The talk was given on the 24th of November at the Young Scientists Workshop 2023, in Ringberg Castle.

Can instantons induce a cosmological bounce?

Finite volume effects in the early universe

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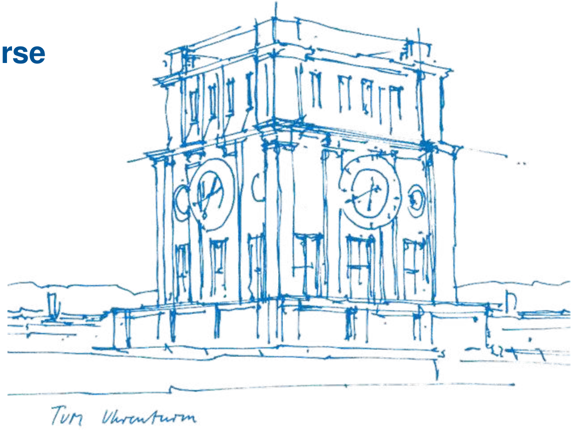
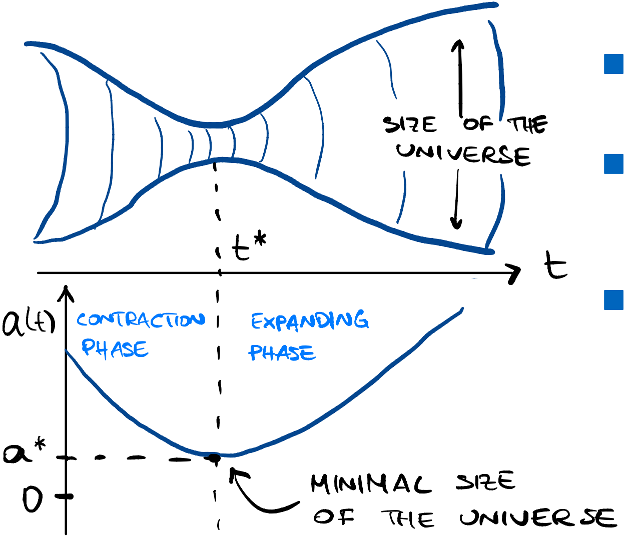
IMPRS Young Scientist Workshop 2023



Outline

1. Bouncing cosmology
   - What is a cosmological bounce?
   - What do we need for a cosmological bounce?
2. Introducing the model
3. Finite volume effects
4. Conclusions and outlook

What is a cosmological bounce?



- bouncing cosmology is an alternative to inflationary cosmology
- idea: the universe begins in a contracting phase, reaches a minimum size, then bounces back and starts expanding
- attempts to address some problems related to inflations, like the Big Bang singularity or the fine tuning needed for "good inflation"

What do we need for a cosmological bounce?

- metric for a flat FLRW universe:

$$ds^2 = -dt^2 + a(t)^2 d\vec{r}^2 \tag{1.1}$$

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- in the vicinity of the bounce one needs

$$\rho + p < 0 \tag{1.4}$$

$\Longrightarrow$ **the Null Energy Condition (NEC) must be violated!**

Outline

The scalar potential

Idea: can we violate the NEC via instantons? [2]

- consider a single real scalar field

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - U(\phi) - \frac{1}{2}\xi R\phi^2 - \frac{\Lambda}{8\pi G}$$

- symmetric double-well potential

$$U(\phi) = \frac{\lambda}{4!}\left(\phi^2 - v^2\right)^2 \qquad (2.1)$$

- in finite volume V there is **no spontaneous symmetry breaking**

$$\Rightarrow \langle\phi\rangle = 0$$

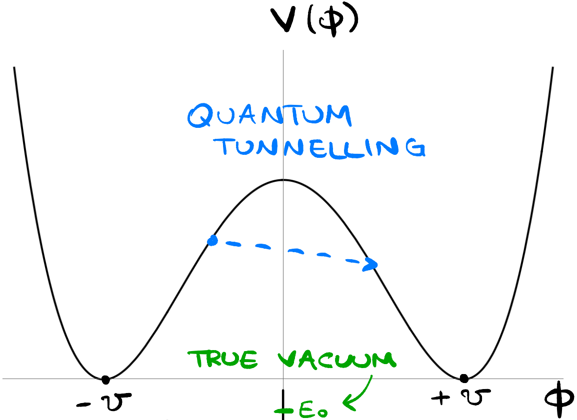

Figure 1 Symmetric double well potential

Our assumptions

A the metric is changing slowly

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 \ll m^2 = \frac{\lambda v^2}{3} \qquad (2.2)$$

