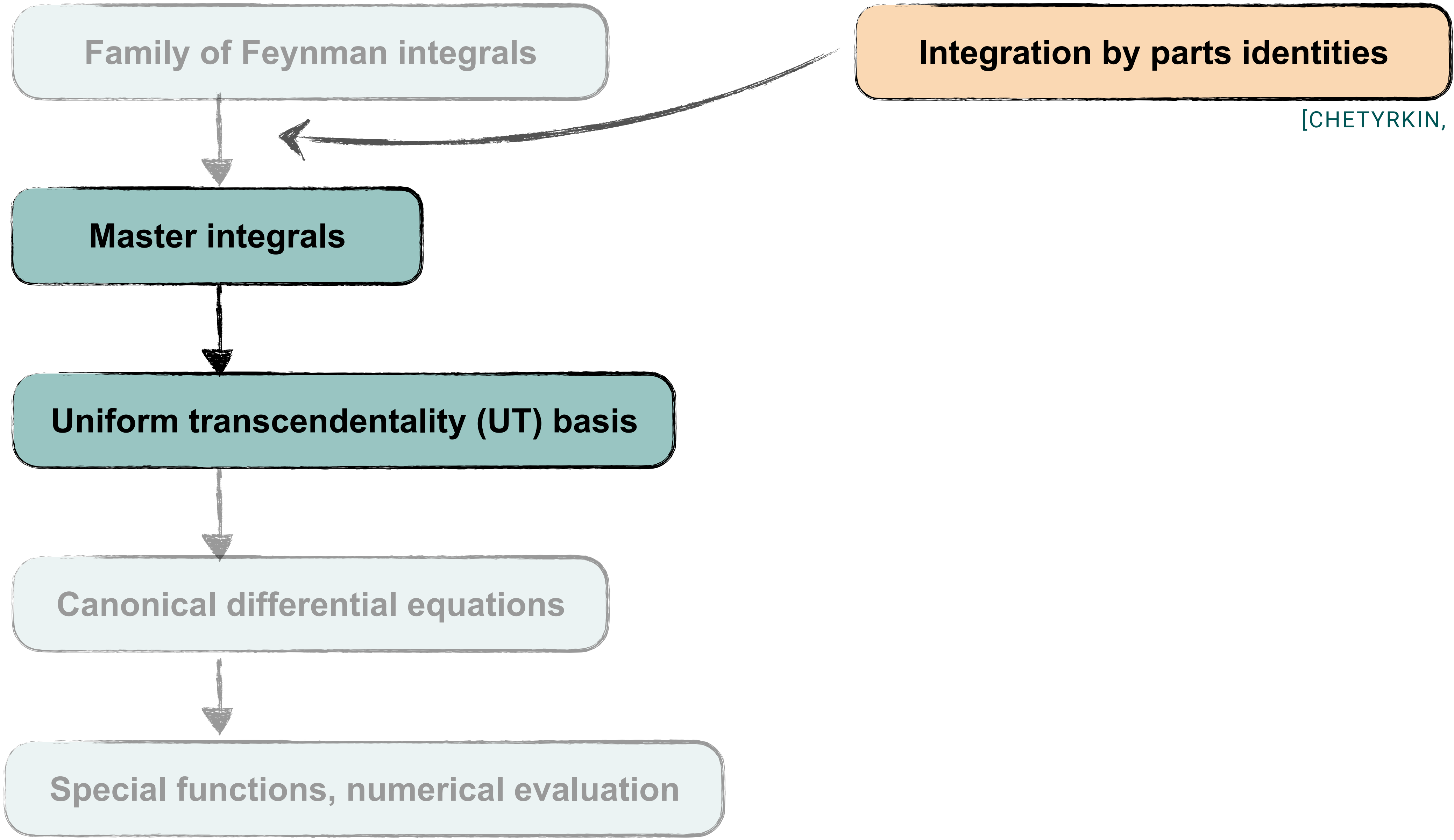
Family of Feynman integrals

Master integrals

Uniform transcendentality (UT) basis

Canonical differential equations