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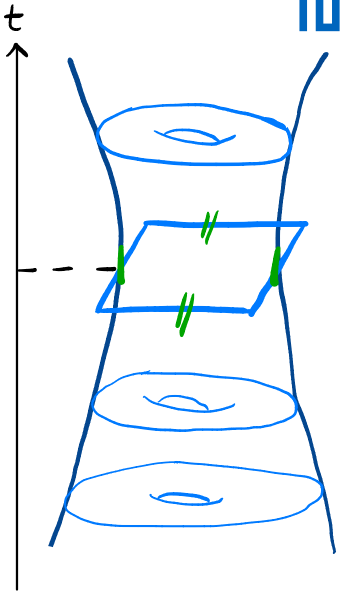
$$R = 0 \qquad \text{and} \qquad V = L^3 \tag{2.3}$$



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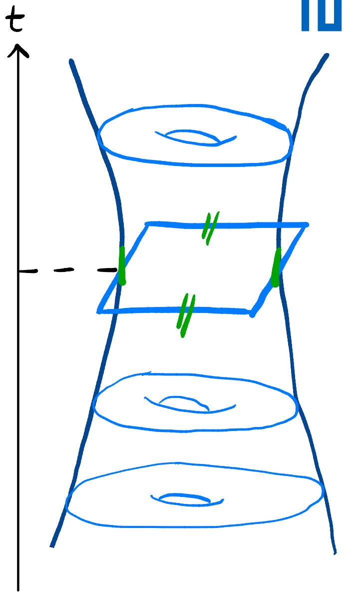
C the universe is empty and "cold" near the bounce

$$T \ll m \qquad (2.4)$$



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The ground state energy

- from thermodynamics

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- the thermal partition function from the Euclidean path-integral

$$Z = \int \mathcal{D}\phi e^{-S_E[\phi]} \tag{3.4}$$

The saddle point approximation

We need to evaluate the partition function from the Euclidean path-integral

$$Z = \int \mathcal{D}\phi e^{-S_E[\phi]} \approx \sum_{\bar{\phi}} e^{-S_E[\bar{\phi}]} \int \mathcal{D}\eta e^{-\int \eta S_E''[\bar{\phi}]\eta} = \sum_{\bar{\phi}} [\bar{\phi}] \tag{3.5}$$

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Saddle points solve the eom and have finite Euclidean action S_E

$$\left(-\partial_\tau^2 - \nabla^2\right)\overline{\phi} + V'(\overline{\phi}) = 0 \qquad (3.6)$$

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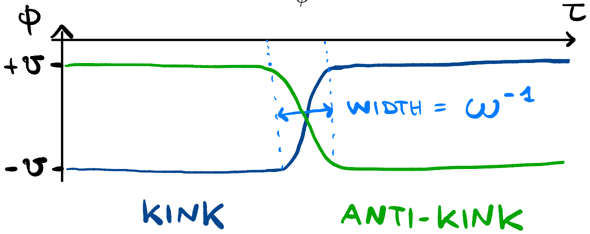

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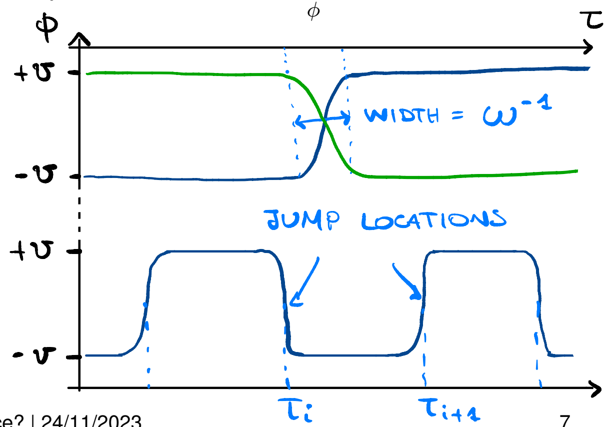

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3. multi-instantons ϕ_{N-I}

The dilute instanton gas approximation

- write $[\overline{\phi}]$ for multi-instantons as a product of one-instanton contributions

$$\left[K_1 \overline{K}_2 \ldots K_{N-1} \overline{K}_N\right] \approx \frac{T^N}{N!} [I]^N [v] \qquad (3.7)$$

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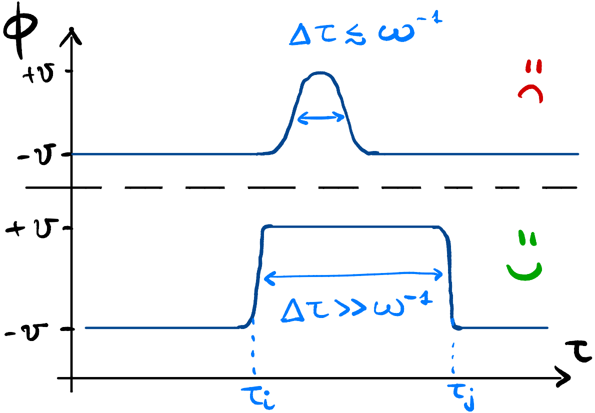
- extract the ground-state energy

$$E_0 = -\lim_{T\to\infty} \frac{1}{T} \log[v] - [I] \quad (3.9)$$



Instantons vs. Casimir effect

We split the energy as the sum of the vacuum energy and the instantons contribution

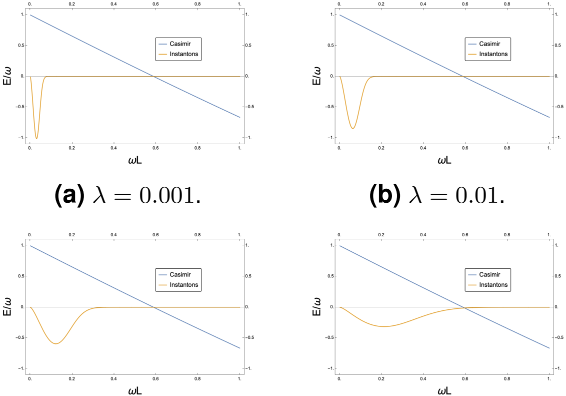
$$E_0 = E_{\text{cas}} + E_{\text{inst}} \tag{3.10}$$

The instantons effect is suppressed exponentially

$$E_{\text{inst}} \sim \sqrt{\frac{K}{2\pi}e^{-K}}\,, \quad K = \frac{16\omega^3 L^3}{\lambda} \tag{3.11}$$

It is only relevant for

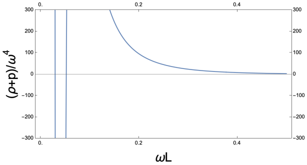
$$\omega L \sim \left(\frac{\lambda}{32}\right)^{\frac{1}{3}} \sim 0.3\,\lambda^{\frac{1}{3}} \tag{3.12}$$



(a) $\lambda = 0.001.$ **(b)** $\lambda = 0.01.$ **(c)** $\lambda = 0.1.$ **(d)** $\lambda = 1.$

Figure 2

Is the NEC violated?



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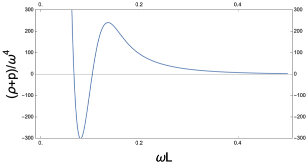

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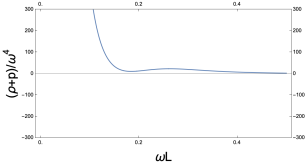
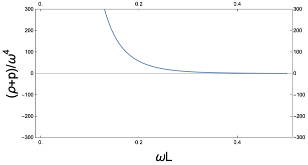


(c) $\lambda = 0.1.$



(d) $\lambda = 1.$

- yes! for small values of the coupling λ and for a narrow window of values of ωL [3]
- it is also violated for large ωL since the Casimir energy on a torus is negative for large ωL
- is the dilute instanton gas approximation valid where the NEC is violated?

Outline

Conclusions and outlook

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What next?

- take into account the competing effect of temperature
- include the slow variation of the metric and back-reaction
- calculate the effect for different spatial topologies

References

Carmen Molina-París and Matt Visser.
Minimal conditions for the creation of a friedman–robertson–walker universe from a "bounce".
Physics Letters B, 455(1–4):90–95, May 1999.

Jean Alexandre and Katy Clough.
Saving the universe with finite volume effects.
Phys. Rev. D, 100(10):103522, 2019.

Jean Alexandre, Matthias Carosi, Björn Garbrecht, and Silvia Pla.
Vacuum energy from instantons and Casimir effect in the early universe.
future publication, 2023-24?

THANK YOU FOR THE ATTENTION

When is the instanton gas really dilute?

- the DIGA is valid if the instantons are widely separated

$$\langle \Delta\tau \rangle \gg \omega^{-1} \tag{5.1}$$

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- the approximation holds if

$$[I] \ll \omega \tag{5.4}$$

Is the instanton gas dilute in the violation window?

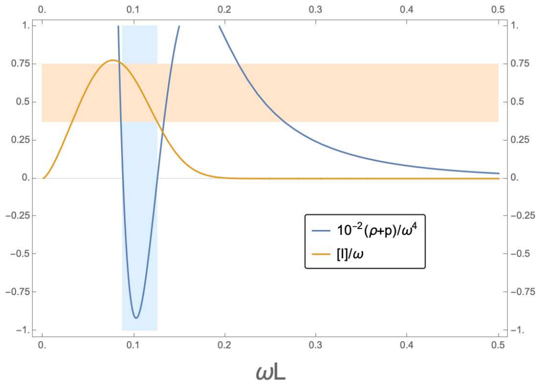

Figure 4 $\lambda = 0.02$

- $\rho + p$ is plotted together with the rescaled instanton rate $[I]/\omega$
- we highlight the region of NEC violation in blue and the corresponding region of values of $[I]/\omega$ in orange
- $[I]/\omega$ ranges from ~ 0.35 to ~ 0.75 in the region of interest