Special functions, numerical evaluation



Family of Feynman integrals

Integration by parts identities

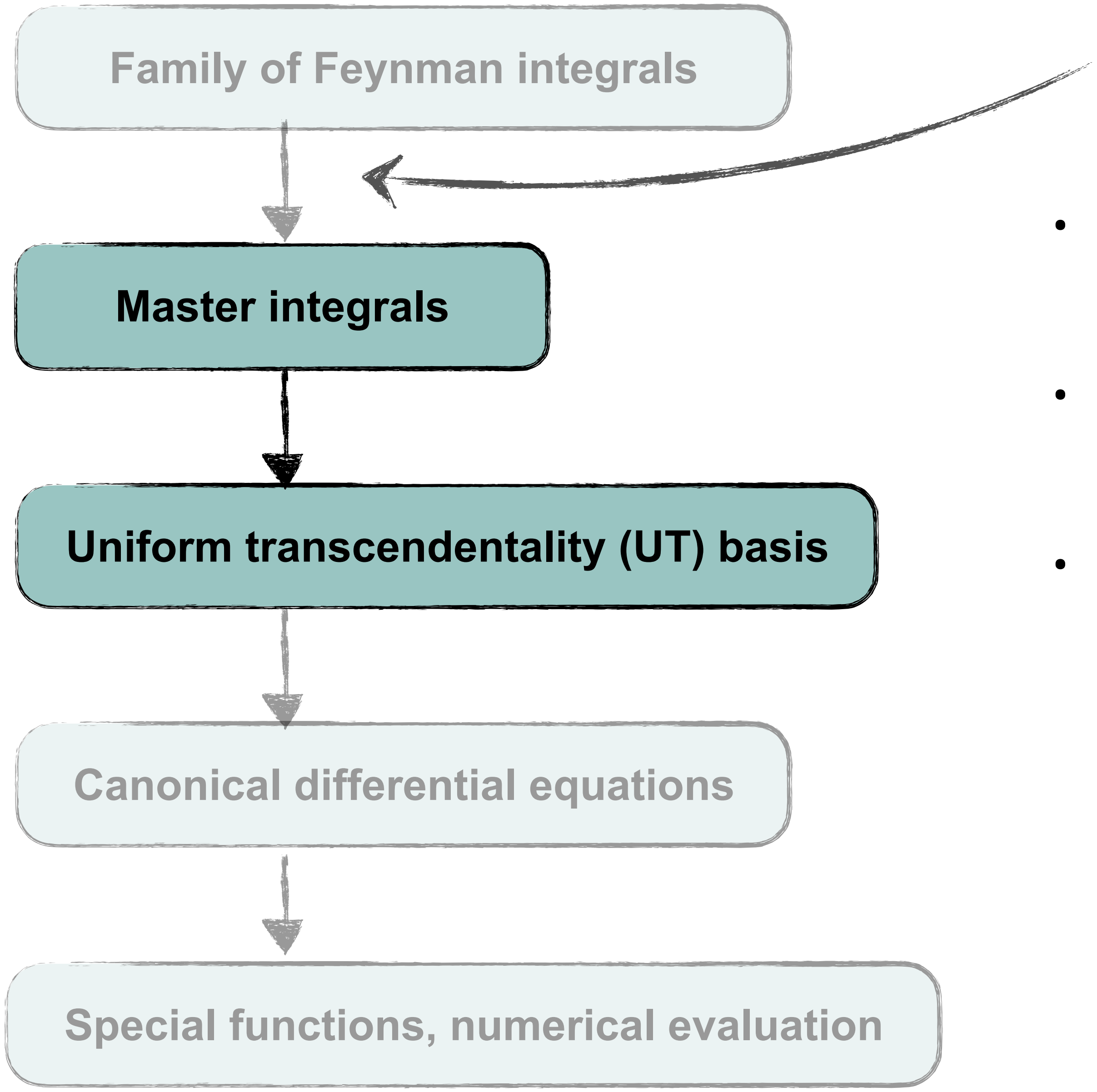
[CHETYRKIN, TKACHOV '81]