Functional determinants

Computing the saddle point approximation requires computing a functional determinant

$$[I] = e^{-K}\sqrt{\frac{K}{2\pi}}\left(\frac{\det'\left(-\partial_\tau^2 - \nabla^2 + 2\omega^2\left(\frac{3\phi_k^2}{v^2} - 1\right)\right)}{\det\left(-\partial_\tau^2 - \nabla^2 + 4\omega^2\right)}\right)^{-\frac{1}{2}} \tag{5.5}$$

There are various ways to compute the ratio of determinants:

- the resolvent method, for which we need to solve analytically for the Green's function of both operators
- computing directly the spectrum of the operators. After Fourier transforming the space variable, we can recognise the Pöschl-Teller operator for each Fourier mode

The resolvent method

- the logarithm of the ratio of determinants is given by a sum

$$\log\left(\frac{\det'\hat{\mathcal{O}}}{\det\hat{\mathcal{O}}_0}\right)^{-\frac{1}{2}} = -\frac{1}{2}\left(\sum_{\lambda}{}' \log\lambda - \sum_{\lambda_0}\log\lambda_0\right) \tag{5.6}$$

- this can be written in terms of the resolvent function, defined by

$$\left(\hat{\mathcal{O}}+s\right)G(s;x,x') = \delta^{(4)}(x-x') \Rightarrow G(s;x,x') = \sum_{\lambda}\frac{\phi_\lambda(x)\phi^*_\lambda(x')}{\lambda+s} \tag{5.7}$$

- thus we can write

$$\log\left(\frac{\det'\hat{\mathcal{O}}}{\det\hat{\mathcal{O}}_0}\right)^{-\frac{1}{2}} = \frac{1}{2}\int_0^\infty ds\int d^4x\left(G^\perp(s;x,x) - G_0(s;x,x)\right) \tag{5.8}$$

Renormalisation

- the 1-loop effective action of the one-instanton solution will read

$$\Gamma[\phi_{\text{kink}}] = K + \frac{1}{2}\log\det'\left(-\partial_\tau^2 - \nabla^2 + 2\omega^2\left(\frac{3\phi_k^2}{v^2} - 1\right)\right) \tag{5.9}$$

- we renormalise this by adding counter terms

$$\Gamma_{\text{ren}}[\phi_{\text{kink}}] = \Gamma[\phi_{\text{kink}}] + \int d^4x\left(V_0 + \frac{\delta m^2}{2}(\phi_{\text{kink}} - v^2) + \frac{\lambda}{4!}(\phi_{\text{kink}}^4 - v^4)\right) \tag{5.10}$$

- with renormalisation conditions

$$U_{\text{eff}}(\phi)|_{\phi=\pm v} = 0\,, \qquad \frac{d^2}{d\phi^2}U_{\text{eff}}(\phi)|_{\phi=\pm v} = m^2\,, \qquad \frac{d^4}{d\phi^4}U_{\text{eff}}(\phi)|_{\phi=\pm v} = \lambda \tag{5.11}$$

- with U_{eff} being the 1-loop Coleman-Weinberg potential in infinite-volume around the minimum $\pm v$

Some explicit expressions

$$E_{\text{cas}} = \frac{1}{2} \sum_{n^2=0}^{N_{\max}^2} r_3(n^2) \left(-k_n + \sqrt{k_n^2 + 4\omega^2} \right) - L^3 \frac{\omega^4}{4\pi^2} \left(\frac{\Lambda^2}{\omega^2} + \frac{1}{2} - \log \frac{\Lambda^2}{\omega^2} \right) , \tag{5.12}$$

$$\begin{aligned} E_{\text{inst}} = & -\sqrt{48\omega^2} \sqrt{\frac{8\omega^3 L^3}{\pi\lambda}} \exp \Bigg\{ -\frac{16\omega^3 L^3}{\lambda} \\ & + \sum_{n^2=1}^{N_{\max}^2} r_3(n^2) \left(\operatorname{arcsech} \sqrt{\frac{k_n^2 + 3\omega^2}{k_n^2 + 4\omega^2}} + \operatorname{arcsinh} \frac{2\omega}{k_n} \right) \\ & - \frac{L^3 \omega^3}{2\pi^2} \left(\frac{3}{2} \frac{\Lambda^2}{\omega^2} - \frac{3}{2} \log \frac{\Lambda^2}{\omega^2} - \frac{63}{2} \right) \Bigg\} , \end{aligned} \tag{5.13}$$

$$k_n = \frac{2\pi n}{L} , \qquad \Lambda = k_{N_{\max}} \tag{5.14}$$