Master integrals

Uniform transcendentality (UT) basis

Canonical differential equations

Special functions, numerical evaluation



Family of Feynman integrals

Master integrals

Uniform transcendentality (UT) basis

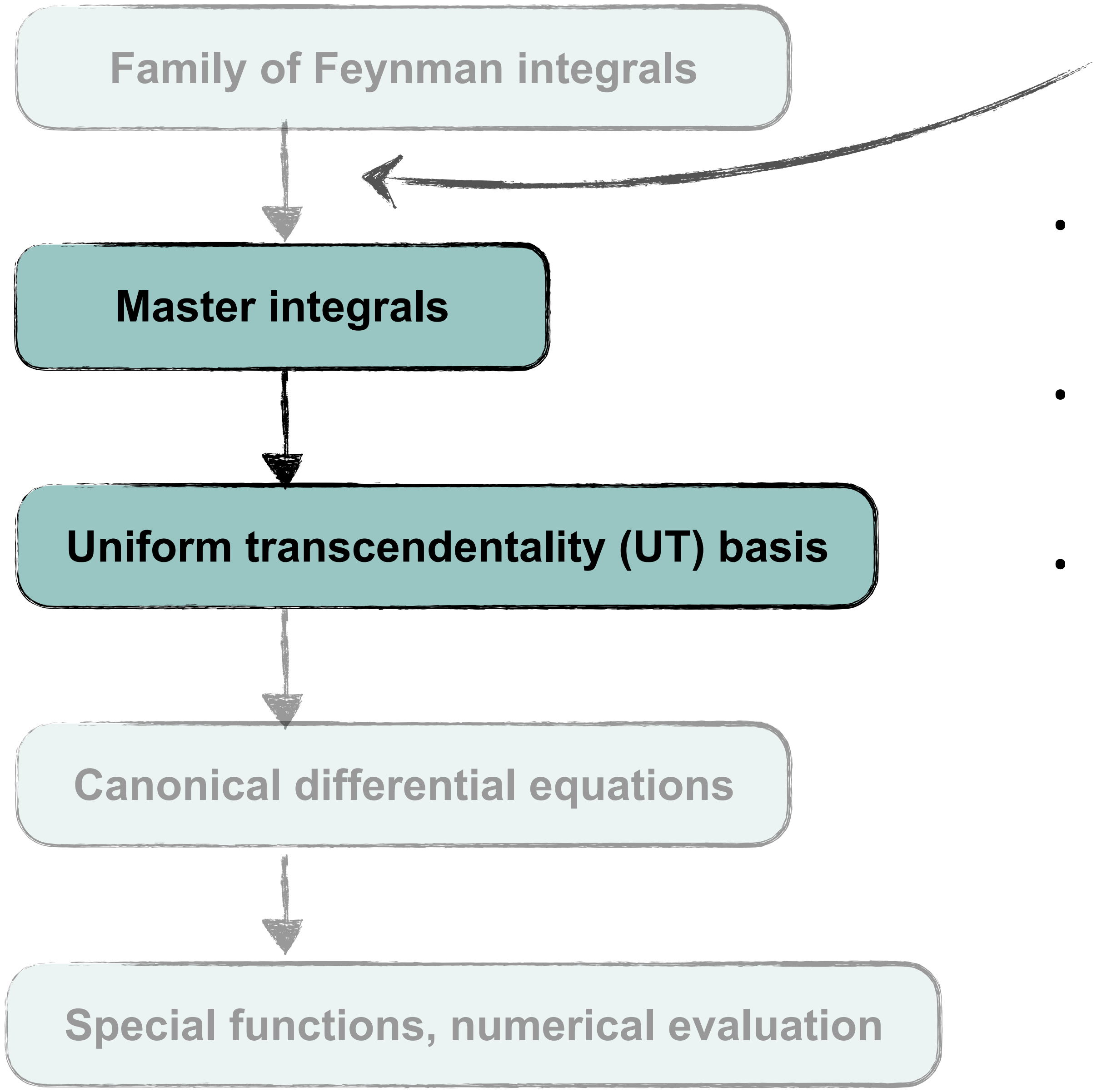
Canonical differential equations

Special functions, numerical evaluation

Integration by parts identities

[CHETYRKIN, TKACHOV ’81]

- Vanishing of total derivatives in dimensional regularization
- Relations between integrals with different powers of propagators
- A finite number of independent integrals → integral basis [Smirnov, Petukhov ‘11]



Family of Feynman integrals

Master integrals

Uniform transcendentality (UT) basis

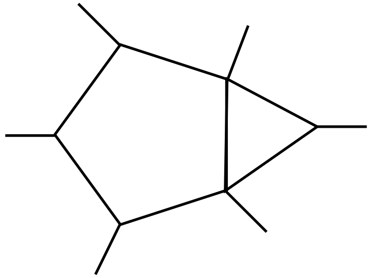
Canonical differential equations

Special functions, numerical evaluation

Integration by parts identities

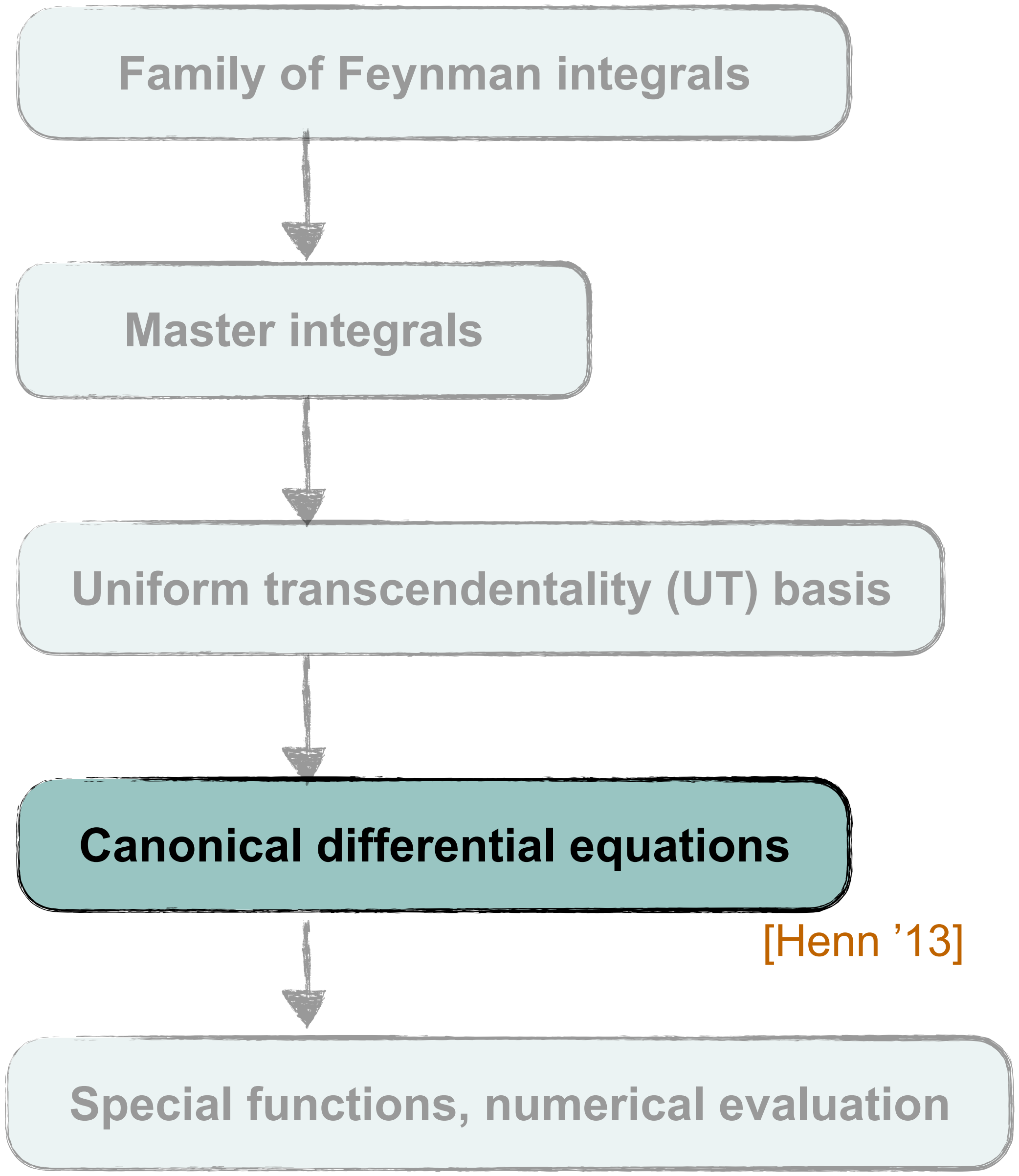
[CHETYRKIN, TKACHOV '81]

- Vanishing of total derivatives in dimensional regularization
- Relations between integrals with different powers of propagators
- A finite number of independent integrals → integral basis [Smirnov, Petukhov '11]



⇒ 45 master integrals

[FiniteFlow, LiteRed]



Family of Feynman integrals

Master integrals

Uniform transcendentality (UT) basis

Canonical differential equations

[Henn '13]

Special functions, numerical evaluation

- By choosing a UT basis, we find canonical differential equations

$$\mathrm{d}\tilde{I} = \epsilon \left[\sum_a c^a_{jk}\,\mathrm{d}\log(W_a)\right]\tilde{I}$$

vector of *N* basis integrals

constant $N \times N$ matrices

$W_a \in \mathbb{A}$

algebraic functions of kinematics

- The alphabet $\mathbb{A}$ encodes the singularity structure and characterizes types of functions that can appear

Family of Feynman integrals

Master integrals

Uniform transcendentality (UT) basis

Canonical differential equations

Special functions, numerical evaluation

Family of Feynman integrals

Master integrals

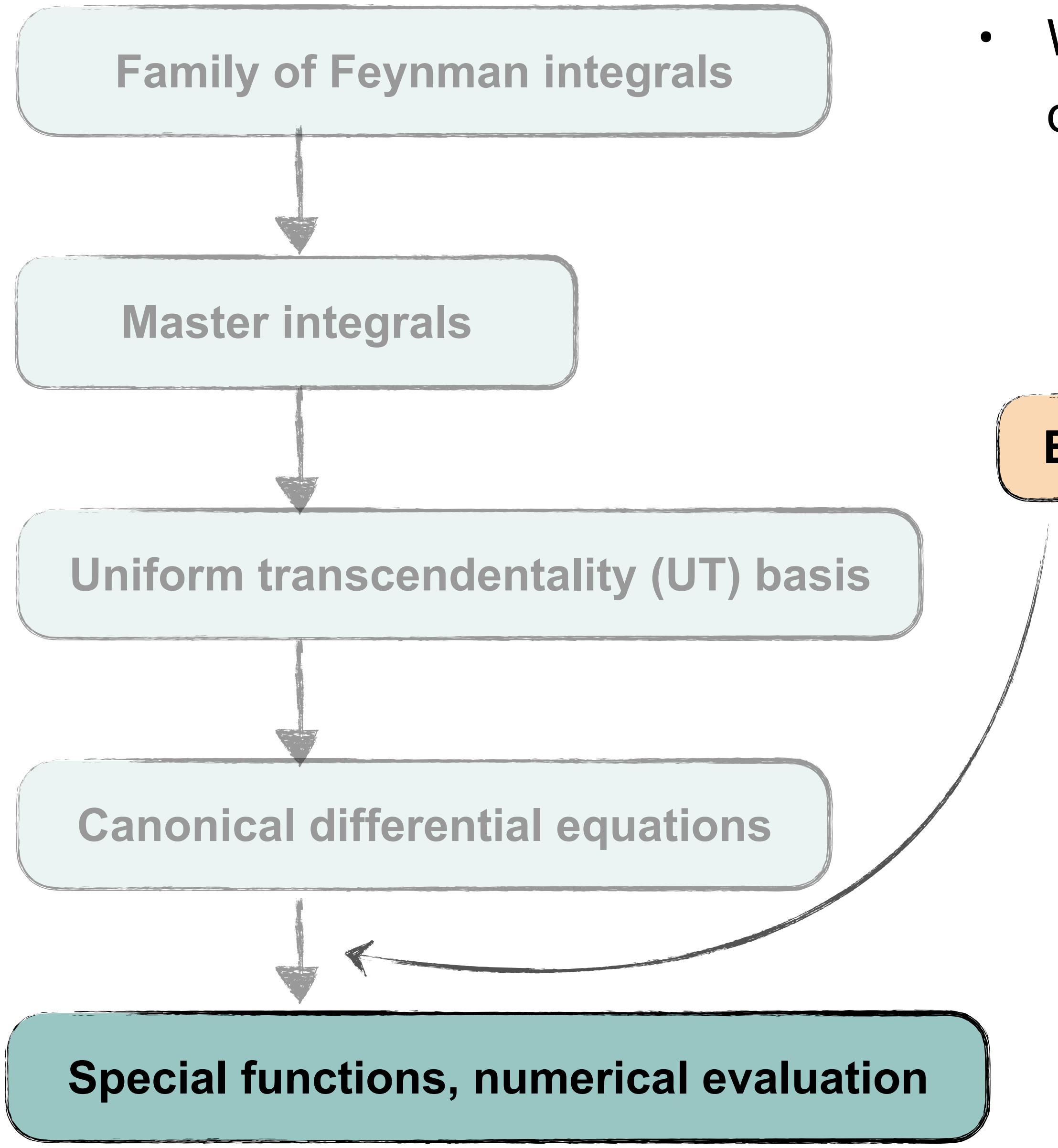
Uniform transcendentality (UT) basis

Canonical differential equations

Special functions, numerical evaluation

- We can solve the canonical differential equation order by order in ϵ

$$\tilde{I}^{(k)}(\vec{v}) = \sum_{k'=0}^{k} \sum_{i_1,\ldots,i_{k'} \in \mathbb{A}} a^{(i_1)} \ldots a^{(i_{k'})} \vec{b}^{(k)} \left[W_{i_1}, \ldots, W_{i_{k'}}\right]_{\vec{v}_0} (\vec{v})$$



Family of Feynman integrals

Master integrals

Uniform transcendentality (UT) basis

Canonical differential equations

Special functions, numerical evaluation

- We can solve the canonical differential equation order by order in ϵ

$$\tilde{I}^{(k)}(\vec{v}) = \sum_{k'=0}^{k} \sum_{i_1,\dots,i_{k'}\in\mathbb{A}} a^{(i_1)}\dots a^{(i_{k'})}\vec{b}^{(k)} \left[W_{i_1},\dots,W_{i_{k'}}\right]_{\vec{v}_0}(\vec{v})$$

Boundary values

Chen iterated integrals

$$\left[W_{i_1},\dots,W_{i_k}\right]_{\vec{v}_0}(\vec{v}) = \int_\gamma \mathrm{d}\log W_k(\vec{v}')\left[W_{i_1},\dots,W_{i_{k-1}}\right]_{\vec{v}_0}$$

Family of Feynman integrals

Master integrals

Uniform transcendentality (UT) basis

Canonical differential equations

Special functions, numerical evaluation

- We can solve the canonical differential equation order by order in ϵ

$$\tilde{I}^{(k)}(\vec{v}) = \sum_{k'=0}^{k} \sum_{i_1,\ldots,i_{k'} \in \mathbb{A}} a^{(i_1)} \ldots a^{(i_{k'})} \vec{b}^{(k)} \left[W_{i_1}, \ldots, W_{i_{k'}}\right]_{\vec{v}_0} (\vec{v})$$

Boundary values

Chen iterated integrals

$$\left[W_{i_1}, \ldots, W_{i_k}\right]_{\vec{v}_0} (\vec{v}) = \int_\gamma \mathrm{d}\log W_k(\vec{v}') \left[W_{i_1}, \ldots, W_{i_{k-1}}\right]_{\vec{v}_0}$$

$$\begin{aligned} I_{27} = {} & \frac{1}{2} - \epsilon \log(v_7) + \epsilon^2 \left[-\frac{1}{2}\zeta_2 + \log^2(v_7^2) + \mathrm{Li}_2\left(1 - \frac{v_5}{v_7}\right)\right] \\ & + \epsilon^3 \left[\frac{16}{3}\zeta_3 + \int_0^1 \mathrm{d}t\partial_t \log(W_7(t))\mathrm{Li}_2\left(1 - \frac{v_5}{v_7}\right) + \ldots\right] \\ & + \epsilon^4 \left[\frac{57}{20}\zeta_2^2 + f^{(4)}\right] + \mathcal{O}(\epsilon^5) \end{aligned}$$

Family of Feynman integrals

Master integrals

Uniform transcendentality (UT) basis

Canonical differential equations

Special functions, numerical evaluation

Family of Feynman integrals

Master integrals

Uniform transcendentality (UT) basis

Canonical differential equations

Special functions, numerical evaluation